Probability theory III solutions exam 2024-12-05

$\boxed{1}$ An integer-valued random variable N is a stopping time wrt $(\mathcal{F}_n)_{n\geq 1}$ if for each $n\geq 1$ we have $\{N=n\}\in\mathcal{F}_n$. 2p

- We may describe $\{T=n\}$ as the event that the random walk visits -1 for the first time in its nth step, which is determined by $S_1,\dots,S_n$ and is hence in $\mathcal{F}_n$.
- U denotes the last visit of the random walk to -1, so the event $\{U=n\}$ is a function of $S_n, S_{n+1},\dots$ and hence <u>not</u> in $\mathcal{F}_n$.
- V denotes the least n for which $(S_0, S_1,\dots,S_n)$ contains k upwards steps in a row. Hence $\{V=n\}$ is contained in $\mathcal{F}_n$. 3p

In conclusion, T and V are stopping times, but not U.

$\boxed{2}$ Since $\mathbb{P}$ and $\mathbb{P}'$ are probability measures on $(\Omega,\mathcal{F})$ it is clear that

$$\mathbb{Q}(A) = \frac{1}{2}\left[\mathbb{P}(A)+\mathbb{P}'(A)\right]$$

is always a number in $[0,1]$. We then need to verify that

(i) $\mathbb{Q}(\Omega)=1$. This holds since

$$\mathbb{Q}(\Omega)=\frac{1}{2}\left[\mathbb{P}(\Omega)+\mathbb{P}'(\Omega)\right]=\frac{1}{2}\left[1+1\right]=1.$$

(ii) For $A_1, A_2,\dots$ pairwise disjoint we have

$$\mathbb{Q}\Big(\bigcup_k A_k\Big)=\sum_k \mathbb{Q}(A_k).$$

This holds since true for $\mathbb{P}$ and $\mathbb{P}'$: 5p

$$\mathbb{Q}\Big(\bigcup_k A_k\Big)=\frac{1}{2}\mathbb{P}\Big(\bigcup_k A_k\Big)+\frac{1}{2}\mathbb{P}'\Big(\bigcup_k A_k\Big)$$

$$=\frac{1}{2}\sum_k \mathbb{P}(A_k)+\frac{1}{2}\sum_k \mathbb{P}'(A_k)=\sum_k \mathbb{Q}(A_k).$$

3 Let $X_1, X_2, \ldots$ be independent taking values ± 1 with equal probability. Set $S_n = X_1 + \ldots + X_n$. For $n \geq 0$ let

$$A_n = \{ X_{n+1} = X_{n+2} = \ldots = X_{n+2m+1} = 1 \}.$$

Note that if $S_n \in [-m, m]$ and A_n occurs, then $S_{n+2m+1} \geq m+1$. Hence it will suffice to show that A_n occurs for some $n \geq 0$ with probability one. 2p

For each n we have $\mathbb{P}(A_n) = \left(\frac{1}{2}\right)^{2m+1} > 0$, and for $k \geq 0$ the events $A_{k(2m+1)}$ are independent. Now set

$$N = \min\{ k \geq 0 : A_{k(2m+1)} \text{ occurs} \}.$$

Then

$$\begin{aligned} \mathbb{P}(\text{never } A_n) &\leq \mathbb{P}(N = \infty) \\ &\leq \mathbb{P}(N > k) = \mathbb{P}(A_0)^k \to 0 \text{ as } k \to \infty. \end{aligned}$$ 3p

Hence $\mathbb{P}(\text{never } A_n) = 0$.

Alt Use Borel-Cantelli.

4 Since $X_n \to X$ in probability, we have for each $\varepsilon > 0$

$$\mathbb{P}(|X_n - X| > \varepsilon) \to 0 \quad \text{as } n \to \infty.$$ 2p

By triangle inequality

$$|X| \leq |X_n - X| + |X_n|.$$

So, if $|X| > K+1$, then $|X_n| > K$ or $|X_n - X| > 1$. It follows that

$$\begin{aligned} \mathbb{P}(|X| > K+1) &\leq \mathbb{P}(|X_n| > K) + \mathbb{P}(|X_n - X| > 1) \\ &= \mathbb{P}(|X_n - X| > 1) \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$ 3p

Hence $\mathbb{P}(|X| > K+1) = 0$.

$\boxed{5}$ Let A_n denote the event that the nth experiment is successful. Then $A_1, A_2, \ldots$ are independent and

$$\mathbb{P}(A_n) = \frac{1}{n^\alpha}$$

for some $\alpha \in (0,1)$. For an integer $m \geq 1$, let

$$B_n = \bigcap_{k=1}^{m} A_{n+k}$$

denote the event that trials $n+1, n+2, \ldots, n+m$ are all successful. By independence

$$\mathbb{P}(B_n) = \prod_{k=1}^{m} \mathbb{P}(A_{n+k}) = \prod_{k=1}^{m} \frac{1}{(n+k)^\alpha} \qquad 4p$$

In particular, for $n \geq m$, we have

$$\frac{1}{(2n)^{\alpha m}} \leq \mathbb{P}(B_n) \leq \frac{1}{n^{\alpha m}}. \qquad 2p$$

For $\alpha m > 1$ we have

$$\sum_{n\geq 1} \mathbb{P}(B_n) \leq \sum_{n \geq 1} \frac{1}{n^{\alpha m}} < \infty,$$

so Borel-Cantelli's first lemma gives $\mathbb{P}(B_n \text{ i.o.}) = 0$.

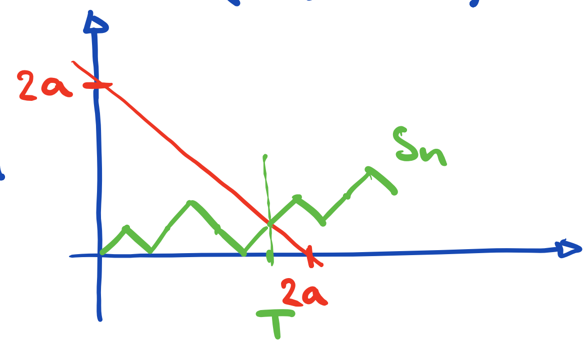
For $k \geq 1$ the events B_{km} are independent and

$$\sum_{k\geq 1} \mathbb{P}(B_{km}) \geq \sum_{k\geq 1} \frac{1}{(2mk)^{\alpha m}} = \infty \qquad 4p$$

for $\alpha m \leq 1$. So Borel-Cantelli's second lemma gives

$$\mathbb{P}(B_n \text{ i.o.}) \geq \mathbb{P}(B_{km} \text{ i.o.}) = 1.$$

$\boxed{6}$ (a) We need to verify that $\{T=n\} \in \sigma(S_1, \dots, S_n)$ for every $n \geq 1$. We have



$$\{T=n\} = \{S_0 < 2a\} \cap \{S_1 < 2a-1\} \cap \dots \cap \{S_{n-1} < 2a-n+1\} \cap \{S_n \geq 2a-n\}$$

Since this event can be expressed in terms of $S_1, \dots, S_n$ it is in $\sigma(S_1, \dots, S_n)$. Hence T is a stopping time.

3p

(b) We next note that in each step

$$Y_n = S_n + n$$

either stays equal or increases by 2 since

$$Y_{n+1} - Y_n = S_{n+1} - S_n + 1 = X_{n+1} + 1$$

which equals 2 if walk jumps up and 0 otw.

Since we may write

$$T = \min\{n \geq 1 : Y_n \geq 2a\}$$

we may describe T as the instant the walk has completed a upwardssteps. Hence T has a negative binomial distribution, so $\mathbb{E}[T] < \infty$. 4p

(c) Let T_1 denote the time of the first upwardsstep. Then T_1 has a geometric distribution with mean 2, and T is distributed as a independent copies of T_1. Hence

$$\mathbb{E}[T] = a\,\mathbb{E}[T_1] = 2a.$$

Alt. Since T is a stopping time with finite mean and $(S_n)_{n\geq 0}$ has bounded increments, the Optional stopping theorem III gives

$$\mathbb{E}[S_T] = \mathbb{E}[S_0] = 0.$$

Moreover, since $(Y_n)_{n\geq 0}$ is a walk on the even integers, we have

3p

$$2a = Y_T = S_T + T.$$

The two equations give $\mathbb{E}[T] = 2a$.

$\boxed{7}$ (a) We note that X is simple, taking the value $k = 0, 1, \dots, 9$ on the interval $I_0, I_1, \dots, I_9$, where

$$I_0 = [0, \tfrac{1}{10}) \quad I_1 = [\tfrac{1}{10}, \tfrac{2}{10}) \quad \dots \quad I_9 = [\tfrac{9}{10}, \tfrac{10}{10})$$

These sets form a finite partition of $[0,1)$. The σ-algebra generated by X consists of all sets of the form

$$X^{-1}(B) = \{\omega \in [0,1) : X(\omega) \in B\}$$

for Borel sets B. These sets can be expressed as

$$X^{-1}(B) = \bigcup_{k \in B} I_k$$

and are thus sets obtained by forming unions of $I_0, I_1, \dots, I_9$. The σ-algebra generated by X thus coincides with the σ-algebra generated by the partition $I_0, I_1, \dots, I_9$. 3p

Since $I_0, I_1, \dots, I_9 \in \mathcal{B}[0,1)$, we have $\sigma(X) \subseteq \mathcal{B}[0,1)$ and X is a random variable. 2p

(b) The conditional expectation of Y given X is, since X is simple, the random variable $\mathbb{E}[Y|X]$ which on $\{X=k\} = I_k$ takes the value $\mathbb{E}[Y|X=k]$. For $[a,b] \subseteq I_k$ we have 2p

$$\mathbb{P}(Y \in [a,b] \mid X=k) = \frac{\mathbb{P}(Y \in [a,b])}{\mathbb{P}(I_k)} = 10(b-a),$$

so the conditional distribution is uniform on I_k. Hence

$$\mathbb{E}[Y|X=k] = \frac{k+1+k}{2\cdot 10} = \frac{2k+1}{20}.$$

3p

Alternatively, use that

$$\mathbb{E}[Y|X=k] = \frac{\mathbb{E}[Y \cdot \mathbb{1}_{I_k}]}{\mathbb{P}(I_k)} = \frac{\int_{I_k} y\,dy}{10} = \frac{2k+1}{20}.$$

$\boxed{5}$ (a) The problem describes a version of Polya's urn. Let Y_n denote the proportion of red balls in the urn after n rounds. We aim to show that $(Y_n)_{n\geq 0}$ is a martingale. Then, since $(Y_n)_{n\geq 0}$ is bounded, it follows from the martingale convergence thm that the sequence has an almost sure limit. 2p

After n steps there are $r+b+2n$ balls in the urn. For $n \geq 1$ let

$$X_n = \begin{cases} 1 & \text{red drawn round } n \\ 0 & \text{otw} \end{cases}$$

Clearly Y_n is $\sigma(X_1, \dots, X_n)$-measurable. Moreover

$$Y_{n+1} - Y_n = \frac{r + 2\sum_{k=1}^{n+1} X_k}{r+b+2(n+1)} - \frac{r + 2\sum_{k=1}^{n} X_k}{r+b+2n}$$

$$= \frac{1}{r+b+2(n+1)} \left[\cancel{r} + 2\sum_{k=1}^{n+1} X_k - \left[\cancel{r} + 2\sum_{k=1}^{n} X_k\right] - 2Y_n \right]$$

$$= \frac{1}{r+b+2(n+1)} \left[2X_{n+1} - 2Y_n \right]$$

3p

Consequently,

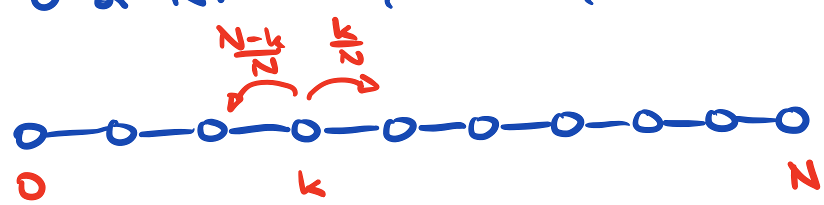
$$\mathbb{E}[Y_{n+1} - Y_n \mid X_1, \dots, X_n] = \frac{1}{r+b+2(n+1)} \left[2\mathbb{E}[X_{n+1} \mid Y_n] - 2Y_n \right]$$

$$= \frac{1}{r+b+2(n+1)} \left[2Y_n - 2Y_n \right] = 0.$$

Hence $(Y_n)_{n\geq 0}$ is a martingale wrt itself.

(b) When the drawn colour is reinforced by 1 and the remaining colour is decreased by 1, the total number of balls in the urn stays constant equal to $N = r+b$.

In this case, the number of red balls Z_n in the urn performs a random walk on $\mathbb{Z}$ that is stopped when it hits 0 or N. Note, however, that the jump probab.



2p

depend on the current position of the walk.

We note further that if the walk ever reaches 0 or N, then it is absorbed, and so is the proportion Y_n of red balls. If not absorbed the walk continues to jump around. Hence convergence a.s. is equivalent to absorbtion.

We want to show absorbtion, and note that if red is drawn N times in a row, then no blue balls may remain. Since the probability of a red draw is bounded from below by $1/N$ as long as there are red balls remaining, the probability of N red draws in a row is bounded from below by $\frac{1}{N^N} > 0$. Hence, as long as there are red balls remaining, we will eventually draw N consecutive reds. Hence $(Y_n)_{n\geq 0}$ converges a.s. to either 0 or 1. 3p

To be more precise, we may distinguish one of the red balls, and decide to keep this ball as long as there are other reds to remove. Finally, when eventually the distinguished red has to be removed, we may replace it by a distinguished blue. Then absorbtion is forced if the distinguished ball is drawn N consecutive times. In each step it occurs independently of previous steps with probability $1/N$.