MT7047 – Probability theory III – exam

Date Friday October 24, 2025
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Tools None.
Grading criteria The exam consists of two parts, which consist of 20 and 40 points respectively. To pass the exam a score of 14 or higher is required on Part I. If attained, then also Part II is graded, and the score on this part determines the grade. Grades are determined according to the following table:

	A	B	C	D	E
Part I	14	14	14	14	14
Part II	32	24	16	8	0

Problems of Part I may give up to five points each, and problems of Part II may give up to ten points each. Complete and well motivated solutions are required for full score. Partial solution may be rewarded with a partial score.

Part I

Problem 1. A regular deck consists of 52 cards divided into four suits and numbered 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King, Ace. In the game of *Black Jack* the cards 2 to 9 are valued accordingly, whereas 10, Jack, Queen and King each have the value 10, and Ace have the value 11 or 1. (Suit is irrelevant.) The goal is to obtain a score as close as possible to, but not exceeding, 21. Two cards which together sums to 21 are referred to as *blackjack*.

(a) Construct a probability space corresponding to the experiment of drawing two cards from a well-shuffled deck.

(b) Let X denote the value of the first card drawn, and compute the conditional probability of obtaining blackjack in the next draw given X.

Problem 2. Let Ω be some set, $\mathcal{F}$ some σ-algebra on Ω and $\mathbb{P}$ and $\mathbb{P}'$ two probability measures on $(\Omega, \mathcal{F})$. For every $A \in \mathcal{F}$ let

$$\mathbb{Q}(A) := \frac{1}{2}\mathbb{P}(A) + \frac{1}{2}\mathbb{P}'(A).$$

Then $\mathbb{Q}$ is again a probability measure on $(\Omega, \mathcal{F})$. Let X be a random variable on $(\Omega, \mathcal{F})$ which under $\mathbb{P}$ is uniformly distributed on $[0, 1]$ and under $\mathbb{P}'$ is uniformly distributed on $[0, 2]$. Determine the distribution function for X with respect to $\mathbb{Q}$.

Problem 3. Let $\Omega = [0, 1)$ and $X : \Omega \to \mathbb{R}$ be given by $X(\omega) = \lfloor 10\omega \rfloor$, where $\lfloor \cdot \rfloor$ denotes the integer part. Describe the σ-algebra generated by X, and determine the number of elements it contains.

Problem 4. Consider the following sequence $(S_n)_{n\geq 0}$ describing the game 'double or nothing': Let $X_1, X_2, \ldots$ be i.i.d. random variables taking values ± 1 with equal probability, and $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$. Set $S_0 = 1$ and let

$$S_{n+1} - S_n = S_n \cdot X_{n+1} \quad \text{for } n \geq 0.$$

Show that $(S_n)_{n\geq 0}$ is a martingale with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$.

Part II

Problem 5. Consider the sequence $(S_n)_{n\geq 0}$ from Problem 4 describing the game 'double or nothing'. Let $T = \min\{n \geq 1 : S_n = 0 \text{ or } S_n = 2^m\}$, where $m \geq 1$ is an integer.

(a) Show that $\mathbb{P}(S_T = 2^m) = 2^{-m}$.

(b) Show that, almost surely, $S_n = 0$ for all large n.

Problem 6. Let $X_1, X_2, \ldots$ be independent Bernoulli-distributed random variables with parameter $p = 1/2$. Set $U = \sum_{k=1}^{\infty} X_k 2^{-k}$, and for $n \geq 1$ let $Z_n = \mathbb{E}[U|X_1, X_2, \ldots, X_n]$.

(a) Show that the limit $Z := \lim_{n\to\infty} Z_n$ exists almost surely.

(b) Determine $\mathbb{E}[Z^2]$.

Problem 7. An urn contains one red and one blue ball initially. In round k a ball is drawn and replaced, together with an additional k balls of the drawn colour. Let N_n denote the number of times the red ball initially present is drawn in the first n rounds, and let $N = N_\infty$.

(a) Show that $\mathbb{P}(N = \infty) = 0$.

(b) Let B_m denote the event that the first m drawn balls are blue. Show that there exists m such that $\mathbb{E}[N_n|B_m] \leq 1/2$ for all $n \geq 1$.

(c) Show that there exists m such that $\mathbb{P}(N \geq 1|B_m) \leq 1/2$.

Problem 8. Consider the probability space $([0,1], \mathcal{B}[0,1], \mathbb{P})$, where $\mathbb{P}$ denotes Lebesgue measure. Let $X(\omega) = 2\omega$ and $Y(\omega) = \mathbf{1}_{[1/2,1]}(\omega)$, and let $\mathbb{E}$ denote expectation with respect to $\mathbb{P}$.

(a) Show that $\mathbb{P}'(B) := \mathbb{E}[X\,\mathbf{1}_B]$ is a probability measure on $([0,1], \mathcal{B}[0,1])$.

(b) Let $\mathbb{E}'$ denote expectation with respect to $\mathbb{P}'$. Show that $\mathbb{E}'[Z] = \mathbb{E}[ZX]$ for every bounded random variable Z.