Probability Theory III exam 2025-10-24

1 (a) We may identify the 52 cards with the numbers $1,2,\dots,52$. The different possible outcomes when two cards are drawn are

$$\Omega = \{(i,j) : i,j \in \{1,2,\dots,52\},\ i \neq j\}.$$

Since Ω is finite we equip Ω with $\mathcal{F}$ being the power set. Finally, since the deck is well-shuffled, we assume each pair of cards are equally likely:

$$\mathbb{P}((i,j)) = \frac{1}{|\Omega|} = \frac{1}{n(n-1)} \qquad \text{for every } (i,j) \in \Omega. \qquad 2\text{p}$$

This defines a probability measure uniquely.

(b) The possible values of X are $2,3,\dots,11$. (When having only two cards we want to count the first Ace as 11.)

Let B be the event that the value of the two cards give 21. The conditional probability $\mathbb{P}(B|X)$ is the rand. var.

$$\mathbb{P}(B|X) = \sum_{k=2}^{11} \mathbb{P}(B|X=k)\,\mathbb{1}_{\{X=k\}}.$$

Hence $\mathbb{P}(B|X)$ is determined by

$$\mathbb{P}(B|X=k) = \begin{cases} 0 & \text{if } X = 2,3,\dots,9 \\ 4/51 & \text{if } X = 10 \\ 16/51 & \text{if } X = 11 \end{cases} \qquad 3\text{p}$$

2 We want to compute

$$F_X(x) := \mathbb{Q}(X \leq x) \qquad \text{for } x \in \mathbb{R}.$$

We have

$$\mathbb{Q}(X \leq x) = \frac{1}{2}\left[\mathbb{P}(X \leq x) + \mathbb{P}'(X \leq x)\right]$$

$$= \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{1}{2}\left[x + \frac{x}{2}\right] = \frac{3}{4}x & \text{if } x \in (0,1] \\ \frac{1}{2}\left[1 + \frac{x}{2}\right] = \frac{1}{2} + \frac{x}{4} & \text{if } x \in (1,2] \\ 1 & \text{if } x > 2 \end{cases} \qquad 5\text{p}$$

[3] We have $\Omega = [0,1)$ and $X(\omega) = \lfloor 10\omega \rfloor$. Hence X takes the values $0, 1, 2, \dots, 9$. Any set of the form

$$X^{-1}(B) = \{\omega \in [0,1) : X(\omega) \in B\},$$

for some Borel set B, is thus the union of sets in

$$\mathcal{H} = \{\{X=0\}, \{X=1\}, \dots, \{X=9\}\}$$

These set coincide with the intervals

$$[0, 0.1), [0.1, 0.2), \dots, [0.9, 1).$$

3p

The σ-algebra generated by X coincides with the σ-algebra generated by this partition of Ω. Since 2^{10} different sets can be generated taking unions of these sets, the σ-algebra contains 2^{10} sets. 2p

[4] Since $S_{n+1} = S_n(1+X_{n+1})$ it follows that either $S_{n+1} = 2S_n$ or $S_{n+1} = 0$. In particular,

$$|S_{n+1}| \le 2|S_n| \le \dots \le 2^n |S_0| = 2^n,$$

so $\mathbb{E}[|S_{n+1}|] < \infty$.

Note further that $S_1 = (1+X_1)$ which is $\mathcal{F}_1$-measurable. Assuming S_n is $\mathcal{F}_n$-measurable, it follows that

$$S_{n+1} = S_n(1+X_{n+1})$$

is $\mathcal{F}_{n+1}$-measurable. Hence, by induction, S_n is $\mathcal{F}_n$-measurable for every $n \ge 1$.

Finally note that

$$\begin{aligned} \mathbb{E}[S_{n+1} - S_n \mid \mathcal{F}_n] &= \mathbb{E}[S_n X_{n+1} \mid \mathcal{F}_n] \\ &= S_n \mathbb{E}[X_{n+1}] \\ &= 0. \end{aligned}$$

5p

Hence $(S_n)_{n \ge 0}$ is a martingale wrt $(S_n)_{n \ge 0}$.

5 (a) We know from the previous exercise that $(S_n)_{n\geq 0}$ is a martingale. We also note that T is a stopping time as

$$\{T=n\} = \{0<S_1<2^m\} \cap \dots \cap \{0<S_{n-1}<2^m\} \cap \{S_n=0 \text{ or } S_n=2^m\}$$

which belongs to $\mathcal{F}_n$. Moreover, once S_n hits 0 it remains at 0 for the future. Hence, either $S_n=2^m$ or $S_n=0$. In particular, $T\leq m$ and is hence bounded.

The first optional stopping theorem gives

$$\begin{aligned} 1 = \mathbb{E}[S_0] &= \mathbb{E}[S_T] \\ &= 0\cdot \mathbb{P}(S_T=0) + 2^m \cdot \mathbb{P}(S_T=2^m) \end{aligned}$$

6p

Hence $\mathbb{P}(S_T=2^m) = 2^{-m}$.

(b) Since $S_n=0$ implies $S_{n+1}=0$, it will suffice to show that $S_n=0$ for some $n\geq 1$. Note that $X_n=0$ implies $S_n=0$, so it will suffice to show that $X_n=0$ for some $n\geq 1$ almost surely. Indeed,

$$\begin{aligned} \mathbb{P}(X_1=X_2=\dots=1) &\leq \mathbb{P}(X_1=\dots=X_n=1) \\ &= \mathbb{P}(X_1=1)\cdots\mathbb{P}(X_n=1) \\ &= (1/2)^n \to 0 \quad \text{as } n\to\infty. \end{aligned}$$

4p

Hence, $S_n=0$ for all n large, almost surely.

6 (a) We recognize $(Z_n)_{n\geq 1}$ as the increasing information martingale: Clearly $|Z_n| \leq \sum_{k=1}^{\infty} 2^{-k} = 1$ and Z_n is $\sigma(X_1,\dots,X_n)$-measurable by definition. Also

$$\mathbb{E}[Z_{n+1} \mid X_1,\dots,X_n] = \mathbb{E}[U \mid X_1,\dots,X_n] = Z_n.$$

Since $(Z_n)_{n\geq 1}$ is bounded it follows from the martingale convergence theorem that $\lim_{n\to\infty} Z_n$ exists almost surely.

4p

(b) Since $(Z_n)_{n\geq 1}$ is bounded, the bounded convergence theorem gives that

$$\mathbb{E}[Z^2] = \mathbb{E}[\lim_{n\to\infty} Z_n^2] = \lim_{n\to\infty} \mathbb{E}[Z_n^2].$$

Using that martingale increments are uncorrelated gives

$$\begin{aligned}\mathbb{E}[Z_n^2] &= \sum_{k=1}^{n} \mathbb{E}[(Z_k - Z_{k-1})^2] + \mathbb{E}[Z_0^2] \\ &= \sum_{k=1}^{n} \mathbb{E}[(X_k - \mathbb{E}[X_k])^2 \cdot \tfrac{1}{2^{2k}}] + \mathbb{E}[U]^2 \\ &= \sum_{k=1}^{n} \tfrac{1}{4}\cdot\tfrac{1}{4^k} + \tfrac{1}{4}\end{aligned}$$

6p

Hence

$$\mathbb{E}[Z^2] = \sum_{k=1}^{\infty} \frac{1}{4^k} = \frac{1}{4}\cdot\frac{1}{1-1/4} = \frac{1}{4}\cdot\frac{4}{3} = \frac{1}{3}.$$

7 (a) After n rounds there are $2+\sum_{k=1}^{n} k = 2+n(n+1)$ balls in the urn, so in round $n+1$ specific ball has probab

$$\frac{1}{2+n(n+1)}, \qquad \text{for } n\geq 0.$$

to be drawn. Let A_k denote the event that the initial red ball is drawn in round k. Then

$$\sum_{k\geq 1} \mathbb{P}(A_k) = \sum_{k\geq 1} \frac{1}{2+k(k-1)} \leq \sum_{k\geq 2} \frac{1}{k^2} < n.$$

3p

By Borel-Cantelli's first lemma $\mathbb{P}(A_k \text{ i.o.}) = 0$.

(b) Note that $N_n = \sum_{k=1}^{n} \mathbb{1}_{A_k}$. The expectation of N_n wrt $\mathbb{P}(\cdot\,|B_m)$ is then

$$\begin{aligned}\mathbb{E}[N_n|B_m] &\overset{\text{lin.}}{=} \sum_{k=1}^{n} \mathbb{E}[\mathbb{1}_{A_k}|B_m] = \sum_{k=1}^{n} \mathbb{P}(A_k|B_m) \\ &= \sum_{k=m+1}^{n} \mathbb{P}(A_k) \leq \sum_{k=m+1}^{\infty} \mathbb{P}(A_k) \leq 1/2,\end{aligned}$$

3p

for large enough values of m.

(c) Since N_n is increasing in n, we have by continuity of measure that

$$\mathbb{P}(N \geq 1 \mid B_m) = \lim_{n \to \infty} \mathbb{P}(N_n \geq 1 \mid B_m) \qquad 2p$$

for each $m \geq 1$ being fixed. Using Markov's inequality we obtain, for m chosen as in (b), that

$$\mathbb{P}(N \geq 1 \mid B_m) \leq \lim_{n \to \infty} \mathbb{E}[N_n \mid B_m] \leq 1/2. \qquad 2p$$

[8] (a) We first note that since $X \geq 0$ we have for every Borel set B that

$$0 \leq \mathbb{E}[X 1_B] \leq \mathbb{E}[X 1_{[0,1]}] = \mathbb{E}[X].$$

Moreover

$$\mathbb{E}[X] = \int_0^1 2x\,dx = [x^2]_0^1 = 1.$$

Hence $0 \leq \mathbb{P}'(B) \leq \mathbb{P}'([0,1]) = 1$. $2p$

Next, for $B_1, B_2, \ldots$ that are pairwise disjoint

$$\mathbb{P}'\Big(\bigcup_{k\geq 1} B_k\Big) = \mathbb{E}\Big[X 1_{\bigcup_{k\geq 1} B_k}\Big] = \mathbb{E}\Big[\sum_{k\geq 1} X 1_{B_k}\Big]$$

$$\xrightarrow{\text{BCT + linearity}} = \sum_{k\geq 1} \mathbb{E}[X 1_{B_k}] = \sum_{k\geq 1} \mathbb{P}'(B_k). \qquad 2p$$

That the order of summation and expectation can be changed follows from the bounded convergence theorem. Hence $\mathbb{P}'$ defines a probability measure.

(b) We first show $\mathbb{E}'[Z] = \mathbb{E}[ZX]$ when Z is simple, e.g. of the form

$$Z = \sum_{k=1}^n a_k \cdot 1_{A_k},$$

for some partition $A_1, \ldots, A_n$ of $[0,1]$. We obtain

$$\mathbb{E}'[Z] = \sum_{k=1}^n a_k \mathbb{P}'(A_k) = \sum_{k=1}^n a_k \mathbb{E}[X 1_{A_k}] \qquad 2p$$

$$\xrightarrow{\text{linearity of } \mathbb{E}} = \mathbb{E}\Big[\sum_{k=1}^n a_k X 1_{A_k}\Big] = \mathbb{E}[XZ].$$

Next suppose Z is bounded. Then by definition

$$\mathbb{E}'[Z] = \sup\{\mathbb{E}'[Y] : Y \le Z, \text{ simple}\}$$
$$= \inf\{\mathbb{E}'[Y] : Y \ge Z, \text{ simple}\}.$$

For $Y \le Z$ that is simple we have from previous step and monotonicity of $\mathbb{E}$ that

$$\mathbb{E}'[Y] = \mathbb{E}[XY] \overset{\text{mon}}{\le} \mathbb{E}[XZ].$$

Hence, from sup definition we get

$$\mathbb{E}'[Z] \le \mathbb{E}[XZ].$$

Similarly, for $Y \ge Z$ that is simple we have

$$\mathbb{E}'[Y] = \mathbb{E}[XY] \overset{\text{mon}}{\ge} \mathbb{E}[XZ].$$

4p

So inf definition gives

$$\mathbb{E}'[Z] \ge \mathbb{E}[XZ].$$

In conclusion: $\mathbb{E}'[Z] = \mathbb{E}[XZ]$ for Z bounded.