MT7047 – Probability theory III – exam

Date Friday December 12, 2025
Examiner Daniel Ahlberg
Tools None.
Grading criteria The exam consists of two parts, which consist of 20 and 40 points respectively. To pass the exam a score of 14 or higher is required on Part I. If attained, then also Part II is graded, and the score on this part determines the grade. Grades are determined according to the following table:

	A	B	C	D	E
Part I	14	14	14	14	14
Part II	32	24	16	8	0

Problems of Part I may give up to five points each, and problems of Part II may give up to ten points each. Complete and well motivated solutions are required for full score. Partial solution may be rewarded with a partial score.

Part I

Problem 1. Let $\mathcal{B}(\mathbb{R})$ denote the Borel σ-algebra on $\mathbb{R}$, i.e. the σ-algebra generated by all sets of the form (a, b) for real numbers $a < b$. Show that $\{x\}$, $(-\infty, x]$ and $[x, \infty)$ belong to $\mathcal{B}(\mathbb{R})$ for all $x \in \mathbb{R}$.

Problem 2. Let X and Y be independent random variables, where X is exponentially distributed with mean 1, and Y is uniformly distributed on the interval $[0, 1]$. Compute the expectation of Z, where

$$Z = \begin{cases} XY & \text{if } Y \leq 1/2, \\ X/2 & \text{if } Y > 1/2. \end{cases}$$

Problem 3. Let $X_1, X_2, \ldots$ be independent random variables taking values ± 1 with equal probability. Let $S_0 = 0$ and set $S_n = \sum_{k=1}^{n} X_k 2^{-k}$. Show that the limit $\lim_{n\to\infty} S_n$ exists almost surely.

Problem 4. Let $X_1, X_2, \ldots$ be independent exponentially distributed random variables with intensity one, i.e. whose distribution function is given by $F(x) = 1 - e^{-x}$ for $x \geq 0$. Let $N_n = \#\{1 \leq k \leq n : X_k > \log k\}$ and set $N = \lim_{n\to\infty} N_n$. (Here and elsewhere log denotes the natural logarithm.)

(a) Prove that $\mathbb{E}[N] = \infty$.

(b) Prove that $\mathbb{P}(N = \infty) = 1$.

Part II

Problem 5. Consider the probability space $([0,1], \mathcal{B}[0,1], \mathbb{P})$, where $\mathbb{P}$ denotes Lebesgue measure. Let $X(\omega) = 2\omega$ and $Y(\omega) = \mathbf{1}_{[1/2,1]}(\omega)$, and let $\mathbb{E}$ denote expectation with respect to $\mathbb{P}$.

(a) Determine the σ-algebra generated by Y.

(b) Determine whether X and Y are independent or not.

(c) Determine $\mathbb{E}[XY]$.

Problem 6. Let $X_1, X_2, \ldots$ be independent random variables with mean zero and where $\mathrm{Var}(X_k) = 2^{-k}$. For $n \geq 1$, let $S_n = X_1 + \ldots + X_n$ and set

$$A_n = \{|X_n| \geq 1\} \quad \text{and} \quad B_n = \{|S_n| \geq \sqrt{n}\}.$$

(a) Determine the probability that A_n occurs for infinitely many $n \geq 1$.

(b) Show that $\lim_{n\to\infty} S_n/\sqrt{n} = 0$ almost surely.

(c) Determine the probability that B_n occurs for infinitely many $n \geq 1$.

Problem 7. Let $X_1, X_2, \ldots$ be i.i.d. exponentially distributed random variables with mean 1. For $n \geq 1$ let $S_n = X_1 + \ldots + X_n$, $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ and set

$$T = \min\{n \geq 1 : S_n \geq 100\}.$$

(a) Show that T is a stopping time with respect to $(\mathcal{F}_n)_{n\geq 1}$.

(b) Show that $\mathbb{E}[T] = \mathbb{E}[S_T]$.

(c) Show that $\mathbb{E}[T] \geq 100$, and argue (informally if you like) why in fact $\mathbb{E}[T] = 101$.

Problem 8. Consider an urn containing one red, one blue and one green ball. In each round a ball is drawn, and replaced together with three additional balls according to the following rule: If a red ball is drawn, then add two red and one green; if blue is drawn, then add two blue and one green; if green is drawn, then add one ball of each colour. Let Y_n denote the proportion of red balls in the urn after n rounds, and let $f(x)$ denote the expected number of red ball is added when the current proportion of red balls equals x.

(a) Show that $f(x) = \frac{1}{3}(6x + 1)$.

(b) Show that $(Y_n)_{n\geq 0}$ is a stochastic approximation process, i.e. satisfies

$$Y_{n+1} - Y_n = \frac{1}{3n+6}[F(Y_n) + \xi_{n+1}],$$

with $F(x) = f(x) - 3x$ and $\mathbb{E}[\xi_{n+1}|Y_1, \ldots, Y_n] = 0$.

(c) Show that the fractions of red, blue and green balls converge, almost surely, to 1/3 each.