MT7047 – Probability theory III – exam

Date Wednesday April 15, 2026
Examiner Daniel Ahlberg
Tools None.
Grading criteria The exam consists of two parts, which consist of 20 and 40 points respectively. To pass the exam a score of 14 or higher is required on Part I. If attained, then also Part II is graded, and the score on this part determines the grade. Grades are determined according to the following table:

	A	B	C	D	E
Part I	14	14	14	14	14
Part II	32	24	16	8	0

Problems of Part I may give up to five points each, and problems of Part II may give up to ten points each. Complete and well motivated solutions are required for full score. Partial solution may be rewarded with a partial score.

Part I

Problem 1. Consider the probability space $([0,1), \mathcal{B}[0,1), \mathbb{P})$, where $\mathbb{P}$ denotes Lebesgue measure. For $k \geq 1$ let $A_k = [2^{-k}, 2^{-k+1})$, and set

$$X(\omega) := \sum_{k\geq 1} k\, \mathbf{1}_{A_k}(\omega) \quad \text{and} \quad Y(\omega) := \sum_{k\geq 1} (-1)^k\, \mathbf{1}_{A_k}(\omega).$$

(a) Determine the σ-algebras generated by X and Y, respectively.

(b) Determine whether X and Y are independent or not.

Problem 2. In a population, two genetic traits A and B are present. In a simplified model for population growth, at discrete time steps an individual is chosen uniformly at random, and replaced by (give birth to) two offspring with the same genetic trait as their parent. Suppose that the population initially consists of one individual of trait A and one of trait B. Show that infinitely many children with each trait will be born.

Problem 3. Let $X_1, X_2, \ldots$ be independent and uniformly distributed random variables on the interval $[0,1]$. Let $Y = \prod_{k\geq 1} X_k$ and $Z = \sum_{k\geq 1} 2^{-k} X_k$. Determine the expectation of Y and Z.

Problem 4. Let $X_1, X_2, \ldots$ be independent random variables taking values ± 1 with equal probability. Let $S_n = X_1 + \ldots + X_n$ and set $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Let $k \geq 1$ be an integer. Determine for which $m \in \mathbb{Z}$ that $T = \min\{n \geq 0 : S_{n+m} = S_{n+m-k} + k\}$ defines a stopping time for the filtration $(\mathcal{F}_n)_{n\geq 1}$.

Part II

Problem 5. Let $(M_n)_{n\geq 1}$ be a square-integrable martingale with respect to some filtration $(\mathcal{F}_n)_{n\geq 1}$. Set $M_0 = 0$ and let $X_k = M_k - M_{k-1}$ denote the martingale increments. Show that $\mathrm{Var}(M_n) = \sum_{k=1}^{n} \mathrm{Var}(X_k)$.

Problem 6. Consider the probability space $([0,1), \mathcal{B}[0,1), \mathbb{P})$, where $\mathbb{P}$ denotes Lebesgue measure. Let $X(\omega) = 2\omega$ and $Y(\omega) = \mathbf{1}_{[1/2,1)}(\omega)$, and let $\mathbb{E}$ denote expectation with respect to $\mathbb{P}$.

(a) Show that $\mathbb{P}'(B) := \mathbb{E}[X\,\mathbf{1}_B]$ is a probability measure on $([0,1), \mathcal{B}[0,1))$.

(b) Let $\mathbb{E}'$ denote expectation with respect to $\mathbb{P}'$. Compute $\mathbb{E}[Y]$ and $\mathbb{E}'[Y]$.

Problem 7. An urn contains initially two red and one blue balls. In each step, a ball is drawn and returned to the urn along with an additional ball of the opposite colour. Let R_n and B_n denote the number of red and blue balls, respectively, in the urn after n steps.

(a) Show that $D_n = (n+2)(R_n - B_n)$ defines a martingale.

(b) For $m \geq 2$ let

$$T_m = \min\{n \geq 1 : R_n - B_n = m\}.$$

Set $T = \min\{T_m, T_0\}$ and show that $\mathbb{P}(T > km) \leq a^{-k}$ for some $a > 0$.

(c) Use optional stopping to show that $\mathbb{P}(T_m < T_0) \leq 2/m^2$.

Problem 8. Let $X_1, X_2, \ldots$ be independent random variables taking values ± 1 with equal probability. Let $\Lambda(\theta) = \log \mathbb{E}[e^{\theta X_1}]$ denote the log-moment generating function of X_1, and set $S_n = X_1 + \ldots + X_n$. Below, let $a, \theta > 0$.

(a) Show that $\mathbb{P}(S_n > an) \leq 1/(a^2 n)$.

(b) Show that $M_n = \exp\big(\theta S_n - n\Lambda(\theta)\big)$ defines a martingale.

(c) Show that $\Lambda(\theta) \leq \theta^2$.

(d) Deduce the stronger bound that $\mathbb{P}(S_n > an) \leq \exp(-a^2 n/4)$.

Hint: Use that $\log(1+x) \leq x$ for all x and $e^{-2x} \leq 1 - 2x + 2x^2$ for $x \geq 0$.