STOCKHOLMS UNIVERSITET
DEPT. OF MATHEMATICS
DIV. OF MATHEMATICAL STATISTICS

MT7052 / MT7012
EXAM
December 10 2024

Exam in:

- **Inference and prediction for life and health processes 7.5 ECTS Credits**
- **Mathematical Methods in Life Assurance I, 7.5 ECTS credits**

December 10 2024

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Allowed to use: Three (3) sheets of (physical) A4 papers with notes (in any fontsize or handwritten and on both sides of each page) that are brought to the exam by the student. Pocket calculator provided by the department in connection with the exam.

Return of exam: No later than three weeks after the written exam.

Each correctly solved problem yields 10 points and the maximum total score is 50 points. All reasoning and notation must be clear. Note that vague comments tend to be interpreted as lack of understanding.

It is OK to answer in Swedish. **Only write your solutions on the front page of each paper!**

The grades A–E are set according to the following lower point limits:

Grade	A	B	C	D	E
Lower limit	45	40	35	30	25

Instructions

Note that this exam covers TWO courses, MT7052 and MT7012. Due to this the problems relevant for MT7052 will be indicated by "♠" and all problems relevant for MT7012 will be indicated by "♣. Note that some problems are valid for both courses and will be indicated by (♠, ♣). Each exam consists of five (5) problems, so make sure that you have answered the correct problems for your exam. These are:

- **MT7052:** P1, P2, P3, P4, P5
- **MT7012:** P3, P4, P5, P6, P7

Indicate in the check box below to which exam you are registered

☐ **MT7052** ("♠") Inference and prediction for life and health processes, 7.5 ECTS Credits

☐ **MT7012** ("♣") Mathematical Methods in Life Assurance I, 7.5 ECTS credits

Note: It is important that you only write your solutions on the front page of each paper, otherwise the scanning of your exam will not work.

Problem 1 (♠)

a) The Gehan-Breslow test is defined in terms of the following two statistics:

$$Z_1(t_0) = \int_0^{t_0} Y_2(t) dN_1(t) - \int_0^{t_0} Y_1(t) \mathrm{d}N_2(t),$$
$$V_{11}(t_0) = \int_0^{t_0} Y_1(t) Y_2(t) \mathrm{d}N_\bullet(t).$$

Under which assumptions is the variance estimator in the test unbiased?

b) Consider the following data (no censorings):

T_j	$Y_1(T_j)$	$Y_2(T_j)$
0.22	16	18
0.51	15	18
0.77	15	17
1.12	15	16

where we in addition know that when the study ends at time $t_0 = 1.5$ we have that $Y_1(t_0) = 14$ and $Y_2(t_0) = 16$.

Calculate $Z_1(t_0)$ and $V_{11}(t_0)$ given the above data when $t_0 = 1.5$ and carry out a test for that the two groups are equal. Clearly specify your alternative hypothesis.

(10p)

Problem 2 (♠)

Let $X_{n+1} := aX_n + b + \epsilon_{n+1}$, where a, b are finite constants and where all ϵ_ns are i.i.d. with $\mathrm{E}[\epsilon_n] = 0$ and $\mathrm{Var}(\epsilon_n) = 1$.

a) Which values of a, b gives us that $\mathrm{E}[X_{n+1} \mid X_n] = X_n$?

b) Given that $\mathrm{E}[X_{n+1} \mid X_n] = X_n$, show that $\mathrm{E}[X_n^2]$ is non-decreasing in n.

(10p)

Problem 3 (♠, ♣)

The Poisson Lee-Carter model is based on the likelihood equivalence corresponding to

$$D_{x,t} \mid W_{x,t} \sim \mathrm{Pois}(W_{x,t}\mu_{x,t}),\ x = x_0, x_1, \ldots, x_\xi; t = t_0, t_1, \ldots, t_\eta,$$

where $D_{x,t}$ corresponds to the number of x-year olds that have died during calendar year t, and where $W_{x,t}$ corresponds to the amount of time that x-year olds have been alive during calendar year t.

a) Express the log-likelihood for a single $\mu_{x,t}$ and interpret the log-likelihood expression in terms of survival and observed deaths.

b) Derive the maximum likelihood estimator of $\mu_{x,t}$ and show that it is unbiased.

c) In the Poisson Lee-Carter model the future mortality rates are modelled as a stochastic process. Describe what the influence of this assumption will have on the the forecasted future death counts.

(10p)

Problem 4 (♠, ♣)

Let $T \geq 0$ be a continuous random variable.

a) Show that

$$\alpha(t) := \lim_{h \to 0} \frac{1}{h} \mathrm{P}(T \leq t + h \mid T > t) = \frac{f(t)}{S(t)},$$

where $f(t)$ is the density function of T and where $S(t)$ is the survival function of T.

b) Based on the previous part, show that

$$S(t) := \exp\{-\int_0^t \alpha(u)\mathrm{d}u\}.$$

(10p)

Problem 5 (♠, ♣)

Consider a continuous time Markov chain $Z(t)$ that can only take on the values 0, 1, or 2, where 0 corresponds to alive and healthy, 1 corresponds to sick, and 2 that corresponds to dead. The only transition rates $\mu_{ij}(t)$ that are non-zero, are $\mu_{01}(t), \mu_{02}(t)$, and $\mu_{12}(t)$.

a) Draw a graph that describes the possible states with allowed transitions indicated by arrows.

b) Let $P_{ij}(s,t) := \mathrm{P}(Z(t) = j \mid Z(s) = i)$. The Kolmogorov forward equations (KFE) are given by

$$\frac{\partial}{\partial t}P_{ij}(s,t) = \sum_{k:k\neq j} P_{ik}(s,t)\mu_{kj}(t) - P_{ij}(s,t)\mu_j(t),$$

where $P_{ij}(s,s) = \delta_{ij}$, $\delta_{ij} = 1$ if $i = j$ and 0 otherwise, and where

$$\mu_j(t) := \sum_{k:k\neq j} \mu_{jk}(t).$$

For the described process $Z(t)$, express the KFE for the relevant transition probabilities $P_{ij}(s,t)$ and simplify.

c) Draw an example trajectory of $Z(t)$ with two jumps over a time interval $[0, t_0]$, when $Z(0) = 1$.

d) Is

$$P_{01}(s,t) = \int_s^t \exp\{-\int_s^w \mu_0(u)\mathrm{d}u\}\mu_{01}(w)P_{11}(w,t)\mathrm{d}w$$

a solution to the KFE?

(10p)

Problem 6 (♣)

The Thiele equations can be written according to

$$\frac{\mathrm{d}}{\mathrm{d}t}V^j(t) = r(t)V^j(t) - b^j(t) - \sum_{k:k\neq j} \mu_{jk}(t)(b^{jk}(t) + V^k(t) - V^j(t)),$$

with the usual boundary conditions.

Consider the dynamics described in Problem 5.

a) If we consider a health insurance, describe which payments are reasonable to include, given that no payments are made in relation to the event of individuals dying.

b) Simplify the Thiele equations for the described system and state relevant sums at risk.

c) The corresponding integral equation to Thiele's equations gives us that

$$V^0(t) = \int_t^\tau \exp\{-\int_t^w (r(u) + \mu_{01}(u) + \mu_{02}(u))\mathrm{d}u\}(\mu_{01}(w)(b^{01}(w) + V^1(w)) + b^0(w))\mathrm{d}w,$$

where τ corresponds to the maximal length of the contract. If you assume that the premium rate is constant, write up the expression that corresponds to the actuarially fair premium.

(10p)

Problem 7 (♣)

Let T denote the lifetime distribution of a newly born individual described by

$$\mathrm{P}(T \leq t) = 1 - \exp\{-\theta t - \frac{\lambda}{\beta}(\mathrm{e}^{\beta t} - 1)\},\ \beta, \lambda, \theta > 0. \tag{1}$$

a) Determine the hazard rate of T.

b) Express p_x and q_x in terms of T defined by (1).

(10p)

Good luck!