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DEPT. OF MATHEMATICS
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MT7052 / MT7012
EXAM
December 10 2024

Sketch solutions: MT7052 (♠)/ MT7012 (♣)

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Note: Problems for MT7052 are marked with "♠" and problems for MT7012 are marked with "♣".

Problem 1 (♠)

a) This is discussed in Ch. 3.3.1 on p. 105-106 in Aalen et al. (2008). We have $H_0 : \alpha_1(t) = \alpha_2(t) = \alpha(t)$ for all $0 < t < t_0$ and the alternative that $H_A : \alpha_1(t) \leq \alpha_2(t)$ or $\alpha_1(t) \geq \alpha_2(t)$ for all $0 < t < t_0$. (One could also consider $H_A : \alpha_1(t) \neq \alpha_2(t)$ for some $0 < t < t_0$, but the strength of this test will be low, see Aalen et al. (2008).)

Under H_0 it follows that

$$\int_0^{t_0} Y_1(t)Y_2(t)\mathrm{d}M_\bullet(t) := \int_0^{t_0} Y_1(t)Y_2(t)\mathrm{d}N_\bullet(t) - \int_0^{t_0} Y_1(t)Y_2(t)Y_\bullet(t)\alpha(t)\mathrm{d}t$$

is a stochastic integral w.r.t. a mean zero martingale, where $Y_\bullet(t) = Y_1(t) + Y_2(t)$, since the $Y_i(t)$s are predictable and bounded (see Aalen et al. (2008) Ch. 2 for details), which implies that

$$\mathrm{E}[\int_0^{t_0} Y_1(t)Y_2(t)\mathrm{d}N_\bullet(t)] = \mathrm{E}[\int_0^{t_0} Y_1(t)Y_2(t)Y_\bullet(t)\alpha(t)\mathrm{d}t],$$

where the expression on the left is the expected value of $V_{11}(t_0)$ and the expression on the right is the expected value of $<Z_1>(t_0)$. For more details, see Ch. 3.1 and Ch. 2 in Aalen et al. (2008).

b) Given the supplied data there are two observed events of type 1 at $t = 0.22, 1.12$, and there are two observed events of type 2 at $t = 0.51, 0.77$, and we can assume that no jumps occur before 0.22 and from the data we see that no jumps occur after 1.12. That is, if we

introduce $\Delta N_k(T_j)$ which is 1 if the jth jump is of type k and 0 otherwise, we can rewrite $Z_1(t_0)$ according to

$$\begin{aligned} Z_1(t_0) &= \int_0^{t_0} Y_2(t)\mathrm{d}N_1(t) - \int_0^{t_0} Y_1(t)\mathrm{d}N_2(t) \\ &= \sum_{j:T_j \le t_0} Y_2(T_j)\Delta N_1(T_j) - \sum_{j:T_j \le t_0} Y_1(T_j)\Delta N_2(T_j) \\ &= (18+16) - (15+15) = 4, \end{aligned}$$

and analogously for $V_{11}(t_0)$ we get

$$V_{11}(t_0) = 16 \cdot 18 + 15 \cdot 18 + 15 \cdot 17 + 15 \cdot 16 = 1053.$$

We could then use the test statistic

$$\frac{Z_1(t_0)}{\sqrt{V_{11}(t_0)}} \sim \text{appr.} \quad \mathrm{N}(0,1)$$

which will not reject H_0 at any reasonable significance level. Alternatively you can square the above statistic and use a χ^2-test instead. For more details, see Ch. 3 in Aalen et al. (2008).

Problem 2 (♠)

a) From the problem formulation it directly follows that

$$\mathrm{E}[X_{n+1} \mid X_n] = \mathrm{E}[a + bX_n + \epsilon_{n+1} \mid X_n] = a + bX_n,$$

which gives us that $a = 0, b = 1$.

b) This part can be shown in different ways. A general argument is to use that

$$\begin{aligned} \mathrm{E}[X_{n+1}^2] &= \mathrm{E}[(X_{n+1} - X_n + X_n)^2] \\ &= \mathrm{E}[(X_{n+1} - X_n)^2] + 2\,\mathrm{E}[X_n(X_{n+1} - X_n)] + \mathrm{E}[X_n^2] \\ &\ge 2\,\mathrm{E}[\mathrm{E}[X_n(X_{n+1} - X_n) \mid X_n]] + \mathrm{E}[X_n^2] \\ &= 2\,\mathrm{E}[X_n\,\mathrm{E}[X_{n+1} - X_n \mid X_n]] + \mathrm{E}[X_n^2] = \mathrm{E}[X_n^2]. \end{aligned}$$

Alternatively, in this particular situation you can use the following direct argument

$$\begin{aligned} \mathrm{E}[X_{n+1}^2] &= \mathrm{E}[(X_n + \epsilon_{n+1})^2] \\ &= \mathrm{E}[X_n^2] + 2\,\mathrm{E}[X_n\epsilon_{n+1}] + \mathrm{E}[\epsilon_{n+1}^2] \\ &= \mathrm{E}[X_n^2] + 2\,\mathrm{E}[\mathrm{E}[X_n\epsilon_{n+1} \mid X_n]] + 1 \\ &= \mathrm{E}[X_n^2] + 2\,\mathrm{E}[X_n\,\mathrm{E}[\epsilon_{n+1}]] + 1 \\ &= \mathrm{E}[X_n^2] + 1 \ge \mathrm{E}[X_n^2]. \end{aligned}$$

Problem 3 (♠, ♣)

a) From the problem formulation we have that $\ell(\mu_{x,t}) := \log L(\mu_{x,t})$ is given by (only keeping the part that depends on $\mu_{x,t}$)

$$\ell(\mu_{x,t}) = \log(\mathrm{e}^{-W_{x,t}\mu_{x,t}} \frac{(W_{x,t}\mu_{x,t})^{D_{x,t}}}{D_{x,t}!}) \propto \log(\mu_{x,t}^{D_{x,t}} \mathrm{e}^{-W_{x,t}\mu_{x,t}}),$$

where $\mu_{x,t}^{D_{x,t}}$ corresponds to the contribution to the log-likelihood from the observed number of deaths, and where $\exp\{-W_{x,t}\mu_{x,t}\}$ corresponds to the contribution to the log-likelihood from from the probability of surviving for $W_{x,t}$ units of time.

b) Direct differentiation of $\ell(\mu)$ w.r.t. μ, dropping the subscripts x, t for notational convenience, yields

$$\ell'(\mu) = \frac{D}{\mu} - W = 0,$$

i.e.

$$\widehat{\mu} = \frac{D}{W},$$

and by using the conditional Poisson assumption we get

$$\mathrm{E}[\frac{D}{W}] = \mathrm{E}[\mathrm{E}[\frac{D}{W} \mid W]] = \mathrm{E}[\frac{W\mu}{\mu}] = \mu.$$

c) By introducing a stochastic time trend in the future mortality rates, which is joint for all ages x, the future mortality evolution for different ages becomes dependent.

Problem 4 (♠, ♣)

See Ch. 1.1.2 in Aalen et al. (2008).

Problem 5 (♠, ♣)

Figure 1: Left panel: Solution to Problem 5**a)**. Right panel: Solution to Problem 5**c)**

a) and **c)** See Fig. 1.

b) First note that

$$\begin{aligned}\mu_0(t) &:= \mu_{01}(t) + \mu_{02}(t),\\ \mu_1(t) &:= \mu_{12}(t),\\ \mu_2(t) &:= 0.\end{aligned}$$

This, together with the boundary conditions $P_{ij}(s,s) = \delta_{ij}$, $\delta_{ij} = 1$ if $i = j$ and 0 otherwise, gives us that $P_{22}(s,t) = 1$ and $P_{2j}(s,t) = 0$ for $j \neq 2$, and that $P_{10}(s,t) = 0$.

By using the above and simplifying the KFEs gives us that

$$\begin{aligned}\frac{\partial}{\partial t}P_{00}(s,t) &= -P_{00}(s,t)\mu_0(t),\\ \frac{\partial}{\partial t}P_{01}(s,t) &= P_{00}(s,t)\mu_{01}(t) - P_{01}(s,t)\mu_1(t),\\ \frac{\partial}{\partial t}P_{02}(s,t) &= P_{00}(s,t)\mu_{02}(t) + P_{01}(s,t)\mu_{12}(t),\\ \frac{\partial}{\partial t}P_{11}(s,t) &= -P_{11}(s,t)\mu_1(t),\\ \frac{\partial}{\partial t}P_{12}(s,t) &= P_{11}(s,t)\mu_{12}(t).\end{aligned}$$

In part **c)** there was a typo: given that $Z(0) = 1$ it is not possible to make two jumps; two jumps are only possible if starting in $Z(0) = 0$, see Fig. **??**.

In Part **d)** the answer is "yes": Start by noting that (check KFE!)

$$P_{00}(s,t) = \exp\{-\int_s^t \mu_0(u)\mathrm{d}u\},$$

which gives us that

$$P_{01}(s,t) = \int_s^t P_{00}(s,w)\mu_{01}(w)P_{11}(w,t)\mathrm{d}w,$$

and direct differentiation of this expression w.r.t. t using the chain rule (Leibniz rule) results in

$$\begin{aligned}\frac{\partial}{\partial t}P_{01}(s,t) &= P_{00}(s,t)\mu_{01}(t)P_{11}(t,t) + \int_s^t P_{00}(s,w)\mu_{01}(w)\frac{\partial}{\partial t}P_{11}(w,t)\mathrm{d}w\\ &= P_{00}(s,t)\mu_{01}(t) + \int_s^t P_{00}(s,w)\mu_{01}(w)\frac{\partial}{\partial t}P_{11}(w,t)\mathrm{d}w,\end{aligned}$$

where we have used that $P_{11}(t,t) = 1$. By inserting the KFE for $P_{11}(t,t)$ then gives us that

$$\begin{aligned}\frac{\partial}{\partial t}P_{01}(s,t) &= P_{00}(s,t)\mu_{01}(t) + \int_s^t P_{00}(s,w)\mu_{01}(w)(-P_{11}(w,t)\mu_1(t))\mathrm{d}w \\ &= P_{00}(s,t)\mu_{01}(t) - \mu_1(t)\int_s^t P_{00}(s,w)\mu_{01}(w)P_{11}(w,t)\mathrm{d}w \\ &= P_{00}(s,t)\mu_{01}(t) - \mu_1(t)P_{01}(s,t).\end{aligned}$$

Problem 6 (♣)

See Chapter 4.2 in Asmussen & Steffensen (2020).

Problem 7 (♣)

a) Direct calculations gives us that $\mu(t) = \theta + \lambda \mathrm{e}^{\beta t}, t \geq 0$, i.e. Gompertz-Makeham.

b) The result follows from plugging in the above into

$$q_x := \mathrm{P}(T_x \leq 1) := \mathrm{P}(T \leq x+1 \mid T > x),$$

and simplify, i.e. T corresponds to the life time distribution of a 0 year old individual.

References

Aalen, O. O., Borgan, Ø. & Gjessing, H. K. (2008), *Survival and Event History Analysis: A Process Point of View*, Springer.

Asmussen, S. & Steffensen, M. (2020), *Risk and insurance*, Springer.