STOCKHOLMS UNIVERSITET,
MATEMATISKA INSTITUTIONEN,
Avd. Matematisk statistik

Exam: Brownian motion and stochastic differential equations (MT7043), 2023-02-20

Examiner:
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Allowed aid: Calculator (provided by the department).

Return of exam: To be announced via the course webpage or the course forum.

The exam consists of five problems. Each problem gives a maximum of 10 points.

- The reasoning should be clear and concise.
- Answers should be motivated (unless otherwise stated).
- Any assumptions should be clearly stated and motivated.
- Start every problem on a new sheet of paper.
- Clearly number each sheet with problem number and sheet order.
- Write your code number (but no name) on each sheet.

Preliminary grading:

A	B	C	D	E
45	40	35	30	25

Good luck!

Problem 1

(A) State a general definition of a one-dimensional controlled SDE. Denote the controlled process by (X_t) and consider Markov controls, i.e., controls of the type $u(t, X_t)$. (5p)

(B) Define a process (X_t) according to $X_t = B_t^4$ (where (B_t) is as usual a Brownian motion). Is this an Itô process? (Do not forget to give a clear motivation). (5p)

Problem 2

Define a process (M_t) according to

$$M_t = B_t^2 - f(t),$$

where $f(t)$ is a deterministic function.

Is it possible to specify $f(t)$ so that (M_t) is a martingale with $M_0 = 0$?

If the answer is yes then specify $f(t)$—regardless, do not forget give to a careful motivation for your answer.

Hint: Use the explicit form of $\mathbb{E}(B_t^2)$ *to find a candidate for* $f(t)$. (10p)

Problem 3

Does the process (X_t) defined according to

$$X_t = 1 + t + (1-t)\int_0^t \frac{1}{1-s}dB_s, \ \ 0 \le t < 1$$

solve the SDE

$$dX_t = \frac{2 - X_t}{1-t}dt + dB_t, \ \ 0 \le t < 1, \ \ X_0 = 1?$$

(10p)

Problem 4

Suppose (X_t) solves a one-dimensional SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t.$$

Consider a function $f : \mathbb{R} \to \mathbb{R}$.

State the definition of the (infinitesimal) generator A of (X_t)—*Hint: it is a limit as* $t \searrow 0$.

Under a certain condition for $f(\cdot)$ it holds that

$$Af(x)$$

is equal to an expression based on $b(\cdot), \sigma(\cdot)$ and derivatives of $f(\cdot)$; state this expression and the condition for $f(\cdot)$.

Suppose f satisfies the condition and state the differential expression for $Af(x)$ in the following cases:

$$\begin{aligned} (i) \;\; dX_t &= bX_t dt + \sigma X_t dB_t \\ (ii) \;\; dX_t &= bX_t dt + \sigma dB_t. \end{aligned}$$

(10p)

Problem 5

State a general optimal stopping problem (based on a one-dimensional SDE) and formulate a corresponding verification theorem. Give a proof sketch for your verification theorem.

Hint: you may formulate the verification theorem considering only threshold solutions. (10p)