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Avd. Matematisk statistik

Exam: Brownian motion and stochastic differential equations (MT7043), 2023-02-20

Problem 1

(A) See Øksendal ch. 11.

(B) Yes, this follows from Itô's formula (Øksendal ch. 4).

Problem 2

In order for (M_t) to be a martingale with $M_0 = 0$ we need that

$$\mathbb{E}(M_t) = 0.$$

which is equivalent to $\mathbb{E}(B_t^2) = f(t)$. But since $\mathbb{E}(B_t^2) = t$ this means that we need that $f(t) = t$. Hence our candidate for being a martingale is

$$M_t = B_t^2 - t.$$

Indeed, it can now be easily be established (and a complete solution should do this) that this process (M_t) satisfies the conditions (Øksendal ch. 3.2) for being a martingale (with respect to the filtration generated by the Brownian motion); this can be done in line with how we have proved similar statements in the lecture (see also Øksendal Example 3.2.3 and Øksendal Exercise 3.5).

Problem 3

This is a version of the Brownian bridge, Øksendal p. 76. Using Itô's formula (see also the hint in the exercise set which contains a question on the Brownian bridge) we obtain

$$
\begin{aligned}
dX_t &= d(1+t) + d\left((1-t)\int_0^t \frac{1}{1-s}dB_s\right) \\
&= dt + (1-t)d\left(\int_0^t \frac{1}{1-s}dB_s\right) + \int_0^t \frac{1}{1-s}dB_s d(1-t) \\
&= dt + (1-t)\frac{1}{1-t}dB_t - \int_0^t \frac{1}{1-s}dB_s dt \\
&= \left(1 - \int_0^t \frac{1}{1-s}dB_s\right) dt + dB_t \\
&= \frac{(1-t)\left(1 - \int_0^t \frac{1}{1-s}dB_s\right)}{1-t}dt + dB_t \\
&= \frac{1 - t - (1-t)\int_0^t \frac{1}{1-s}dB_s}{1-t}dt + dB_t \\
&= \frac{2 - \left(1 + t + (1-t)\int_0^t \frac{1}{1-s}dB_s\right)}{1-t}dt + dB_t \\
&= \frac{2 - X_t}{1-t}dt + dB_t.
\end{aligned}
$$

It is obvious that $X_0 = 1$. We conclude that the answer to the question is yes.

Problem 4

This is based on Øksendal ch. 7. The generator is defined by

$$
Af(x) := \lim_{t \searrow 0} \frac{\mathbb{E}^x(f(X_t)) - f(x)}{t}.
$$

The condition (stated in Øksendal) is $f \in C_0^2(\mathbb{R})$ and given this it holds that

$$
Af(x) := b(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x).
$$

It directly follows that

$$
\begin{aligned}
(i) \;\; Af(x) &= bxf'(x) + \frac{1}{2}x^2\sigma^2 f''(x) \\
(ii) \;\; Af(x) &= bxf'(x) + \frac{1}{2}\sigma^2 f''(x).
\end{aligned}
$$

Problem 5

Compare Øksendal ch. 10.