STOCKHOLMS UNIVERSITET,
MATEMATISKA INSTITUTIONEN,
Avd. Matematisk statistik

Brownian motion and stochastic differential equations (MT7043), 2024-12-17

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Allowed aid: Calculator (provided by the department).

Return of exam: To be announced via the course webpage or the course forum.

The exam consists of five problems. Each problem gives a maximum of 10 points.

- The reasoning should be clear and concise.
- Answers should be motivated (unless otherwise stated).
- Any assumptions should be clearly stated and motivated.
- Start every problem on a new sheet of paper.
- Clearly number each sheet with problem number and sheet order.
- Write your code number (but no name) on each sheet.

Preliminary grading:

A	B	C	D	E
45	40	35	30	25

Good luck!

Problem 1

(A) State the definition of an n-dimensional SDE. (5p)

(B) State the definition of a stopping time. (5p)

Problem 2

Let (B_t) be a Brownian motion (with $B_0 = 0$).

(i) Express B_t^4 as the sum of two integrals.

(ii) Using your answer in (i), find an expression for $\mathbb{E}(B_t^4)$.

Hints: For the first part use Itô's formula. For the second part, consider taking the expected value of your expression from (i), and then changing the order of expectation and integration (for one of the integrals).

(10p)

Problem 3

Let (Y_t) be an Itô process. Show that

$$d(e^{-rt}Y_t) = -re^{-rt}Y_t dt + e^{-rt}dY_t. \tag{1}$$

Hint: use Itô's formula for $f(t, Y_t) = e^{-rt}Y_t$.

(10p)

Problem 4

Consider the stochastic process

$$X_t = e^{-rt}\left(x + \int_0^t \sigma e^{rs} dB_s\right), t \geq 0.$$

(i) For an arbitrary fixed $t > 0$ find the expectation and the variance of the random variable X_t. *Hint: Itô isometry may be helpful (for the second part).*

(ii) Find the SDE that the process (X_t) satisfies. *Hint:* consider the process

$$Y_t = x + \int_0^t \sigma e^{rs} dB_s, \ \ t \geq 0,$$

with which our process (X_t) can be written as

$$X_t = e^{-rt}Y_t.$$

Use this and the relationship in equation (1).

(10p)

Problem 5

Consider a constant $\sigma > 0$ and the SDE

$$dX_t = \sigma X_t dB_t, \;\; X_0 = x > 0.$$

(i) Which is the differential operator associated to the SDE? (You do not have to motivate your answer to this question.)

(ii) Solve the SDE.

(iii) Let $D = (a, b)$ with $b > a > 0$ and consider

$$\tau_D = \inf\{t \geq 0 : X_t \notin D\},$$

(i.e., τ_D is the first exit time of (X_t) from D). Derive an expression for $\mathbb{P}^x(X_{\tau_D} = b)$ for $x \in D$. *Hint: identify a corresponding boundary value problem.*

(10p)