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Suggested solutions

Exam: Brownian motion and stochastic differential equations (MT7043), 2024-12-17

Problem 1

See Øksendal Ch. 5 and 7.

Problem 2

(i) With $f(x) = x^4$ we have $f'(x) = 4x^3$ and $f''(x) = 12x^2$. Hence, Itô's formula yields

$$\begin{aligned} d(B_t^4) &= df(B_t) \\ &= f'(B_t)dB_t + \frac{f''(t, B_t)}{2}dt \\ &= 4B_t^3 dB_t + 6B_t^2 dt, \end{aligned}$$

where moreover $B_0^2 = 0$. But this is simply short hand notation for

$$B_t^4 = 4\int_0^t B_s^3 dB_s + 6\int_0^t B_s^2 ds.$$

(ii) The Itô integral has zero expectation and hence

$$\mathbb{E}(B_t^4) = \mathbb{E}\left(6\int_0^t B_s^2 ds\right) = 6\int_0^t \mathbb{E}(B_s^2)ds.$$

Using $\mathbb{E}(B_s^2) = s$ (a basic property of Brownian motion) we obtain

$$\mathbb{E}(B_t^4) = 6\int_0^t s ds = 6\frac{t^2}{2} = 3t^2.$$

(Which we note corresponds to the well-known fourth moment for the distribution $N(0,t)$.)

Problem 3

With $f(t, y) = e^{-rt}y$ we have $f_t(t, y) = -re^{-rt}y$, $f_y(t, y) = e^{-rt}$, and $f_{yy}(t, y) = 0$. Hence Itô's formula (Ch. 4 in Øksendal) yields

$$\begin{aligned} d(e^{-rt}Y_t) &= df(t, Y_t) \\ &= f_t(t, Y_t)dt + f_y(t, Y_t)dY_t + \frac{f_{yy}(t, Y_t)}{2}(dY_t)^2 \\ &= -re^{-rt}Y_t dt + e^{-rt}dY_t + \frac{0}{2}(dY_t)^2, \end{aligned}$$

which yields the result.

Problem 4

This is a so-called Ornstein-Uhlenbeck process.

(i) The expectation of the Itô integral is zero and we hence obtain

$$\mathbb{E}(X_t) = e^{-rt}x.$$

With Itô isometry we find (with some calculations and basic probability theory) the variance

$$\begin{aligned}
\mathbb{V}(X_t) &= \mathbb{V}\left(e^{-rt}\left(x + \int_0^t \sigma e^{rs} dB_s\right)\right) \\
&= (e^{-rt}\sigma)^2 \mathbb{V}\left(\int_0^t e^{rs} dB_s\right) \\
&= e^{-2rt}\sigma^2 \mathbb{V}\left(\int_0^t e^{rs} dB_s\right) \\
&= \sigma^2 e^{-2rt} \mathbb{E}\left(\left(\int_0^t e^{rs} dB_s\right)^2\right) \\
&= \sigma^2 e^{-2rt} \left(\int_0^t (e^{rs})^2 ds\right) \\
&= \sigma^2 e^{-2rt} \left(\int_0^t e^{2rs} ds\right) \\
&= \sigma^2 e^{-2rt} \frac{1}{2r}\left(e^{2rt} - 1\right) \\
&= \frac{\sigma^2}{2r}\left(1 - e^{-2rt}\right).
\end{aligned}$$

(We remark that it can furthermore be shown that X_t is normally distributed, for each fixed $t > 0$).

(ii) Let us find the SDE: The first hint means that

$$dY_t = \sigma e^{rt} dB_t.$$

Using this as well as $X_t = e^{-rt}Y_t$ and the second hint we find

$$\begin{aligned}
dX_t &= d(e^{-rt}Y_t) \\
&= -re^{-rt}Y_t dt + e^{-rt} dY_t \\
&= -rX_t dt + e^{-rt}\sigma e^{rt} dB_t \\
&= -rX_t dt + \sigma dB_t.
\end{aligned}$$

It is moreover directly seen that $X_0 = x$. We conclude that the SDE is

$$dX_t = -rX_t dt + \sigma dB_t, \quad X_0 = x.$$

Problem 5

(i) The differential operator associated to our process corresponds to

$$Lf(x) = \frac{\sigma^2 x^2}{2} f''(x).$$

(ii) This is a specification of the well-known GBM and how it can be solved can be seen in Øksendal Ch 5.1. The solution is

$$X_t = xe^{-\frac{1}{2}\sigma^2 t + \sigma B_t}$$

(iii) (Compare with Øksendal Exercise 9.14). This part of the problem can be studied with our usual BVP approach (cf. Øksendal Ch 9 and Corollary 9.1.2): Note that

$$\mathbb{P}^x(X_{\tau_D} = b) = \mathbb{E}^x(I_{\{X_{\tau_D}=b\}}) = \mathbb{E}^x(\phi(\{X_{\tau_D=b}))$$

for $\phi(a) := 0$ and $\phi(b) := 1$. The associated BPV is

$$\begin{aligned} \frac{1}{2}\sigma^2 x^2 w''(x) = 0, \ \ x \in (a, b) \\ w(b) = \phi(b) = 1 \\ w(a) = \phi(a) = 0. \end{aligned}$$

Since $a > 0$ we see that the BVP is equivalent to

$$\begin{aligned} w''(x) = 0, \ \ x \in (a, b) \\ w(b) = 1 \\ w(a) = 0. \end{aligned}$$

With basic ODE theory we find that the ODE solution is $w(x) = C_1 x + C_2$. Using the boundary condition $w(b) = 1$ and $w(a) = 0$ we can, with basic calculations, determine C_1 and C_2, yielding the BVP solution

$$w(x) = \frac{x - a}{b - a}.$$

The mentioned result (Corollary 9.1.2) now gives

$$\mathbb{P}^x(X_{\tau_D} = b) = \frac{x - a}{b - a}.$$