STOCKHOLMS UNIVERSITET,
MATEMATISKA INSTITUTIONEN,
Avd. Matematisk statistik

Brownian motion and stochastic differential equations (MT7043), 2026-01-30

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Allowed aid: Calculator (provided by the department).

Return of exam: To be announced via the course webpage or the course forum.

The exam consists of five problems. Each problem gives a maximum of 10 points.

- The reasoning should be clear and concise.

- Answers should be motivated (unless otherwise stated).

- Any assumptions should be clearly stated and motivated.

- Start every problem on a new sheet of paper.

- Clearly number each sheet with problem number and sheet order.

- Write your code number (but no name) on each sheet.

Preliminary grading:

A	B	C	D	E
45	40	35	30	25

Good luck!

Problem 1

(A) State the definition of the strong Markov property.

(5p)

(B) Suppose (X_t) is a solution to a one-dimensional SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \ \ X_0 = x.$$

Explain briefly what it means that (X_t) is a solution to the SDE. Your answer should include mathematical expression(s). (5p)

Problem 2

Consider two constants μ and $\sigma > 0$ and the one-dimensional SDE

$$dX_t = \mu X_t dt + \sigma X_t dB_t, \ \ X_0 = x > 0.$$

Solve the SDE and give, for a fixed $t > 0$, a formula for

$$\mathbb{E}^x \left(I_{\{X_t \leq 1\}} \right).$$

Hints. (i) The indicator function I is here defined according to

$$I_{\{X_t \leq 1\}} = \begin{cases} 1, & \text{if } X_t \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

(ii) Base your formula on the CDF (Cumulative distribution function) of the normal distribution (which we may denote by $\Phi(x)$). (10p)

Problem 3

Consider a one-dimensional SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \ \ X_0 = x,$$

(where you may assume that $b(\cdot)$ and $\sigma(\cdot)$ satisfy the usual conditions of Øksendal).
   Consider a function $f : \mathbb{R} \to \mathbb{R}$ with $f \in C^2$.
   To the SDE there is a differential operator $Lf(x)$ defined.
   (i) Give a definition for $Lf(x)$. *Hint: if you do not know it by heart, then use your answer to (iii) below.*
   (ii) Consider
$$dX_t = bdt + \sigma X_t dB_t, \ \ X_0 = x,$$
where b and σ are constants. Let $f(x) = x^3$. Calculate $Lf(x)$.
   (iii) Suppose $f \in C_0^2$ and give a proof (sketch) of the result

$$\lim_{t \downarrow 0} \frac{\mathbb{E}^x \left(f(X_t) - f(x) \right)}{t} = Lf(x). \tag{1}$$

Hint: use Itô's formula in the numerator in (1).

(10p)

Problem 4

Let (X_t) and (Y_t) be Itô processes. Show that

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t dY_t,$$

or prove that this does not hold.

Hint: use the two-dimensional Itô's formula for $f(X_t, Y_t) := X_t Y_t$. (10p)

Problem 5

(i) State a general stochastic control problem based on a one-dimensional SDE.
(ii) Formulate a (possibly simplified) corresponding verification theorem.
(iii) Give a proof (sketch) for your verification theorem. (10p)