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Suggested solutions

Exam: Brownian motion and stochastic differential equations (MT7043), 2026-01-30

Problem 1

(A) See chapter 7.2 in Øksendal.
(B) It means that the process (X_t) is such that the following equality is satisfied

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s,$$

for each $t \geq 0$, and that these integrals are well-defined; see chapter 5 (in particular 5.1) in Øksendal.

Problem 2

The solution of the SDE is the well-known GBM (geomteric Brownian motion)

$$X_t = x^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}, \tag{1}$$

see Øksendal Chapter 5.1 for details.

Using this and some basic math we find

$$\begin{aligned}
\mathbb{E}^x\left(I_{\{X_t \leq 1\}}\right) &= \mathbb{E}^x\left(I_{\{xe^{(\mu-\frac{1}{2}\sigma^2)t+\sigma B_t} \leq 1\}}\right) \\
&= \mathbb{E}^x\left(I_{\{B_t \leq \frac{ln(1/x) - (\mu - \frac{1}{2}\sigma^2)t}{\sigma}\}}\right).
\end{aligned}$$

Let $Z \sim N(0,1)$ so that B_t and $\sqrt{t}Z$ have the same distribution. Using the above and some basic probability we find

$$\begin{aligned}
\mathbb{E}^x\left(I_{\{X_t \leq 1\}}\right) &= \mathbb{E}^x\left(I_{\{B_t \leq \frac{ln(1/x) - (\mu - \frac{1}{2}\sigma^2)t}{\sigma}\}}\right) \\
&= \mathbb{E}^x\left(I_{\{\sqrt{t}Z \leq \frac{ln(1/x) - (\mu - \frac{1}{2}\sigma^2)t}{\sigma}\}}\right) \\
&= \mathbb{E}^x\left(I_{\{Z \leq \frac{ln(1/x) - (\mu - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\}}\right) \\
&= \mathbb{P}^x\left(Z \leq \frac{ln(1/x) - (\mu - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right).
\end{aligned}$$

If we denote the distribution function of a standard normal random variable by Ψ, then the above can be written as

$$\mathbb{E}^x\left(I_{\{X_t \leq 1\}}\right) = \Psi\left(\frac{ln(1/x) - (\mu - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right).$$

The interested student may note that the above corresponds to a pricing formula for a *cash-or-nothing/binary* option; see e.g., the following book for a comprehensive study of the connection between the mathematical content of the present course and mathematical finance: *Björk, Tomas, Arbitrage theory in continuous time. Oxford university press.*

Problem 3

(i) The differential operator is

$$Lf(x) = b(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x),$$

(see e.g., Theorem 7.5.4 in Øksendal).

(ii) Simple calculations give

$$\begin{aligned} Lf(x) &= b(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x) \\ &= bf'(x) + \frac{1}{2}(\sigma x)^2 f''(x) \\ &= b3x^2 + \sigma^2 3x^3. \end{aligned}$$

(iii) See chapter 7.3 e.g., Theorem 7.3.3 (as well as chapters 7.4-7.5) in Øksendal.

Problem 4

With $f(x, y) = xy$ we have $f_x(x, y) = y$, $f_y(x, y) = x$, $f_{xx}(x, y) = f_{yy}(x, y) = 0$ and $f_{xy}(x, y) = f_{yx}(x, y) = 1$. Hence Itô's formula (Ch. 4 in Øksendal) yields

$$\begin{aligned} &d(X_tY_t) \\ &= df(X_tY_t) \\ &= f_x(X_t, Y_t)dX_t + f_y(X_t, Y_t)dY_t + \frac{f_{xx}(X_t, Y_t)}{2}(dX_t)^2 + \frac{f_{yy}(X_t, Y_t)}{2}(dY_t)^2 \\ &\quad + \frac{f_{xy}(X_t, Y_t)}{2}dX_tdY_t + \frac{f_{yx}(X_t, Y_t)}{2}dY_tdX_t \\ &= Y_tdX_t + X_tdY_t + \frac{0}{2}(dX_t)^2 + \frac{0}{2}(dY_t)^2 \\ &\quad + \frac{1}{2}dX_tdY_t + \frac{1}{2}dY_tdX_t. \end{aligned}$$

Simplifying the above yields

$$d(X_tY_t) = Y_tdX_t + X_tdY_t + dX_tdY_t.$$

Problem 5

This can be done in many different ways; see e.g., Chapters 10.1, and Theorem 11.2.2 in Øksendal.