Introduction to Real Analysis

Lecture 4: Sequences and Series

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Questions?

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Lecture Plan

- Sequences (Rudin 3.1-3.20)
- Series (3.22-3.51)

Section 1
Sequences

Convergent Sequences

Definition

A sequence $\{p_n\}$ in a metric space X is said to be convergent if there is $p \in X$ such that, for all $\varepsilon > 0$ there is a $N = N(\varepsilon) \in \mathbb{Z}$ such that

$$d(p_n, p) < \varepsilon$$

for all $n \geq N$.

In this case we write

$$p = \lim_{n \to +\infty} p_n, \quad \text{or} \quad p_n \to p$$

Some properties

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Theorem

Let $\{p_n\}$ a sequence in a metric space (X, d).

1. we have that $p_n \to p$ if, and only if, every neighbourhood of p contains all but finitely many elements of the sequence.
2. The limit of a convergent sequence is unique
3. If $\{p_n\}$ is convergent, then it is bounded.
4. Given $E \subseteq X$ and p a limit point of E, then there exist a sequence in E converging to p.

Subsequences

Let $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ a strictly increasing function. Given a sequence $\{p_n\}$, the sequence $\{p_{f(k)}\}$ is called subsequence of $\{p_n\}$.

We usually denote $f(k)$ by n_k, and thus a subsequence is denoted by $\{p_{n_k}\}$

Subsequences

$$\begin{tikzcd} & \mathbb{Z}^+ \arrow[dl, "f"'] \arrow[d, dotted] \\ p : \mathbb{Z}^+ \arrow[r] & X \end{tikzcd}$$

subsequence

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Proposition

A sequence converges to p if, and only if, all its subsequences converge to p.

Nice exercise to understand limits & subsequence

Ex $a_n = (-1)^n$ has two convergent subse

$a_{2n} = 1$ $a_{2n+1} = -1$

Sequences and compact sets

Theorem

Let $\{p_n\} \subseteq \{K\}$ a compact set, then there is a subsequence $\{p_{n_k}\}$ converging to a point $p \in K$. In particular every bounded sequence in $\mathbb{R}^n$ admits a convergent subsequence.

Proposition

Let $\{p_n\}$ a sequence in a metirc space X, then the set

$$E := \{x \in X \mid \text{there is } p_{n_k} \to x\}$$

is closed.

Proof $E = \{p_n\} \subseteq K$

$|E| < +\infty \implies \exists\, x \in K$ such that

$p_n = x$ for ∞ many n

$$\begin{cases} n_1 = \min\{n \mid p_n = x\} \\ n_2 = \min\{n \mid n > n_1 \;\; p_n = x\} \\ \vdots \\ n_k = \min\{n \mid n > n_{k-1} \;\; p_n = x\} \end{cases}$$

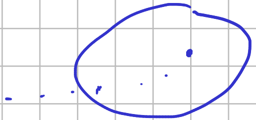
$\hookrightarrow$ strictly increasing

$p_{n_k} = x$ for all $k \in \mathbb{Z}^+$ $\quad p_{n_k} \xrightarrow[k \to +\infty]{} x$

$|E| = +\infty$

$E \subseteq K$ compact $\quad E$ not finite

Last $\quad \exists\, p \in K$ a limit point for E



$\forall\, r>0$

$$N_r(p) \cap E \neq \emptyset$$

$$\begin{cases} n_1 = \min\{n \mid p_n \in N_1(p)\} \\ \vee \\ n_2 = \min\{n > n_1 \mid p_n \in N_{\frac{1}{2}}(p)\} \\ \vee \\ \vdots \\ \vee \\ n_k = \min\{n > n_{k-1} \mid p_n \in N_{\frac{1}{k}}(p)\} \end{cases}$$

p_{n_k} subsequence

Fix $\varepsilon > 0$ $\mathbb{R}$ archimedean

find N such that $\frac{1}{N} < \varepsilon$

$d(p_{n_k}, p) < \frac{1}{k} < \varepsilon$

for all $k \geq N$

$\Rightarrow$ $p_{n_k} \to p$

Cauchy sequence

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Defintion

A sequence (p_n) in a metric space (X, d) is called Cauchy if for every $\epsilon > 0$, there exists an $N \in \mathbb{Z}$ such that for all $m, n \geq N$, we have $d(x_m, x_n) < \epsilon$.

Proposition

A convergent sequence is Cauchy

We say that a metric space (X, d) is complete if every Cauchy sequence is convergent.

Proof $p_n \to p$

Fix $\varepsilon > 0$ find N such that

$m > N \quad d(p_n, p) < \frac{\varepsilon}{2}$

if $\boxed{m, n > N}$

$$d(p_n, p_m) \overset{\Delta}{\leq} d(p_n, p) + d(p, p_m)$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Complete spaces

Theorem

1. Compact spaces are complete.
2. The space $\mathbb{R}^n$ with the Euclidean metric is complete.

Proof:

(a) X compact $\{p_n\} \subseteq X$ Cauchy

$\exists\ p_{n_k} \longrightarrow p \in X$ want to show that $\boxed{p_n \longrightarrow p}$

<u>Claim</u> $f: \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$ increasing

$f(k) \geq k$

(by induction)

Fix $\varepsilon > 0$ we choose N_1 such that

for all $k > N_1$

$d(p_{n_k}, p) < \varepsilon/2$

· choose N_2 $n, m > N_2$ $d(p_n, p_m) < \frac{\varepsilon}{2}$

(p_n Cauchy)

$N = \max\{N_1, N_2\}$

$n_k \geq \boxed{k > N}$

$$d(p_k, p) \overset{\triangle}{<} \underbrace{d(p_k, p_{n_k})}_{} + d(p_{n_k}, p) \quad \begin{matrix} <\varepsilon/2 & <\varepsilon/2 \end{matrix}$$

$\leq \varepsilon$

$$p_k \xrightarrow[k \to +\infty]{} p$$

(b) (a) + $\boxed{\text{Cauchy} \Rightarrow \text{bounded}}$

$\hookrightarrow$ A Cauchy S is contained in in a compact subspace of $\mathbb{R}^n$

p_n Cauchy

N $\quad n, m \geq N$

$d(p_n, p_m) < 1$

$\rho = \max \{1, d(p_n, p_N)' \; n < N\}$

$\{p_n\} \subseteq N_{\rho+1}(p_N)$

$\Rightarrow$ p_n is bounded.

Monotonic sequence

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Definition

In a metric space $\mathbb{R}$, with the Euclidean metric, a sequence (p_n) is said to be monotonic if it satisfies one of the following conditions:

1. It is monotonically increasing, meaning that $p_{n+1} \geq p_n$ for all $n \in \mathbb{Z},\ n \geq 0$.
2. It is monotonically decreasing, meaning that $p_{n+1} \leq p_n$ for all $n \in \mathbb{N},\ n \geq 0$.

Proposition

A monotonic sequence is convergent if, and only if, it is bounded.

Bounded $\Rightarrow$ convergent subsequence

Use MONOTONICITY $p_n \to p$. some limit

Monotonic sequence

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Proposition

A monotonic sequence is convergent if, and only if, it is bounded.

Lim sup and lim inf

Let $\{p_n\}$ a sequence in $\mathbb{R}$, and consider the (closed) set

$$E := \{x \in \mathbb{R} \mid \text{there is } p_{n_k} \to x\},$$

We define

$$p^* := \sup E =: \limsup_{n \to +\infty} p_n$$

$$p_* := \inf E =: \liminf_{n \to +\infty} p_n$$

which are both element of the extended real line $\overline{\mathbb{R}}$.

Lim sup and lim inf

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Let $\{p_n\}$ a sequence in $\mathbb{R}$, and consider the (closed) set

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If one consider

$$s_k = \sup\{p_n \,|\, n \geq k\}, \searrow$$

we have that s_k is an ~~increasing~~ dec. sequence, so it has limit in $\overline{\mathbb{R}}$, and we have that $s_k \to p^*$.
We have a similar characterization with for the lim inf.

$$t_k = \inf\{p_n \,|\, n \geq k\} \nearrow$$

Lim sup and lim inf

Let $\{p_n\}$ a sequence in $\mathbb{R}$, and consider the (closed) set

$$E := \{x \in \mathbb{R} \mid \text{there is } p_{n_k} \to x\},$$

Theorem

The lim sup p^* belongs to the extended real line and it is the only element having the following two properties $\overline{\mathbb{R}}$

1. $s^* \in E = \{ x \mid \exists \; s_{n_k} \to x \}$
2. if $x > s^*$, then there is $N \in \mathbb{N}$ such that $s_n < x$ for $n > N$.

Proof

$s^* = +\infty \implies E$ is not bounded above

$\implies$ you have a subsequence with lim. $+\infty = s^*$

$s^* \in E$ $\xrightarrow{}$ closed

$\sup E \in E$

If $s^* \overset{\sup E}{=} \in \mathbb{R}$ $\Rightarrow$ E bounded above

you can construct a subsequence with limit s^* $\boxed{s^* \in \overline{E}}$

$\exists\, s_{n_k} \to s^*$

is a limit point element of E

$s^* = -\infty$ $\qquad E = \{-\infty\}$

since $\sup E = -\infty$

every subsequence $\to -\infty$ $\qquad -\infty \in E$

$x > s^*$ suppose that

$s_n \geq x$ for ∞ many n

$\Rightarrow$ subsequence $s_{n_k} \longrightarrow y \geq x > s^*$

$y \in E$ $\quad y > \sup E$?!

$\Rightarrow$ $s_n \geq x$ for finitely many n. which is exactly (b).

UNICITY p satisfies (a) and (b)

$\Rightarrow p = s^*$ $\quad p \in \quad p \leq \sup E = s^*$

If $p < s^*$ $\quad x$ in the middle

Since (b) holds for p

There are only finitely many

$s_n \geqslant x$

$\Rightarrow$ contradict that s^*

is a subsequence limit for s_n.

☺

Section 2
Series

Convergent series

$$\mathbb{Z}_{\geq k} \longrightarrow X$$
$$\mathbb{N} \longrightarrow X$$

Let a_n a sequence in $\mathbb{C}$ and consider

$$s_k := \sum_{n=0}^{k} a_{\cancel{m}\,n}$$

This is a sequence in $\mathbb{C}$, and if it converges we denote the limit by

$$\boxed{\sum_{n=0}^{\infty} a_n} \to a \neq 0$$

Convregnce Criteria

- If $|a_n| \leq c_n$ for $n >> 0$ and $\sum_{n=0}^{+\infty} c_n$ converges, then $\sum a_n$ converges.
- If $\sum a_n^2$ and $\sum b_n$ converge, then $\sum |a_n b_n|$, $\sum (a_n + b_n)^2$ and $\sum \frac{a_n}{n}$ converges.

  $\sum \frac{1}{n}$ does not converge

  $na_n = 1 \quad na_n \to 1 \neq 0$
- If $na_n \to a \neq 0$, then $\sum a_n$ diverges.
- (root test) Let $\sigma = \limsup \sqrt[n]{|a_n|}$, then we have the following cases

  not needed to converge
  1. if $\sigma < 1$ then $\sum a_n$ converges
  2. if $\sigma > 1$ then $\sum a_n$ diverges
  3. if $\sigma = 1$ then test gives no answer.

  not needed to conv.
- (ratio test) Let $\sigma = \limsup \left| \frac{a_{n+1}}{a_n} \right|$, then we have the following cases
  1. if $\sigma < 1$ then $\sum a_n$ converges
  2. if $\sigma > 1$ then $\sum a_n$ diverges
  3. if $\sigma = 1$ then test gives no answer.
- $\sum a_n$ converges iff $\sum 2^n a_n$ converges.

Power Series

function.

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Let c_n be a sequence in $\mathbb{C}$ and $z \in \mathbb{C}$, consider the series

$$\sum_{n=0}^{+\infty} \boxed{c_n z^n},$$

series of functions

This is called power series and its convergence depends of z

Theorem

Let $\alpha = \limsup \sqrt[n]{|c_n|}$, then the series converges for every z such that

$$|z| \leq R < \frac{1}{\alpha}$$

the radius of converge.

Def The radius of convergence of a power se

$\sup \{ r \in \mathbb{R}^+ \ / \ \sum c_n z^n$ converges for $|z| \leq r \}$

Example $\quad C_n = 1$ for all n

$$\sum C_n z^n = \sum_{n=0}^{+\infty} z^n$$

Geometric series converge

$\iff |z| < 1$

1 is the radius of convergence of the sequence

$$c_n = 1$$

$$\sqrt[n]{|c_n|} = 1 \qquad \forall n$$

$$\limsup \sqrt[n]{|c_n|} = 1$$

$\Rightarrow$ the series converges for all

$$|z| \leq R < 1$$

$$|z| < 1.$$

Questions?

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$a_n \longrightarrow p$

$\iff \quad \limsup a_n = \liminf a_n$

$\limsup a_n \geq \liminf a_n$

$\sup E \qquad \inf E$

$\limsup a_n = \liminf a_n =: p$

$\iff \quad E = \{p\} \iff$ all convergent subsequences of $a_n \to p$

$\iff \quad a_n \longrightarrow p$

Thank you for your attention!