MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Beräkningsmatematik
Examinator: Marc Hellmuth

Tentamensskrivning i
Algoritmer och komplexitet
7.5 hp
2025-12-11

General Information

- Time 08:00 - 12:00
- No books, internet, smartphones, notebooks etc allowed.
- In total, you can get 52 points and you need to get at least 26 points to pass the exam.
- There are statements "Prove *in detail* [..]".

  In detail means that arguments "as shown in the lecture" are not sufficient however and that a rigorous proof is required. However, you are allowed to apply the standard definitions like the axioms for matroids or terms like "connected", "cycle", etc.
- Make sure not to overlook any of the **8** problems.

  On average, you have 30 minutes for each of the 8 problems.

Problem 1 (*Turing Machine (TM)*) 5p

Let $L = \{0^n 1^n \mid n \in \mathbb{N}_{\geq 1}\} \subseteq \Sigma^*$ with $\Sigma = \{0, 1\}$ be the language that consists of all strings that start with n 0's followed by the same number n of 1's,

Provide a TM (in simplified finite state representation) that recognizes L.

Show the behaviour of your TM on the strings 0011, 011 and 001.

Problem 2 (*Runtime, Big-$\mathcal{O}$ notation and Graphs*) 3+(2+2)=7p

(a) Let S be a finite set of size $|S| = N$ and $k \leq N$ be a positive integer. Show that the number of all subsets S' of S of size $|S'| = k$ is in $\mathcal{O}(N^k)$.

(b) Let $G = (V, E)$ be an undirected graph and define the neighborhood of v as $N(v) = \{w \in V \mid \{v, w\} \in E\}$ for all $v \in V$. Moreover, put $\xi(G) := \sum_{v \in V} |N(v)|$

Prove in detail that

$$\xi(G) \in O(|E|).$$

for all undirected graphs G.

Provide a class of connected undirected graphs $G = (V, E)$ for which $\xi(G) \in \mathcal{O}(|V|)$ and where $|V|$ is not treated as a constant. Briefly explain your results.

Problem 3 (*Complexity*) (2+1)+5=8p

(a) Define the class P. Is the statement "NP $\setminus$ P $= \emptyset$" true?

(b) Consider the following two problems:

Clique:

Given an undirected graph G and an integer $k \in \{1, \ldots, |V(G)|\}$

Does there exists a subgraph $H \subseteq G$ such that H is isomorphic to a complete graph K_ℓ on $\ell \geq k$ vertices?

IndependentSet:

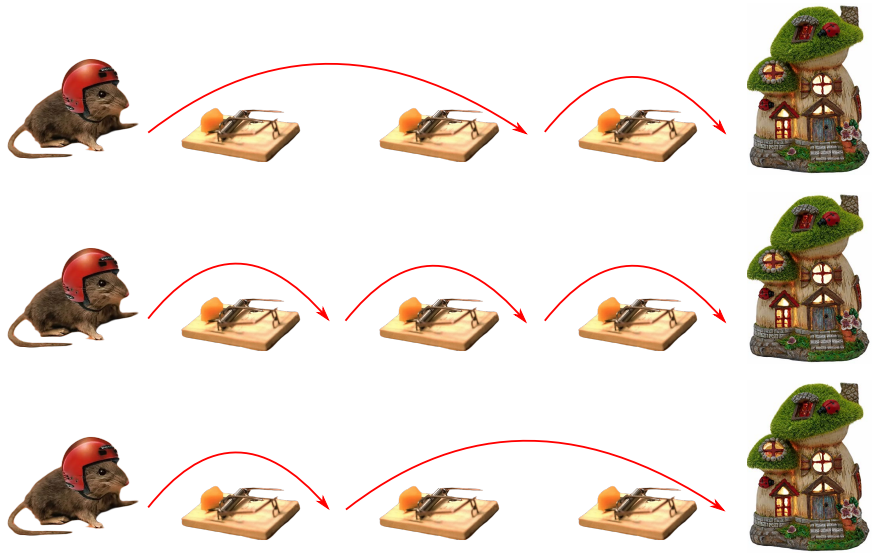
Given an undirected graph $G = (V, E)$ and an integer $k \in \{1, \ldots, |V(G)|\}$

Does there exists a subset $I \subseteq V$ of size $|I| \geq k$ such that for all $u, v \in I$ it holds that $\{u, v\} \notin E$. In other words, the subgraph of G induced by the vertices in I is edge-less.

Assume that we know that Clique is NP-complete. Prove that IndependentSet is NP-complete by reduction from Clique.

Problem 4 (*Dynamic Programming*) 3+4 = 7p

You are a mouse and you want to go home. There is only one path to your home and this path is filled with n mousetraps. Luckily, you can jump. In one jump, you can jump over 1 or 2 mousetraps. The following figure shows all three different ways to jump over 3 mousetraps.



(a) What is the number of different ways you can reach your home for $n = 5$ mousetraps?

(b) Give a recursive formula for counting the number of different ways you can reach your home. Explain, why your formula works (don't forget the base case).

Problem 5 (*Greedy and Matroids*) 3+5=8p

In the following, E is a finite set and $\mathbb{F}$ a collection of subsets of E.

(a) Give the properities that must be satisfied for $(E, \mathbb{F})$ to be a matroid.

(b) Consider the following problem. Let S be a collection of students and let $c\colon S \to \{1, \ldots, m\}$, $m \geq 1$ be a map that assigns to every student $s \in S$ a particular skill $c(s)$.

The simple aim is to find a maximum-sized subset $S' \subseteq S$ of students such that all students in S' have different skills, i.e., $c(s) \neq c(s')$ for all distinct $s, s' \in S'$

Define the independence system $(E, \mathbb{F})$ that describes this problem and also prove that $(E, \mathbb{F})$ is a matroid.

Problem 6 (*Approximation Algorithms*) 2+3=5p

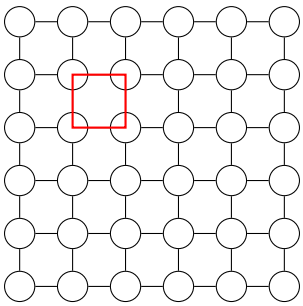
(a) Give the definition of a ρ-approximation algorithm for an optimization problem.

(b) Provide an arbitary optimization problem for which a polynomial ρ-approximation algorithm with $\rho = 1$ exists. Explain your results.

Problem 7 (*Fixed Parameter Tractability (FPT)*) 2+6 = 8p

(a) Given is the branching vector $(2, 1, 1)$. Provide a bound for the number of leaves in a search tree to which this branching vector applies.

(c) Shown is a sketch of graph G that reflects an $n \times n$ grid (here $n = 6$).



A square in G is a cycle consisting of exactly 4 vertices. One of such squares is highlighted in G. Consider the following problem.

DeleteSquares:

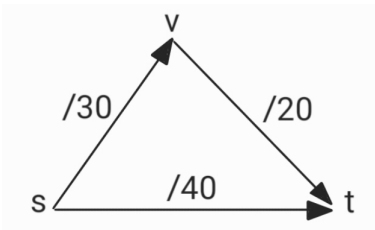
Given an $n \times n$ grid $G = (V, E)$ and an integer $k \in \{1, \ldots, |V(G)|\}$
Does there exists a subset $D \subseteq V$ of size $|D| \leq k$ such that removing all vertices in D (and their incident edges) from G results in a graph H that does not contain squares?

Show that DeleteSquares if FPT when parameterized by k.

Hint: you may show first that finding squares in any graph G can always be achieved in polynomial time. Then, you may use Bounded Search Tree.

Problem 8 (*Maximum Flow*) 4p

Given is the following flow network $G = (V, E)$ together with capacities along its edges. Non-edges (u, v) have capacity $c(u, v) = 0$.



Apply the Ford-Fulkerson-Algorithm to compute a maximum flow in G. Draw for each individual step the residual network and provide the value of a maximum flow.