Lecture 11: Stochastic Calculus

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1. Definition and Properties of Stochastic Integrals

1.1. Model assumptions

Our goal is to define so-called stochastic integrals of the following form,

\[ I(t) = \int_0^t g(s)ds + \int_0^t h(s)dW(s) \]

Riemann integral \hspace{1cm} Stochastic integral
for stochastic \hspace{1cm} for stochastic function \( h \)
function \( g \) \hspace{1cm} with respect to BM

Let \( \{W(t), t \geq 0\} \) be a standard Brownian Motion (Wiener process) defined on a probability space \( <\Omega, \mathcal{F}, \mathcal{P}> \).

Let also \( \{\mathcal{F}_t, t \geq 0\} \) be a filtration for the same probability space, i.e., a family of \( \sigma \)-algebras satisfying the following conditions:

A: \( \mathcal{F}_t \subseteq \mathcal{F}, t \geq 0 \).

B: \( \mathcal{F}_t \subseteq \mathcal{F}_s \) if \( t \leq s \).

We also assume that the following additional conditions hold:

C: \( W(t) \) is \( \mathcal{F}_t \)-measurable for every \( t \geq 0 \), i.e. process \( W(t) \) is adopted to the filtration \( \{\mathcal{F}_t\} \).

D: \( W(t + s) - W(t) \) is independent of \( \mathcal{F}_t \), for every \( t, s \geq 0 \).
The simplest filtration that satisfies conditions A - D is

$$\mathcal{F}_t = \sigma(W(s), 0 \leq s \leq t), \; t \geq 0$$

Let also $H_2[0, T]$ be a space of stochastic functions $\{f(t), \; t \geq 0\}$ defined on the same probability space $<\Omega, \mathcal{F}, \mathcal{P}>$ and satisfying the following conditions:

E: $f(t)$ is $\mathcal{F}_t$ - measurable for every $t \geq 0$ ($f(t)$ is adopted to the filtration $\{\mathcal{F}_t\}$).

F: $\int_0^T E f^2(t) dt < \infty$.

1.2 Definition of stochastic integral

We define stochastic integral,

$$I = \int_0^T f(t) dW(t)$$

in two steps, first for stochastic step functions (integral sums), and then for functions $\{f(t)\} \in H_2[0, T]$ using the corresponding limit transition.

Let $f(t)$ be a stochastic step function, i.e., a stochastic function from the space $H_2[0, T]$ (in this case, condition E holds if $f(t_k)$ is $\mathcal{F}_{t_k}$-measurable for every $k = 0, 1, \ldots, m$) satisfying the following condition:

$$f(t) = f(t_k) \text{ for } t_k \leq t < t_{k+1}, \; k = 0, \ldots, m \quad (1)$$

where $0 = t_0 < t_1 < \ldots < t_m = T$. 

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Definition 11.1. The stochastic integral for a step function \( f(t) \) from the space \( H_2[0, T] \) is defined by the formula,

\[
I_T(f) = \int_0^T f(t) dW(t) = \sum_{k=0}^{m-1} f(t_k) \left( W(t_{k+1}) - W(t_k) \right). \quad (2)
\]

Henceforth, use notation \( E_t X = E\{X / \mathcal{F}_t\} \).

\[ (1) \quad E I_T(f) = 0. \]

(a) Using conditions D and E, we get,

\[
E \int_0^T f(t) dW(t) = E \left[ \sum_{k=0}^{m-1} f(t_k) \left( W(t_{k+1}) - W(t_k) \right) \right]
\]

\[
= \sum_{k=0}^{m-1} E \left\{ E_{t_k} \left\{ f(t_k) \left( W(t_{k+1}) - W(t_k) \right) \right\} \right\}
\]

\[
= \sum_{k=0}^{m-1} E \left\{ f(t_k) E_{t_k} \left( W(t_{k+1}) - W(t_k) \right) \right\}
\]

\[
= \sum_{k=0}^{m-1} E \left\{ f(t_k) E \left\{ W(t_{k+1}) - W(t_k) \right\} \right\} =
\]

\[
= \sum_{k=0}^{m-1} E \left\{ f(t_k) \times 0 \right\} = 0.
\]

\[ (2) \quad E(I_T(f))^2 = \int_0^T E f(t)^2 dt. \]
(a) Using conditions D and E, we get,
\[
E\left(\int_0^T f(t) dW(t)\right)^2 = E\left(\sum_{k=1}^{m-1} f(t_k) (W(t_{k+1}) - W(t_k))\right)^2
\]
\[
= E\sum_{k=1}^{m-1} f(t_k)^2 (W(t_{k+1}) - W(t_k))^2
\]
\[
+ 2E\sum_{k<r} f(t_k)f(t_r) (W(t_{k+1}) - W(t_k))(W(t_{r+1}) - W(t_r)) =
\]
\[
= \sum_{k=1}^{m-1} E\left\{f(t_k)^2 E_{t_k}\left\{(W(t_{k+1}) - W(t_k))^2\right\}\right\} +
\]
\[
+ 2\sum_{k<r} E\left\{f(t_k)f(t_r) (W(t_{k+1}) - W(t_k))E_{t}\left\{(W(t_{r+1}) - W(t_r))\right\}\right\}
\]
\[
= \sum_{k=1}^{m-1} Ef(t_k)^2(t_{k+1} - t_k) + 0 = \int_0^T Ef(t)^2 dt.
\]

Let \( f(t) \) be a stochastic function from the space \( H_2[0,T] \).

**Theorem 11.1.** If conditions A – F holds, then:

I. There exist step functions \( f_n(t) \) from the space \( H_2[0,T] \) such that
\[
\lim_{n \to \infty} \int_0^T E|f(t) - f_n(t)|^2 dt = 0. \tag{3}
\]

II. Random variables \( I_T(f_n) = \int_0^T f_n(t) dW(t) \) converge in mean-square sense to some random variable \( I \), i.e.,
\[
E(I - \int_0^T f_n(t) dW(t))^2 \to 0 \text{ as } n \to \infty. \tag{4}
\]
III. The limit \( I \) does not depend on the choice of approximating sequence \( f_n(t) \), for which relation (3) holds, i.e., any two such limits a.s. coincide.

Definition 11.2. The stochastic integral for a function \( f(t) \) from the space \( H_2[0, T] \) is defined as a random variable \( I \) that penetrates Theorem 11.1, i.e.,

\[
I = I_T(f) = \int_0^T f(t)dW(t). \tag{5}
\]

(a) \( H_2[0, T] \) is a linear space with a norm squared \( \|f\|^2 = \int_0^T \mathbb{E}f(t)^2dt \). Proposition I follows from the corresponding general property of this space.

(b) Relation (3) implies that

\[
\lim_{n,m \to \infty} \int_0^T \mathbb{E}|f_n(t) - f_m(t)|^2dt = 0. \tag{6}
\]

(c) Thus, by property (2) for stochastic integrals for step functions,

\[
\lim_{n,m \to \infty} \mathbb{E}\left( I_T(f_n(t)) - I_T(f_m(t)) \right)^2 = \lim_{n,m \to \infty} \mathbb{E}\left( I_T(f_n(t) - f_m(t)) \right)^2 \\
= \lim_{n,m \to \infty} \int_0^T \mathbb{E}|f_n(t) - f_m(t)|^2dt = 0. \tag{7}
\]

(d) Thus, the sequence \( I_T(f_n(t)) \) is fundamental, and, therefore, exists a random variable \( I \) such that \( \mathbb{E}(I_T(f_n(t)) - I)^2 \to 0 \) as \( n \to \infty \).
(e) If also $E(I_T(f_n'(t)) - I')^2 \to 0$ as $n \to \infty$ then $E(I - I')^2 = E\left((I - I_T(f_n(t)) + (I_T(f'_n(t)) - I) + I_T(f_n(t) - f'_n(t))\right)^2 \leq 3(E(I_T(f_n(t)) - I)^2 + E(I_T(f'_n(t)) - I')^2 + E(\int_0^T (f_n(t) - f'_n(t))dt)^2) \to 0$ as $n \to \infty$ since $E(\int_0^T (f_n(t) - f'_n(t))dt)^2 = \int_0^T E(f_n(t) - f'_n(t))^2dt \leq 2(\int_0^T E(f_n(t) - f(t))^2dt + \int_0^T E(f'_n(t) - f(t))^2dt \to 0$ as $n \to \infty$. Thus, $P\{I = I'\} = 1$.

\[ \text{(1)} \] The condition $F$ can be weaken to the assumption that $\int_0^T f^2(t)dt < \infty \text{ a.s.}$ in the definition of stochastic integrals $\int_0^T f(t)dW(t)$. See SK: Chapter 30.

1.3 Properties of stochastic integral

The properties listed below can be first by checking them for step stochastic functions from the space $H_2[0, T]$ and then by translating them to any stochastic function from the space $H_2[0, T]$ with the use of limiting transition based on approximation relations given in Theorem 11.1.

\[ \text{(1)} \] If $f_1, f_2 \in H_2[0, T]$ and $\alpha_1, \alpha_2$ are random variables such that $\alpha_1 f_1(t), \alpha_2 f_2(t) \in H_2[0, T]$, then $\alpha_1 f_1(t) + \alpha_2 f_2(t) \in H_2[0, T]$ and,

\[
\int_0^T (\alpha_1 f_1(t) + \alpha_2 f_2(t))dW(t) = \alpha_1 \int_0^T f_1(t)dW(t) + \alpha_2 \int_0^T f_2(t)dW(t). \tag{8}
\]
(2) The following formula takes place for any \([a, b] \subseteq [0, T]\),
\[
\int_0^T I_{[a,b]}(t)dW(t) = W(b) - W(a).
\]  

(3) If \(f \in H_2[0, T]\), then,
\[
\mathbb{E}I_T(f) = 0,
\] \(E(I_T(f))^2 = \int_0^T \mathbb{E}f(t)^2 dt.\)

(4) Stochastic integral \(I_t(f) = \int_0^t f(s)dW(s)\), as function of the upper limit \(t\), is a continuous stochastic process.

(5) Stochastic process \(I_t(f)\) is a martingale.

2. Stochastic differential

2.1. Definition of stochastic differential

**Definition 11.3** Let a stochastic process \(X(t), t \geq 0\) satisfy the following relation
\[
X(t_2) - X(t_1) = \int_{t_1}^{t_2} a(t)dt + \int_{t_1}^{t_2} b(t)dW(t), 0 \leq t_1 \leq t_2 \leq T,
\] \(\sqrt{|a(t)|}\) and \(b(t)\) are stochastic functions from \(H_2[0, T]\).

Then we say that the process \(X(t)\) has Itô differential,
\[
dX(t) = a(t)dt + b(t)dW(t).
\]
\[ F(t) = \int_0^t f(x)dg(x) = \]
\[ = \lim_{n \to \infty} \sum_k f(t_{nk})(g(t_{nk+1}) - g(t_{nk})) \]
\[ F(t_2) - F(t_1) = \int_{t_1}^{t_2} f(x)dg(x) \]
\[ dF(x) = f(x)dg(x) \]

If \( g(x) = x \) then
\[ F(t) = \int_0^t f(x)dx \]
\[ F(t_2) - F(t_1) = \int_{t_1}^{t_2} f(x)dx \]
\[ dF(x) = f(x)dx \] that means \( F'(x) = f(x) \).

2.2 Simple stochastic calculus

Simple calculus

\[ dx = 1 \cdot dx \ (f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}, \ x' = 1) \]
\[ df(x)g(x) = f(x)dg(x) + g(x)df(x) \]
\[ dx^2 = d(x \cdot x) = dx + dx = 2xdx \]
\[ \ldots \]
\[ \left\{ \begin{array}{l}
  dx^m = mx^{m-1}dx \\
  \int_{x_1}^{x_2} mx^{m-1}dx = x^m
\end{array} \right. \]
Lemma 11.1. The following formula takes place,
\[ dW^2(t) = dt + 2W(t)dW(t) \] \hspace{1cm} (14)

Lemma 11.2. The following formula takes place,
\[ d(tW(t)) = W(t)dt + tdW(t) \] \hspace{1cm} (15)

(a) Let \( t_1 = t_{n0} < t_{n1} < \ldots < t_{nn} = t_2 \) and \( \Delta_n = \max_k (t_{nk+1} - t_{nk}) \to 0 \) as \( n \to \infty \).

(b) By the definition of stochastic integral,
\[
\int_{t_1}^{t_2} W(t)dW(t) = \lim_{n \to \infty} (m.s.) \sum_{k=0}^{n-1} W(t_{nk})(W(t_{nk+1}) - W(t_{nk})). \hspace{1cm} (16)
\]

(c) The following formulas take place,
\[
\sum_{k=0}^{n-1} W(t_{nk})(W(t_{nk+1}) - W(t_{nk})) = \frac{1}{2} \sum_{k} \left( (W(t_{nk+1})^2 - W(t_{nk})^2) - (W(t_{nk+1}) - W(t_{nk}))^2 \right)
\]
\[
= \frac{1}{2} (W(t_2)^2 - Z(t_1)^2) - \frac{1}{2} \sum_{k=p}^{n-1} (W(t_{nk+1}) - W(t_{nk}))^2
\]
\[
= \frac{1}{2} (W(t_2)^2 - W(t_1)^2) - \frac{1}{2} \sum_{k=p}^{n-1} (W(t_{nk+1}) - W(t_{nk}))^2. \hspace{1cm} (17)
\]
(d) We also should prove that the so-called quadratic variation for BM,

\[ [W, W]_{t_2} = \lim_{n \to \infty} (m.s.) \sum_{k=0}^{n-1} (W(t_{nk+1}) - W(t_{nk}))^2 = t_2 - t_1. \]  

(18)

(e) Relations (16) – (18) imply that

\[ W(t_2)^2 - W(t_1)^2 = \int_{t_1}^{t_2} 1 \cdot dt + \int_{t_1}^{t_2} 2W(t)dW(t) \]  

(19)

(f) This is equivalent to \( dW^2(t) = dt + 2W(t)dW(t) \).

(g) Indeed,

\[ \mathbb{E}\left(\sum_k (W(t_{nk+1}) - W(t_{nk}))^2\right) = \sum_k (t_{nk+1} - t_{nk}) = t_2 - t_1. \]  

(20)

and

\[
\text{Var}\left(\sum_k (W(t_{nk+1}) - W(t_{nk}))^2\right)
= \sum_k \text{Var}\left((W(t_{nk+1}) - W(t_{nk}))^2\right)
\leq \sum_k \mathbb{E}\left((W(t_{nk+1}) - W(t_{nk}))^4\right)
= \sum_k 3(t_{nk+1} - t_{nk})^2
\leq 3\Delta_n \sum_k (t_{nk+1} - t_{nk})
= 3\Delta_n (t_2 - t_1) \to 0 \text{ as } n \to \infty.
\]  

(21)
(h) Thus,
\[ E\left| \sum_k \left( W(t_{nk+1}) - W(t_{nk}) \right)^2 - (t_2 - t_1)^2 \right| \rightarrow 0 \text{ as } n \rightarrow \infty. \] (22)

(i) By the definition of stochastic integral,
\[
\int_{t_1}^{t_2} W(t)dt + \int_{t_1}^{t_2} tdW(t) = \lim_{n \to \infty} \left( \sum_{k=0}^{n-1} W(t_{nk+1})(t_{nk+1} - t_{nk})
+ \sum_{k=0}^{n-1} t_{nk}(W(t_{nk+1} - W(t_{nk})) \right). \] (23)

(j) But,
\[
\sum_{k=0}^{n-1} W(t_{nk+1})(t_{nk+1} - t_{nk}) + \sum_{k=0}^{n-1} t_{nk}(W(t_{nk+1} - W(t_{nk}))
= \sum_{k=0}^{n-1} \left( t_{nk+1}W(t_{nk+1}) - t_{nk}W(t_{nk}) \right)
= t_2W(t_2) - t_1W(t_1). \] (24)

(k) Relations (23) and (24) imply that
\[
t_2W(t_2) - t_1W(t_1) = \int_{t_1}^{t_2} W(t)dt + \int_{t_1}^{t_2} tdW(t). \] (25)

(l) This is equivalent to relation \( d(tW(t)) = W(t)dt + tdW(t). \)
2.3 Product and chain rules

**Theorem 11.2 (Product rule).** Let

\[
\begin{align*}
  dX_1(t) &= a_1(t)dt + b_1(t)dW(t), \\
  dX_2(t) &= a_2(t)dt + b_2(t)dW(t),
\end{align*}
\]

(26)

that is equivalent to

\[
\begin{align*}
  X_1(t) &= X_1(0) + \int_0^t a_1(s)ds + \int_0^t b_1(s)dW(s), \\
  X_2(t) &= X_2(0) + \int_0^t a_2(s)ds + \int_0^t b_2(s)dW(s),
\end{align*}
\]

(27)

then

\[
\begin{align*}
  d\left(X_1(t) \cdot X_2(t)\right) &= X_1(t)dX_2(t) + X_2(t)dX_1(t) + b_1(t)b_2(t)dt.
\end{align*}
\]

(28)

(a) Let assume that stochastic coefficients \(a_1(t) = a_1, \ b_1(t) = b_1, \ a_2(t) = a_2, \ b_2(t) = b_2\) do not depend on \(t\). Then,

\[
\begin{align*}
  X_1(t) &= a_1t + b_1W(t) \\
  X_2(t) &= a_2t + b_2Z(t)
\end{align*}
\]

(29)

(b) So,

\[
X_1(t)X_2(t) = a_1a_2t^2 + (a_1b_2 + a_2b_1)tW(t) + b_1b_2W(t)^2.
\]

(30)

(c) Thus, using Lemmas 11.1 and 11.2, we get,

\[
\begin{align*}
  d\left(X_1(t)X_2(t)\right) &= 2a_1a_2t dt \\
  &\quad + (a_1b_2 + a_2b_1)(W(t)dt + tdW(t)) \\
  &\quad + b_1b_2(dt + 2W(t)dt) \\
  &= (a_1t + b_1W(t))(a_2dt + b_2dW(t)) \\
  &\quad + (a_2t + b_2W(t))(a_1dt + b_1dW(t)) + b_1b_2dt \\
  &= X_1(t)dX_2(t) + X_2(t)dX_1(t) + b_1b_2dt.
\end{align*}
\]

(31)
(d) The corresponding limiting transition let one translate the
formula to the case of step-wise coefficients and then to coeffi-
cients depending on $t$ satisfying conditions imposed on them in
the definition of stochastic differentials.

Theorem 11.3. The following formula takes place for $m \geq 2$,

$$dW(t)^m = mW(t)^{m-1}dW(t) + \frac{m(m-1)}{2}W(t)^{m-2}dt. \quad (32)$$

(a) $m = 2$:

$$dW^2(t) = 2W(t)dW(t) + dt. \quad (33)$$

(b) $m = 3$:

$$dW^3(t) = d(W(t)^2 \cdot W(t))$$
$$= (2W(t)dW(t) + dt)W(t) + W(t)^2dW(t) + 2W(t) \cdot 1dt$$
$$= 3W^2(t)dW(t) + 3W(t)dt. \quad (34)$$

(c) $\cdots$.

Theorem 11.4. Let $P(t) = a_0 + a_1 + \ldots + a_m t^m$ be a polynom
of degree $m$. Then,

$$dP(W(t)) = P'(W(t))dW(t) + \frac{1}{2}P''(W(t))dt. \quad (35)$$

Theorem 11.5 (chain rule). Let $f(t)$ be a function that
has the derivative $f''(t)$ which is continuous. Then,

$$df(W(t)) = f'(W(t))dW(t) + \frac{1}{2}f''(W(t))dt. \quad (36)$$
(a) It can be conducted by mean-square approximation of functions $f(t)$, $f'(t)$, $f''(t)$ by polynomials.

(1) It is useful to compare formula (36) with the corresponding deterministic formula,

$$df(g(x)) = f'(g(x))dg(x) = f'(g(x))g'(x)dx.$$  \hspace{1cm} (37)

3. Itô’s formula

3.1. Variants of Itô’s formulas

**Theorem 11.6 (1st Itô formula).** Let $f(t, x)$ be a function that have continuous derivatives $f'_t$, $f'_x$ and $f''_{xx}$. Then,

$$df(t, W(t)) = \left(f'_t(t, W(t)) + \frac{1}{2}f''_{xx}(t, W(t))\right)dt$$

$$+ f'_x(t, W(t))dW(t).$$  \hspace{1cm} (38)

(a) Let consider first the case when $f(t, x) = h(t)g(x)$ then Theorem 11.5 implies that

$$d\left(h(t)g(W(t))\right) = g(W(t))h'(t)dt + h(t)dg(W(t))$$

$$= \left(g(W(t))h'(t) + \frac{1}{2}h(t)g''(W(t))\right)dt$$

$$+ h(t)g'(W(t))dW(t).$$  \hspace{1cm} (39)

(b) So, (38) holds for functions of the form $f(t, x) = h(t)g(x)$.  

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(c) The general case can be proved by approximation of functions $f$, $f'_t$, $f'_x$ and $f''_{xx}$ by functions of the form $\sum a_k f_k(t)g_k(x)$ and their derivatives.

\textbf{Theorem 11.6 (Itô formula).} Let $dx(t) = a(t)dt + b(t)W(t)$ and $f(t, x)$ be a function that has continuous derivatives $f'_t$, $f'_x$ and $f''_{xx}$. Then, the following formula takes place,

$$
df(t, X(t)) = \left( f'_t(t, X(t)) + \frac{1}{2} f''_{xx}(t, X(t))b^2(t) \right) dt + f'_t(t, X(t))b(t)dW(t). \tag{40}$$

(a) Let $X(t) = x_0 + at + bW(t)$. Then, $f(t, x_0 + at + bW(t)) = \Phi(t, W(t))$, where $\Phi(t, x) = f(t, x_0 + at + bx)$.

(b) In this case,

$$
\Phi'_t(t, x) = f'_t(t, x_0 + at + bx) + f'_x(t, x_0 + at + bx)a,
\Phi'_x(t, x) = f'_x(t, x_0 + at + bx)b,
\Phi''_{xx}(t, x) = f''_{xx}(t, x_0 + at + bx)b^2. \tag{41}
$$

(c) Now application of Theorem 11.6 to function $\Phi(t, x)$ yields (40) in the case where $a(t) = a$, $b(t) = b$ are random variables which do not depend on $t$.

(d) Formula (40) for such $a(t)$ and $b(t)$ implies this formula for step stochastic function.

(e) The last step is based on the corresponding limiting transition from step stochastic function $X(t)$ to the case of general $X(t)$ with Itô differential.
4.2 Examples

(1)

(1.1) Consider a Geometrical Brownian Motion

\[ S(t) = S(0)e^{\mu t + \sigma W(t)}, \quad t \geq 0. \] (42)

(1.2) In this case, (a) \( f(t, x) = S(0)e^{\mu t + \sigma x} \); (b) \( f'_t(t, x) = S(0)e^{\mu t + \sigma x} \mu \); (c) \( f'_x(t, x) = S(0)e^{\mu t + \sigma x} \sigma \); (d) \( f''_{xx}(t, x) = S(0)e^{\mu t + \sigma x} \sigma^2 \).

(1.3) Application of Itô formula yields the following relation,

\[
\begin{align*}
&dS(t) = \left( S(0)\mu e^{\mu t + \sigma W(t)} + \frac{1}{2} S(0)\sigma^2 e^{\mu t + \sigma W(t)} \right) dt \\
&\quad + S(0)\sigma e^{\mu t + \sigma W(t)} dW(t). 
\end{align*}
\] (43)

(1.4) In this way, we have get a stochastic differential equation for the geometrical Brownian Motion,

\[
\begin{align*}
&dS(t) = (\mu + \frac{1}{2} \sigma^2)S(t)dt + \sigma S(t)dW(t). 
\end{align*}
\] (44)

(2)

(2.1) Let there exist the stochastic differential \( dX(t) = a(t)dt + b(t)dW(t) \) and function \( f(x) = \frac{1}{x} \).

(2.2) In this case, \( f(t, x) = f(x) = \frac{1}{x} \) and, thus, (a) \( f'_t = 0 \); (b) \( f'_x = -\frac{1}{x^2} \); (c) \( f''_{xx} = \frac{2}{x^3} \).
(2.3) Application of Itô formula yields the following relation,

\[
\begin{aligned}
d\frac{1}{X(t)} &= \left( 0 + \frac{1}{2} \frac{2}{X(t)^3} b^2(t) - \frac{a(t)}{X(t)^2} \right) dt \\
&\quad - \frac{1}{X(t)^2} b(t) dW(t) = \frac{b^2(t)}{X(t)^3} - \frac{dX(t)}{X(t)^2}.
\end{aligned}
\]  

(45)

(3)

(3.1) Let there exist stochastic differentials,

\[
\begin{align*}
dX_1(t) &= a_1(t) dt + b_1(t) dW(t), \\
\begin{align*}
dX_2(t) &= a_2(t) dt + b_2(t) dW(t).
\end{align*}
\end{align*}
\]  

(46)

(3.2) Then,

\[
\begin{aligned}
d\frac{X_1(t)}{X_2(t)} &= d\left( X_1(t) \cdot \frac{1}{X_2(t)} \right) \\
&= X_1(t) d\frac{1}{X_2(t)} + \frac{1}{X_2(t)} dX_1(t) + \left[ - \frac{b_2(t)}{X_2(t)^2} \right] b_1(t) dt \\
&= X_1(t) \left[ - \frac{dX_2(t)}{X_2(t)^2} + \frac{b_2(t)^2}{X_2(t)^3} dt \right] \\
&\quad + \frac{1}{X_2(t)} dX_1(t) - \frac{b_1(t) b_2(t)}{X_2(t)^2} dt \\
&= \frac{X_2(t) dX_1(t) - X_1(t) dX_2(t)}{X_2(t)^2} \\
&\quad + \frac{b_2(t)^2 - b_1(t) b_2(t) X_2(t)}{X_2(t)^3} dt.
\end{aligned}
\]  

(47)
4. LN Problems

11.1. Let $dX(t) = a(t)dt + b(t)dW(t)$ find $df(t, X(t))$ for the cases:
(a) $f(t, x) = \ln x$;
(b) $f(t, x) = e^{tx}$;
(c) $f(t, x) = \frac{x}{x+t}$;
(d) $f(t, x) = \sin(tx)$;
(e) $f(t, x) = x^\alpha$.

11.2. Let $dX(t) = a(t)dt + b(t)dW(t)$. Prove that $X(t)$ is a martingale, i.e., $E_t\{X(t+s)\} = X(t)$, $s, t \geq 0$ if $Ea(s) = 0$, $s \geq 0$.

11.3. Let $X(t) = \exp\{bW(t) - \frac{b^2t}{2}\}$. Using Itô formula, prove that $X(t)$ is a martingale.