

How many trees can one see in a Poisson forest?

Lauge Gregers Hedegaard

Kandidatuppsats i matematisk statistik Bachelor Thesis in Mathematical Statistics

Kandidatuppsats 2019:14 Matematisk statistik Juni 2019

www.math.su.se

Matematisk statistik Matematiska institutionen Stockholms universitet 106 91 Stockholm

Matematiska institutionen



Mathematical Statistics Stockholm University Bachelor Thesis **2019:14** http://www.math.su.se

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June 2019

Abstract

We let each point in a two dimensional homogeneous Poisson process with intensity λ be the center of a disk of radius r. By an argument of scaling, we can assume r = 1 without loss of generality. We ask for how many disks it is possible to draw a straight line from its boundary to the origin with crossing through other disks. We find the expectation to be $e^{-\lambda \pi} \left(\frac{\pi}{\lambda} + \frac{\pi^2}{2}\right)$ and discuss why the variance is difficult to find.

^{*}Postal address: Mathematical Statistics, Stockholm University, SE-106 91, Sweden. E-mail: lauge@gregershedegaard.dk. Supervisor: Pieter Trapman and Abhishek Pal Majumder.

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1 Introduction

1.1 Overview

Imagine standing in a forest of randomly placed trees all being perfectly circular having radius one. How many trees are visible from our location, if we can look in all 360 degrees?

A more precise definition of the model is as follows:

Let each point in a two dimensional homogeneous Poisson process with rate λ be the center of a disk with radius r. Let a tree be visible, if a straight line can be drawn from some point on its boundary to the origin without passing through the interior of any tree.

The question is then, from the boundary of how many trees do we expect that, we can draw a straight line, not passing through the interior of any tree, to the origin? Or simply, how many trees do we expect to be visible from the origin? Also what is the variance in the number of visible trees? It will also be of interest to ask what happens, when the intensity grows to infinity.

Following directly from the definition of the model, if the interior of a tree overlaps with the origin, we will not be able to see anything in the forest. Trees overlapping with the origin can be handled in other ways, including the following three. Modify the definition such that the line drawn may not pass through the interior of any *other* tree, thus if a single tree overlaps with the origin it will be the only visible. Let all trees overlapping with the origin be visible, this model wouldn't be very interesting, as more trees overlap with the origin, this can be interpreted as "not standing inside of a tree", however the following argument of scaling will not work, as the area conditioned on not containing any points changes, when the radii change.

A way to imagine, what happens to the process as λ and r change, is to realise, that an equivalence exists between changing the radii and changing the intensity. If we imagine having the unit intensity of 1 tree per m² with unit trees, r = 1 m, then this can also be expressed as having the intensity of 1,000,000 trees per km² and trees with radii r = 0.001 km. Thus if we ignore the units, when we increase the intensity we will have to decrease the radii to not alter the process. It should be quite intuitive that increasing the radii of the trees, without decreasing the intensity of trees, will lead to stochasticly less trees being visible. Thus increasing the intensity, without decrasing the radii, will have the same effect, that is less trees being visible. Thus we do not have to include the radii, r, in the calculations but can assume that we have r = 1.

Using the above argument we expect the mean number of visible trees to tend to zero and $\lambda \to \infty$. Another way to imagine this, is to realise that as the intensity grows it will be more and more likely, that a tree overlaps with the origin, and thus nothing is visible.

1.2 Motivation

A quite gruesome motivation for this model is the 2005 London tube bombings. It was speculated, that a reason for the low number of deaths and injuries was, quite counter-intuitively, the high number of people on the train and the fact, that they were packed very tightly, essentially making the people closer to the bomb human shields for those further away. Thus it is of interest to know, if it is actually true, that if you pack more people into a space fewer will be exposed to the blast of a bomb.

Of course, another much less gruesome motivation is imagining the process as trees in a (very idealised) forest and asking how many one can see, when standing in the forest.

1.3 Similar models

This model is quite similar to the dead leaves model in two dimensions, proposed first in 1968 by G. Matheron [2]. The two dimensional dead leaves model consist of a Poisson process on $\mathbb{R}^2 \times [0, \infty)$, where the each point is, as in our model, the center of a disk (parallel to the plane), the third dimension is used to keep track of "the order" of the arrivals. In higher dimensions than two, the disks are instead replaced by *d*-dimensional balls. Since an extra dimension is added, the dead leaves model provides a way to tessellate the *d*-dimensional space. In two dimensions one might imagine the process as dead leaves falling onto a table, if the leaves fall for all eternity the table will almost surely be completely covered, when looking from above, the model can be thought of as evolving in time since the set of visible leaves are changing. If the table is imagined as being a glass table, one can look from below and the leaves will form the same stationary distribution. The model gets its name from just this example of dead leaves falling from above and stacking on top of each other.

In [4] the dead leaves model is imagined in both one and two dimensions as leaves on the line and in the plane respectively. The leaves are allowed to have different sizes. The process that occurs by counting the visible parts of the leaves is then studied. A special case of the dead leaves model is the confetti percolation model, in which each disk is coloured black with probability p and white with probability 1-p. It is then asked for what values of p do we get unbounded curves to infinity consisting of points of one colour only (basically a path to infinity)? T. Müller showed in 2015 [3] that for p > 1/2 such curves of black points almost surely exist and for $p \le 1/2$ almost surely all connected components of black points are bounded. The confetti percolation model gets its name from imagining two different colours of confetti raining down on a glass table and looking below. The percolation is when p = 1/2 where neither the white nor the black points create unbounded curves to infinity.

A model, which would be more realistic for the interpretation of trees in a forest, is the Poisson disk distribution, where each point produced cannot be too close to any other point. It is a Poisson process with disks, with the condition that the disks cannot overlap. The study of this distribution however is focused on sampling from it in an efficient manner [1] and not much about its theoretical properties. One might imagine that working with the distribution theoretically would be quite the challenge, as the placement of the points in the process no longer are independent.

1.4 Notation

We will denote the Poisson processes both by the number of points in a region, $\{N(t), t \ge 0\}$ for the number of points in [0, t) for the process on $[0, \infty)$, and by the placements of the points of the process, $\{S_n, n \ge 0\}$ for the process on $[0, \infty)$ or $\{(X_n, Y_n), n \ge 0\}$ for the two dimensional process. These two notation will be used interchangeably when it is convenient.

In [5], Resnick introduces the notation $N(A) = \sum_{n} \epsilon_{X_n}$ for the number of points in the area A for a general point process. Here X_n is the distribution of points, and ϵ_{X_n} is an indicator variable being 1 if $X_n \in A$. Resnick then uses $N = \sum_{n} \epsilon_{X_n}$ as notation for the process in general. This notation is quite flexible and can easily be expanded to more than one dimension, e.g. $N = \sum_{n} \epsilon_{(X_n,Y_n)}$ if we want a process in two dimensions. The notation also allows for multiple points at each location, then $N = \sum_{n} \xi_n \epsilon_{(X_n,Y_n)}$, where $\{\xi_n\}$ indicates the number of points.

We will however use the notation $\{N(t), t \ge 0\}$ and $\{S_n, n \ge 0\}$, as they are versatile enough for the uses, we need them for. Similar notation is used in [6].

1.5 Theory

The theory in this section is mostly rewriting of [5, ch. 4] and [6, ch. 5.3].

Definition 1: (Poisson random variable)

The random variable X is said to be Poisson (distributed) with mean λ , if it takes the values k = 0, 1, 2, 3, ... with probability $\mathbb{P}(X = k) = \frac{\lambda^k}{k!}e^{-\lambda k}$, it is denoted $X \sim \text{Po}(\lambda)$.

A Poisson variable has mean $\mathbb{E}[X] = \lambda$ and variance $\operatorname{Var}(X) = \lambda$.

Definition 2: (Poisson process)

For
$$d \in \mathbb{N}_+$$
, let $m(\cdot)$ be the d-dimensional measure for subsets of \mathbb{R}^d , that is

$$m(A) = \int_A 1 d\boldsymbol{a}.$$

Then let a random process produce points in $E \subseteq \mathbb{R}^d$, and for $A \subseteq E$ let N(A) denote the number of points in A. The collection of random variables $\{N(A) : A \subseteq E\}$ is a Poisson process with rate (or intensity) $\lambda > 0$, if the following axioms are satisfied:

- N(A) is Poisson distributed with mean $\lambda \cdot m(A)$
- For all sequences of pairwise disjoint subsets of E, (A_1, A_2, \ldots) , $(N(A_1), N(A_2), \ldots)$ is a sequence of independent random variables.

If $\lambda > 0$ is constant on all of E, the process is said to be homogeneous. The definition can be extended to include the inhomogeneous case, where $\lambda = \lambda(a)$ is a non-constant, non-negative function over E. Then we need the mean measure of the region A, which is defined as

$$\mu(A) = \mathbb{E}[N(A)] = \int_A \lambda(a) da,$$

and the mean in the first axiom then becomes $\mu(A)$. The inhomogeneous Poisson process is said to have local intensity $\lambda(a)$ at a. To simplify notation, we will often not write λ as a function if it is not specifically needed.

The second axiom is called *complete randomness*, or *independent incre*ments when $E \subseteq \mathbb{R}$.

Definition 3: (o-notation)

A function $f(\cdot)$ is said to be o(h) (read little o of h) if

$$\lim_{h \to 0} \frac{f(h)}{h} = 0.$$

An important property of the Poisson process is that

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- $\mathbb{P}(N(A) = 1) = \lambda \cdot m(A) + o(m(A))$
- $\mathbb{P}(N(A) \ge 2) = o(m(A))$
- $\mathbb{P}(N(A) = 0) = 1 \lambda \cdot m(A) + o(m(A)).$

This holds for the homogeneous case. For the inhomogeneous case, $\lambda m(A)$ is substituted for $\mu(A)$, but for m(A) close to 0, the integral simply becomes $\lambda m(A)$. The property is realised by Taylor expanding the Poisson probability once with respect to m(A).

For the one-dimensional Poisson process on $E = [0, \infty)$, this can be used in the definition of the process, (together with N(0) = 0 and that $\{N(t), t \ge 0\}$ has independent increments), after which it is then deduced that the number of points in an interval must be Poisson distributed.

Proposition 1: (Transformation of the Poisson process) From [5, proposition 4.3.1, p. 310].

Let

$$T: E \mapsto E'$$

be a bijective transformation of one (Euclidian) space, E, to another, E', with the property that if $B' \subset E'$ is bounded in E', so is $T^{-1}(B')$ in E, where $T^{-1}: E' \mapsto E$ is the inverse transformation. Suppose that $\{N(\cdot)\}$ is a Poisson process on E with intensity $\lambda(\cdot)$, then $\{N'(\cdot)\} := \{N(T^{-1}(\cdot))\}$ is a Poisson process on E' with intensity $\lambda' := \lambda(T^{-1}(\cdot))$.

Proof: In the proof we will use that $\mu(A) = \int_A \lambda da$ instead of the intensity to prove both the homogeneous and inhomogeneous case. We have that

$$\mathbb{P}(N'(B') = k) = \mathbb{P}(N(T^{-1}(B)) = k) = \frac{\mu(T^{-1}(B))^k}{k!} e^{-\mu(T^{-1}(B))},$$

so N' is Poisson, and we have proved the first axiom. The complete randomness property follows from the fact that if B'_1, \ldots, B'_n are disjoint, so are $T^{-1}(B'_1), \ldots, T^{-1}(B'_n)$, and thus

$$(N'(B'_1),\ldots,N'(B'_n)) = (N(T^{-1}(B'_1)),\ldots,N(T^{-1}(B'_n)))$$

are independent random variables. Having shown that both properties are satisfied, we are done. $\hfill\blacksquare$

We need to show, how we can construct an inhomogeneous Poisson process on $E = [0, \infty)$ by transforming the homogeneous unit Poisson process.

Construction 1: (Inhomogeneous Poisson process) From[5, p. 312]. We want to construct $\{N(t), t \ge 0\}$ with mean measure μ , which is absolutely continuous, with density $\lambda(t)$. Let

$$\Lambda(t) = \mu([0,t]) = \int_0^t \lambda(s) ds,$$

and let it have the inverse

$$\Lambda^{\leftarrow}(x) = \inf\{u : \Lambda(u) \ge x\}.$$

If we also let $\Lambda(\infty) = \infty$, then $\{u : \Lambda(u) \ge x\}$ is non-empty for all x. We see that $\Lambda(x)$ is continuous, and thus $\Lambda^{\leftarrow}(x)$ is strictly increasing. We also see that $\Lambda^{\leftarrow} : [0, \infty) \mapsto [0, \infty)$. If $\{S_n, n \ge 1\}$ are the points of a homogeneous Poisson process with $\lambda = 1$, then by Proposition 1 $\{\Lambda^{\leftarrow}(S_n), n \ge 1\}$ are points of a Poisson process $\{N'(t), t \ge 0\}$. The mean measure of N' is

$$\mu'([0,t]) = |\{x : \Lambda^{\leftarrow}(x) \le t\}| = |\{x : x \le \Lambda(t)\}| = |[0,\Lambda(t)]| = \Lambda(t) = \mu([0,t]),$$

so we have constructed a Poisson process with mean measure μ , which is what we wanted.

In the above construction, the assumption of μ being absolutely continuous is often made, but it is more than is needed, it is only needed that $\Lambda(t)$ is continuous and $\Lambda(\infty) = \infty$. One might realise, that $S_n = \sum_{i=1}^n T_i \sim Gamma(n,1)$, where $T_i \sim Exp(1)$ are the interarrival times, which can be used in simulations.

Construction 2: (Two dimensional Poisson process) From [5, example 4.4.2, p. 319].

Let $\{(X_n, Y_n), n \ge 0\}$ be the points of a two dimensional Poisson process with intensity λ . Let $T(x, y) = (r, \theta) = (\sqrt{x^2 + y^2}, \arctan(\frac{y}{x}))$ be the transformation to polar coordinates. By Proposition 1 $\{T(X_n, Y_n), n \ge 0\}$ is also a Poisson process. The mean measure of $([0, r] \times [0, \theta]), \mu'$, is

$$\begin{split} \mu'([0,r]\times[0,\theta]) &= \iint_{\{(x,y):T(x,y)\in[0,r]\times[0,\theta]\}} \lambda dx dy = \iint_{\{s\leq r,\eta\leq\theta\}} \lambda s ds d\eta \\ &= \lambda \frac{1}{2} r^2 \theta = \pi r^2 \frac{\theta}{2\pi} \left(= \frac{\theta}{2\pi} \int_0^r \lambda \alpha(s) ds \right), \end{split}$$

where $\alpha(s) = 2\pi s$. Here we use $\alpha(s)$ instead of $\lambda(s)$ to minimize confusion in notation. We know from Construction 1, that we can construct the inhomogeneous Poisson process, $\{S_n, n \ge 0\}$, with local intensity $\alpha(r) = 2\pi\lambda r$ by transforming the unit Poisson process by $\Lambda^{\leftarrow}(x) = \sqrt{\frac{x}{\lambda\pi}}$.

Thus if we let $\{U_n\}$ be iid. uniform on $[0, 2\pi)$, independent of $\{S_n, n \ge 0\}$, we can construct

$$\{(\sqrt{\frac{S_n}{\lambda\pi}}, U_n), n \ge 0\}$$

and transform by

$$(r,\theta) \mapsto (x,y) = (r\cos\theta, r\sin\theta)$$

to get a two dimensional homogeneous Poisson process with intensity λ .

Above, $\sqrt{\frac{S_n}{\lambda\pi}}$, is the distances from the origin to the n-th point. We are going to use this construction not by transforming back into Cartesian coordinates, but by using the two independent processes in polar coordinates.

Lastly let us define an indicator variable, as this will be quite central in the calculations further on.

Definition 4: (Indicator variable)

A random variable, $\mathbb{1}_E$, is called an indicator variable of some event, E, if $\mathbb{1}_E = \begin{cases} 1, & \text{if } E \text{ happens} \\ 0, & \text{if } E \text{ does not happen}. \end{cases}$

If E happens with probability p, then $\mathbb{1}_E$ is a random variable with $\mathbb{P}(\mathbb{1}_E = 1) = \mathbb{E}[\mathbb{1}_E] = p \text{ and } \operatorname{Var}(\mathbb{1}_E) = p(1-p).$

$\mathbf{2}$ Calculations

We will only look at the process where the trees have radii r = 1. By the argument in the introduction this will be sufficient. We say that we cannot see any trees, if a tree overlaps with the origin.

2.1The expected number of visible trees

To find the mean number of visible trees, we are going to start with finding the probability of seeing a tree placed at a certain point. Then generalise this to trees placed at any point and integrate over the whole plane to yield the mean.

Let us place a tree at Y = (0, y) and a point $X = (\sin \alpha, y - \cos \alpha)$ on the boundary of the disk with center Y at angle α , when measuring counterclockwise from the y-axis. What is the probability of X not being covered by trees to the left of the y-axis and to the right of the y-axis?

We start with trees on the left. In Figure 1 we want to find the areas of

$$(\Box OXBA \land \triangle OXD) \cup (\heartsuit XYB \land \triangle XYD) = (\Box OXBA \cup \heartsuit XYB) \land \triangle OXY,$$

as well as the wedge in the lower circle with angle $\frac{\pi}{2} + \beta$. The angle β is the angle which X forms in a circle of radius ||X|| measured clockwise from the y-axis, it will have the same sign as α .



Figure 1: Sketch of the situation for the left-side area. Placing a tree inside OYBA or the part of the unit circle to the left of the y-axis will block the view to X from the left.

Since $\Box OXBA$ has length ||X|| and width 1,

$$A_{\Box OXBA} = ||X|| = \sqrt{(\sin \alpha)^2 + (y - (\cos \alpha)^2)} = \sqrt{1 + y^2 - 2y \cos \alpha}.$$

The area of $\triangle OXY$ is found as $A_{\triangle OXY} = \frac{|CX| \cdot |YO|}{2} = \frac{\sin \alpha \cdot y}{2}$. We know that a wedge of angle θ in a circle of radius r has area $\frac{\theta}{2}r^2$, so our wedges have areas $A_{\nabla \frac{\pi}{2} + \beta} = \frac{\frac{\pi}{2} + \beta}{2}$ and $A_{\nabla XYB} = \frac{\frac{\pi}{2} - \alpha - \beta}{2}$. Now we can find the area on the left of the y-axis, where placing a tree

will block X at angle α on Y, let us call it the forbidden area to the left.

$$A_{fl} = A_{\Box OXBA} + A_{\nabla XYB} - A_{\triangle OXY} + A_{\nabla \frac{\pi}{2} + \beta}$$

= $||X|| + \frac{\frac{\pi}{2} - \alpha - \beta}{2} - \frac{\sin \alpha \cdot y}{2} + \frac{\frac{\pi}{2} + \beta}{2}$
= $\sqrt{1 + y^2 - 2y \cos \alpha} - \frac{y \sin \alpha}{2} + \frac{\pi - \alpha}{2}$ (1)

Let us move on to the right side of the y-axis. Here we want to find the area of $\Box OEFX$ and the wedges with angles $\frac{\pi}{2} + \alpha + \beta$ and $\frac{\pi}{2} - \beta$. The situation is drawn up in Figure 2.

The area of $\Box OEFX$ is the same as $\Box OXBA$, so $A_{\Box OEFX} = ||X||$. The two wedges have areas $A_{\nabla \frac{\pi}{2}-\beta} = \frac{\frac{\pi}{2}-\beta}{2}$ and $A_{\nabla \frac{\pi}{2}+\alpha+\beta} = \frac{\frac{\pi}{2}+\alpha+\beta}{2}$. Thus the forbidden area to the right is

$$A_{fr} = \Box OEFX + \Delta OXY + A_{\nabla \frac{\pi}{2} - \beta} + A_{\nabla \frac{\pi}{2} + \alpha + \beta}$$

= $||X|| + \frac{y \sin \alpha}{2} + \frac{\frac{\pi}{2} - \beta}{2} + \frac{\frac{\pi}{2} + \alpha + \beta}{2}$
= $\sqrt{1 + y^2 - 2y \cos \alpha} + \frac{y \sin \alpha}{2} + \frac{\pi + \alpha}{2}.$ (2)

If we want to be more specific we might write the forbidden areas as $A_{fl}((x,y),\alpha)$ and $A_{fr}((x,y),\alpha)$, where α indicates the angle at which X lies and (x, y) the coordinates of the center of the tree. Though to minimize notation, we will simply write A_{fl} and A_{fr} if we do not need to be specific.

Now, because we have a Poisson process, we know the probability of Xnot being blocked by a tree from the left is given by

$$\mathbb{P}(N(A_{fl})=0) = \exp\left\{-\lambda\left(\sqrt{1+y^2-2y\cos\alpha} - \frac{y\sin\alpha}{2} + \frac{\pi-\alpha}{2}\right)\right\}.$$

Since

 $\mathbb{P}(N(A_{fl}) = 0) = \mathbb{P}(X \text{ not blocked by tree from the left})$

= \mathbb{P} (Tree being blocked at most up to an angle α from the left)

= $\mathbb{P}(\max \text{ angle blocked by tree from the left} < \alpha)$, (3)



Figure 2: Sketch of the situation for the right-side area. Placing a tree inside OEFY or the part of the unit circle to the right of the y-axis will block the view to X from the right.

we also have the distribution function for the angle of a tree being blocked from the left. This holds for $-\arccos\left(\frac{1}{y}\right) \leq \alpha \leq \arccos\left(\frac{1}{y}\right)$ and $y \geq 1$, by the argument, that if $\alpha < 0$ we make α positive and switch to the formula for the right side, but for that one, when $\alpha < 0$ we also make it positive and switch the formula for the left side. Of course $|\alpha| \leq \arccos\left(\frac{1}{y}\right)$, as we cannot see X if it is placed behind the tangent point with the origin. If y < 1 we are going to block the origin and thus we cannot see X.

The same of course holds for the distribution of being blocked from the right, so

$$\mathbb{P}(N(A_{fr})=0) = \exp\left\{-\lambda\left(\sqrt{1+y^2-2y\cos\alpha} + \frac{y\sin\alpha}{2} + \frac{\pi+\alpha}{2}\right)\right\},\$$

and

 $\mathbb{P}(N(A_{fr}) = 0) = \mathbb{P}(\text{min angle blocked by tree from the right} > \alpha), \quad (4)$

also for $\operatorname{arccos}\left(\frac{1}{y}\right) \geq \alpha \geq -\operatorname{arccos}\left(\frac{1}{y}\right), y \geq 1$. The above probability is a bit tough to put into words, it can be thought of as "angle at which blocking by tree from the right starts $> \alpha$ ". Note that this probability grows as α decreases.

We know that $\mathbb{P}(N(A_{fr}) = 0)$ and $\mathbb{P}(N(A_{fl}) = 0)$ are independent, as A_{fl} and A_{fr} are disjoint areas, thus we also have the probability of seeing the point X:

$$\mathbb{P}(X \text{ not blocked by trees from left or right}) = \mathbb{P}(X \text{ visible})$$
$$= \mathbb{P}(N(A_{fr}) = 0) \mathbb{P}(N(A_{fl}) = 0)$$
$$= \exp\{-\lambda (A_{fl} + A_{fr})\} = \exp\{-\lambda \left(2\sqrt{1 + y^2 - 2y\cos\alpha} + \pi\right)\}$$
(5)

These probabilities are for a given point X placed at an angle α on a tree at placed Y. We would like to know, if we can see a tree placed at Y at all. Thus we want to integrate α out of the formulas. From (3) and (4) we can find the probability density functions of left and right side blockage as their derivatives with respect to α . That is

$$f_{\text{left},y}(\alpha) = \frac{\partial}{\partial \alpha} \mathbb{P}\left(N(A_{fl}) = 0\right) = \lambda \left(\frac{1}{2} + \frac{y \cos \alpha}{2} - \frac{y \sin \alpha}{\sqrt{1 + y^2 - 2y \cos \alpha}}\right)$$
$$\cdot \exp\left\{-\lambda \left(\sqrt{1 + y^2 - 2y \cos \alpha} - \frac{y \sin \alpha}{2} + \frac{\pi - \alpha}{2}\right)\right\} \quad (6)$$

and

$$f_{\text{right},y}(\alpha) = \frac{\partial}{\partial \alpha} \mathbb{P}\left(N(A_{fr}) = 0\right) = \lambda \left(\frac{1}{2} + \frac{y \cos \alpha}{2} + \frac{y \sin \alpha}{\sqrt{1 + y^2 - 2y \cos \alpha}}\right)$$
$$\cdot \exp\left\{-\lambda \left(\sqrt{1 + y^2 - 2y \cos \alpha} + \frac{y \sin \alpha}{2} + \frac{\pi + \alpha}{2}\right)\right\}. \quad (7)$$

Notice that the exponentials are $\mathbb{P}(N(A_{fl}) = 0)$ and $\mathbb{P}(N(A_{fr}) = 0)$ respectively, we are going to use this in a bit.

The tree at Y is visible if X at angle $\alpha = -\arccos \frac{1}{y}$ is visible or if for some $\alpha \in (-\arccos \frac{1}{y}, \arccos \frac{1}{y}], \alpha$ is the maximum angle being blocked by a tree from the left and α is not blocked from the right. For $\alpha \in (-\arccos \frac{1}{y}, \arccos \frac{1}{y}]$ we can multiply the left-side density with the right-side probability. However for $\alpha = -\arccos \frac{1}{y}$ we have to use the left-side probability multiplied with the right side probability. The left-side density multiplied with the right-side probability is

$$f_{\text{left},y}(\alpha) \mathbb{P} \left(N(A_{fr}) = 0 \right)$$

$$= \lambda \left(\frac{1}{2} + \frac{y \cos \alpha}{2} - \frac{y \sin \alpha}{\sqrt{1 + y^2 - 2y \cos \alpha}} \right) \mathbb{P} \left(\text{X visible} \right)$$

$$= \lambda \left(\frac{1}{2} + \frac{y \cos \alpha}{2} - \frac{y \sin \alpha}{\sqrt{1 + y^2 - 2y \cos \alpha}} \right)$$

$$\cdot \exp \left\{ -\lambda \left(2\sqrt{1 + y^2 - 2y \cos \alpha} + \pi \right) \right\}, \quad (8)$$

which is the density of the left side angle of blockage while not being blocked from the right. For $\alpha = -\arccos \frac{1}{y}$ we have

$$\mathbb{P}\left(N(A_{fl}((0,y), -\arccos\frac{1}{y})) = 0\right) \mathbb{P}\left(N(A_{fr}((0,y), -\arccos\frac{1}{y})) = 0\right) \\
= \mathbb{P}\left(X \text{ visible}\right)|_{\alpha = -\arccos\frac{1}{y}} \\
= \exp\left\{-\lambda\left(2\sqrt{y^2 - 1} + \pi\right)\right\}.$$
(9)

This argument of course also holds for the right-side probabilities and densities, but for angles in the other direction.

We can now find the probability of seeing a tree placed at Y.

 \mathbb{P} (seeing a tree placed at Y)

$$= \mathbb{P}(X \text{ visible})|_{\alpha = -\arccos \frac{1}{y}} + \int_{(-\arccos \frac{1}{y}, \arccos \frac{1}{y}]} f_{\operatorname{left}, y}(\alpha) \mathbb{P}(N(A_{fr}) = 0) d\alpha$$

$$= \int_{-\arccos\frac{1}{y}}^{\arccos\frac{1}{y}} \lambda \left(\frac{1}{2} + \frac{y\cos\alpha}{2} - \frac{y\sin\alpha}{\sqrt{1+y^2 - 2y\cos\alpha}} \right) e^{-\lambda \left(2\sqrt{1+y^2 - 2y\cos\alpha} + \pi \right)} d\alpha$$
$$+ e^{-\lambda \left(2\sqrt{y^2 - 1} + \pi \right)} \tag{10}$$

$$= \int_{0}^{\arccos \frac{1}{y}} \lambda \left(\frac{1}{2} + \frac{y \cos \alpha}{2} - \frac{y \sin \alpha}{\sqrt{1 + y^2 - 2y \cos \alpha}} \right) e^{-\lambda \left(2\sqrt{1 + y^2 - 2y \cos \alpha} + \pi \right)} d\alpha$$
$$+ \int_{0}^{\arccos \frac{1}{y}} \lambda \left(\frac{1}{2} + \frac{y \cos \alpha}{2} + \frac{y \sin \alpha}{\sqrt{1 + y^2 - 2y \cos \alpha}} \right) e^{-\lambda \left(2\sqrt{1 + y^2 - 2y \cos \alpha} + \pi \right)} d\alpha$$
$$+ e^{-\lambda \left(2\sqrt{y^2 - 1} + \pi \right)} \tag{11}$$

$$= \int_0^{\arccos\frac{1}{y}} \lambda (1 + y \cos \alpha) e^{-\lambda \left(2\sqrt{1 + y^2 - 2y \cos \alpha} + \pi\right)} d\alpha + e^{-\lambda \left(2\sqrt{y^2 - 1} + \pi\right)}$$
(12)

Where we in (11) use the argument from earlier, that for negative α we make it positive and switch to the right-side formula, that is $f_{\text{right},y}(\alpha) \cdot \mathbb{P}(N(A_{fl}) = 0)$. By the symmetry of the set up

$$\int_0^{\operatorname{arccos} \frac{1}{y}} f_{\operatorname{right},y}(\alpha) \mathbb{P}\left(N(A_{fl})=0\right) d\alpha = \int_{-\operatorname{arccos} \frac{1}{y}}^0 f_{\operatorname{left},y}(\alpha) \mathbb{P}\left(N(A_{fr})=0\right) d\alpha.$$

Now using the change of variables $t = y \cos \alpha$ in (12), implying $\alpha = \arccos \frac{t}{y}$ and $\left|\frac{d\alpha}{dy}\right| = \frac{1}{\sqrt{y^2 - t^2}}$, we get

 \mathbb{P} (seeing a tree placed at Y)

$$= e^{-\lambda \left(2\sqrt{y^{2}-1}+\pi\right)} + \lambda \int_{1}^{y} (1+t) e^{-\lambda \left(2\sqrt{1+y^{2}-2t}+\pi\right)} \frac{1}{\sqrt{y^{2}-t^{2}}} dt$$
$$= e^{-\lambda \pi} \left(e^{-2\lambda \sqrt{y^{2}-1}} + \lambda \int_{1}^{y} \frac{1+t}{\sqrt{y^{2}-t^{2}}} e^{-2\lambda \sqrt{1+y^{2}-2t}} dt \right), \quad (13)$$

where the limits have been switched as $y \cos \alpha$ is decreasing on $\alpha \in [0, \arccos \frac{1}{n}]$.

As shown in Construction 2, we can construct a homogeneous two dimensional Poisson process from two independent processes. This can also be thought of in reverse, that we can split the two dimensional Poisson process in two. We can use this by realising that the area A_{fl} and A_{fr} , and thus the subsequent probabilities (3), (4) and (13), do not actually depend on Y being placed on the y-axis, but rather that $Y = (y, \theta)$ in polar-coordinates, as we only need Y to be a distance y from the origin. The angle α of course will not be measured from the y-axis, but rather that the axis going through the origin and Y. In Construction 2, also notice that the 2π in the local intensity arises from integrating over θ to get the right number of points in the plane. When we integrate over the whole plane to calculate the mean, we are also going to integrate over θ . We can thus let the distances from the origin be an inhomogeneous Poisson process with local intensity λy at y.

Using the properties of the Poisson process, that the probability of having exactly one point in an interval of length h is $\lambda h + o(h)$, we will have exactly one point in a small interval of length dy around y with probability $\lambda y dy$ for some small dy.

 $\mathbb{P}(a \text{ visible tree existing at } (y, \theta)) = \mathbb{P}(\text{seeing a tree placed at } Y) \cdot \lambda y dy$

$$= \lambda y e^{-\lambda \pi} \left(e^{-2\lambda \sqrt{y^2 - 1}} + \lambda \int_1^y \frac{1 + t}{\sqrt{y^2 - t^2}} e^{-2\lambda \sqrt{1 + y^2 - 2t}} dt \right) dy \quad (14)$$

Define

$$\mathbb{1}(y,\theta) = \begin{cases} 1 & \text{when a visible tree exists at } (y,\theta) \\ 0 & \text{else} \end{cases}$$
(15)

to be the indicator of a visible tree existing at distance y from the origin and at angle θ with the x-axis. Notice that the indicator will be zero even when a tree exists at (y, θ) , but it cannot be seen. In equation (14) we found the probability of the indicator being 1 as

$$\mathbb{P}(\mathbb{1}(y,\theta)=1) = \mathbb{P}(a \text{ visible tree existing at } (y,\theta)),$$

and by the properties of indicator variables

 $\mathbb{E}[\mathbb{1}(y,\theta)] = \mathbb{P}(\text{a visible tree existing at } (y,\theta)).$

If we integrate the indicator variable of seeing a tree over the whole plane, we will get the number of trees, that are visible from the origin. Taking the mean of this of course yields the expected number of visible trees. By the properties of expectations we can integrate over the expected value, since $\mathbb{E}[\mathbb{1}(y,\theta)] < \infty$ for all points on the plane.

As the unit disk, by definition, will not add any visible trees, we do not have to include it in the integration, thus we are going to integrate θ over 0 to 2π and y over 1 to ∞ .

$$\mathbb{E}\left[\text{Number of visible trees}\right] = \mathbb{E}\left[\int_{\theta} \int_{y} \mathbb{1}(y,\theta) dy d\theta\right] = \int_{\theta} \int_{y} \mathbb{E}\left[\mathbb{1}(y,\theta)\right] d\theta$$
$$= 2\lambda \pi e^{-\lambda \pi} \int_{1}^{\infty} y \left(e^{-2\lambda \sqrt{y^{2}-1}} + \lambda \int_{1}^{y} \frac{1+t}{\sqrt{y^{2}-t^{2}}} e^{-2\lambda \sqrt{1+y^{2}-2t}} dt\right) dy \quad (16)$$

In $\int_{\theta} \int_{y} \mathbb{E}[\mathbb{1}(y,\theta)] d\theta$ the dy is left out, as it is already present in the expectation.

Using the change of variables $z = \sqrt{y^2 - 1}$, implying $\left|\frac{dy}{dz}\right| = \frac{y}{z}$, we get

$$\int_{1}^{\infty} y e^{-2\lambda\sqrt{y^2-1}} dy = \int_{0}^{\infty} z e^{-2\lambda z} dz = \frac{1}{4\lambda^2}.$$
(17)

For the second part of the integral, we first change the order of integration,

$$\int_{1}^{\infty} \int_{1}^{y} y \frac{1+t}{\sqrt{y^{2}-t^{2}}} e^{-2\lambda\sqrt{1+y^{2}-2t}} dt dy$$
$$= \int_{1}^{\infty} \int_{t}^{\infty} y \frac{1+t}{\sqrt{y^{2}-t^{2}}} e^{-2\lambda\sqrt{1+y^{2}-2t}} dy dt \quad (18)$$

then perform the change of variables $z = \sqrt{1 + y^2 - 2t}$, implying $y = \sqrt{z^2 + 2t - 1}$, $\sqrt{y^2 - t^2} = \sqrt{z^2 - (t - 1)^2}$ and $\left|\frac{dy}{dz}\right| = \frac{z}{y}$, we get (18) to be

$$= \int_{1}^{\infty} \int_{t-1}^{\infty} z \frac{1+t}{\sqrt{z^2 - (t-1)^2}} e^{-2\lambda z} dz dt = \int_{0}^{\infty} \int_{t}^{\infty} z \frac{2+t}{\sqrt{z^2 - t^2}} e^{-2\lambda z} dz dt, \quad (19)$$

where the lower limit in the first integral is $\sqrt{t^2 - 2t + 1} = \sqrt{(t-1)^2} = t - 1$, when $t \ge 1$, and a change of variable has happened in the second with t' = t - 1, but we haven't changed the name of the variable. Changing the order of integration again, (19) becomes:

$$\begin{split} &= \int_{0}^{\infty} z e^{-2\lambda z} \int_{0}^{z} \frac{2+t}{\sqrt{z^{2}-t^{2}}} dt dz \\ &= \int_{0}^{\infty} z e^{-2\lambda z} \left(\int_{0}^{z} \frac{2}{\sqrt{z^{2}-t^{2}}} dt + \int_{0}^{z} \frac{t}{\sqrt{z^{2}-t^{2}}} dt \right) dz \\ &= \int_{0}^{\infty} z e^{-2\lambda z} \left(2 \int_{0}^{z} \frac{1}{z\sqrt{1-\frac{t^{2}}{z^{2}}}} dt + \int_{0}^{z} \frac{t}{\sqrt{z^{2}-t^{2}}} dt \right) dz \\ &= \int_{0}^{\infty} z e^{-2\lambda z} \left(2 \arcsin\left(\frac{t^{2}}{z^{2}}\right) \Big|_{t=0}^{z} - \sqrt{z^{2}-t^{2}} \Big|_{t=0}^{z} \right) dz \\ &= \int_{0}^{\infty} z e^{-2\lambda z} \left(\pi + z \right) dz = \pi \int_{0}^{\infty} z e^{-2\lambda z} dz + \int_{0}^{\infty} z^{2} e^{-2\lambda z} dz \\ &= -\pi \frac{e^{-2\lambda z} (2\lambda z + 1)}{4\lambda^{2}} \Big|_{z=0}^{\infty} - \frac{e^{-2\lambda z} (2\lambda^{2} z^{2} + 2\lambda z + 1)}{4\lambda^{3}} \Big|_{z=0}^{\infty} \end{split}$$

$$(20)$$

We can now substitute (17) and (20) into (16) to get:

$$\mathbb{E}\left[\text{Number of visible trees}\right] = 2\lambda\pi e^{-\lambda\pi} \left(\frac{1}{4\lambda^2} + \lambda\left(\frac{\pi}{4\lambda^2} + \frac{1}{4\lambda^3}\right)\right)$$
$$= e^{-\lambda\pi} \left(\frac{\pi}{\lambda} + \frac{\pi^2}{2}\right)$$
(21)

This confirms our argument in the introduction about seeing fewer trees as the intensity grows.

If we want to condition on no trees overlapping with the origin, we will have to subtract $\frac{\pi}{2}$ from both A_{fl} and A_{fr} , which corresponds to the part in the unit circle. We can also use the properties of conditional expectation and divide (21) by the probability of not having any trees in the unit disk, which is $e^{-\lambda\pi}$.

 \mathbb{E} [Number of visible trees | No trees overlap with the origin] = $\frac{\pi}{\lambda} + \frac{\pi^2}{2}$ (22)

This approaches $\frac{\pi^2}{2}$ as the intensity grows to infinity.

2.2 Problems with calculating the variance

In this section we are going to present some problems, one might encounter when trying to compute the variance.

Using some of the properties of sums of variances for stochastic variables, which are not independent, we can write the variance as:

$$\operatorname{Var}\left(\operatorname{Number of visible trees}\right) = \operatorname{Var}\left(\int_{y} \int_{\theta} \mathbb{1}(y,\theta) d\theta dy\right)$$
$$= \int_{y_{1}} \int_{\theta_{1}} \int_{y_{2}} \int_{\theta_{2}} \operatorname{Cov}\left(\mathbb{1}(y_{1},\theta_{1}),\mathbb{1}(y_{2},\theta_{2})\right) d\theta_{2} d\theta_{1} dy_{2} dy_{1}$$
$$= \int_{y} \int_{\theta} \operatorname{Var}\left(\mathbb{1}(y,\theta)\right) d\theta dy$$
$$+ 2 \int_{y_{1}} \int_{y_{2}>y_{1}} \int_{\theta_{1}} \int_{\theta_{2}>\theta_{1}} \operatorname{Cov}\left(\mathbb{1}(y_{1},\theta_{1}),\mathbb{1}(y_{2},\theta_{2})\right) d\theta_{2} d\theta_{1} dy_{2} dy_{1} \qquad (23)$$

We can expand the first double integral a bit.

$$\int_{\theta} \int_{y} \operatorname{Var}\left(\mathbb{1}(y,\theta)\right) dy d\theta = \int_{\theta} \int_{y} \mathbb{E}\left[\mathbb{1}(y,\theta)^{2}\right] - \mathbb{E}\left[\mathbb{1}(y,\theta)\right]^{2} dy d\theta$$
$$= e^{-\lambda \pi} \left(\frac{\pi}{\lambda} + \frac{\pi^{2}}{2}\right) - \int_{\theta} \int_{y} \mathbb{E}\left[\mathbb{1}(y,\theta)\right]^{2} dy d\theta$$

The first problem, that we encounter, is the last integral, which will be tough to compute.

We also need to find the covariance. Generally

$$\operatorname{Cov}(X,Y) = \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right] = \mathbb{E}\left[XY\right] - \mathbb{E}\left[X\right]\mathbb{E}\left[Y\right].$$

In our case

$$\mathbb{E}\left[\mathbb{1}(y_1,\theta_1)\cdot\mathbb{1}(y_2,\theta_2)\right] = \mathbb{P}\left(\mathbb{1}(y_1,\theta_1)\cdot\mathbb{1}(y_2,\theta_2)=1\right)$$
(24)

and

$$\mathbb{E}\left[\mathbb{1}(y_1,\theta_1)\right] \cdot \mathbb{E}\left[\mathbb{1}(y_2,\theta_2)\right] = \mathbb{P}\left(\mathbb{1}(y_1,\theta_1) = 1\right) \cdot \mathbb{P}\left(\mathbb{1}(y_2,\theta_2) = 1\right).$$
(25)

So we can expand the integral over the covariance in (23) as

$$\int_{y_1} \int_{y_2 > y_1} \int_{\theta_1} \int_{\theta_2 > \theta_1} \operatorname{Cov} \left(\mathbb{1}(y_1, \theta_1), \mathbb{1}(y_2, \theta_2) \right) d\theta_2 d\theta_1 dy_2 dy_1
= \int_{y_1} \int_{y_2 > y_1} \int_{\theta_1} \int_{\theta_2 > \theta_1} \mathbb{P} \left(\mathbb{1}(y_1, \theta_1) = 1 \right) \cdot \mathbb{P} \left(\mathbb{1}(y_2, \theta_2) = 1 \right) d\theta_2 d\theta_1 dy_2 dy_1
- \int_{y_1} \int_{y_2 > y_1} \int_{\theta_1} \int_{\theta_2 > \theta_1} \mathbb{P} \left(\mathbb{1}(y_1, \theta_1) = 1 \right) \cdot \mathbb{P} \left(\mathbb{1}(y_2, \theta_2) = 1 \right) d\theta_2 d\theta_1 dy_2 dy_1.$$
(26)

We can find (25), just apply (14) for two different values of y and multiply them together, the integral over them most likely is not nice. However (24) is quite a lot more difficult. We will need to find the probability of seeing any pair of two trees simultaneously.

Let us call the two trees of interest T_1 and T_2 , for simplicity this will both refer to the center of the tree and the tree itself. Let $T_1 = (y_1, \theta_1) = (y_1, 0)$ and $T_2 = (y_2, \theta_2)$ in polar coordinates, and assume that $y_2 > y_1$ and $\theta_2 > \theta_1$. We need to specify how we measure the angles. We are going to measure θ counter-clockwise starting at the x-axis, we also measure β in Figure 1 and 2 clockwise from the axis going through the tree, this such that α and β have the same sign.

Let us sketch the situation for which we can see both T_1 and T_2 . This is done in Figure 3, the area inside of the blue lines is $A_{fr}(T_2, \alpha_2)$ and inside of the orange lines $A_{fl}(T_1, \alpha_1)$. We want for T_1 and T_2 to not be completely blocked from "the outside", that is by trees to the right of T_1 and trees to the left of T_2 . We also do not want T_1 and T_2 to completely block each other. Let us say, that T_2 is blocked up to an angle $\alpha_2 < \arccos \frac{1}{y_2}$ from the left, then T_1 may not block from α_2 from the right. T_1 might also be blocked from an angle $\alpha_1 > -\arccos \frac{1}{y_1}$ from the right, such that it is not completely blocked. If this is the case, we will need to find the forbidden area "between" the two trees, such that neither of the trees are blocked completely.



Figure 3: Sketch of the situation for the variance. Placing a tree in the area that is both inside of the blue and orange lines will block the view to both T_1 and T_2 completely. Placing within either the orange or the blue will block T_1 or T_2 respectively.

From earlier we already have the density for the blockage on the two trees from the left and the right, equation (6) and (7) respectively. Ensuring that T_1 does not block from an angle α_2 on T_2 is equivalent to $T_1 \notin A_{fr}(T_2, \alpha_2)$. A thought on how to ensure this, is to look at the angles, since $y_1 < y_2$. It seems that if $\theta_2 - \beta_2 < \theta_1 - \beta_{1,min}$ then $T_1 \in A_{fr}(T_2, \alpha_2)$, where $\beta_{1,min}$ is β for the point $X_{1,min}$ in Figure 3, which is when X lie at $\alpha_1 = -\arccos \frac{1}{y_1}$. This however is not the case. It becomes the most obvious for values of α_2 close to zero, but it also does not hold for larger angles. Thus the second problem is knowing when one tree is in the forbidden area of the other. It will suffice to to find the formula one way, as $T_1 \notin A_{fr}(T_2, \alpha_2)$ implies $T_2 \notin A_{fl}(T_1, \alpha_1)$, when $y_1 < y_2$.

Let us assume we have the situation where $T_1 \notin A_{fr}(T_2, \alpha_2)$. We then need to find the area between the trees, where placing a tree would lead to at least one being blocked completely. This is $A_{fl}(T_1, \alpha_1) \cup A_{fr}(T_2, \alpha_2) \cup$ {the unit disk}. To find the area of this, an intuitive thought would be to find the area of the polygon OEGA and subtract it from $A_{fl}(T_1, \alpha_1) + A_{fr}(T_2, \alpha_2)$, while also making sure to not double count the area in the unit circle. This approach will not work, as when T_1 and T_2 are moved close together in their angle, the point G will lie further from the origin than both B and F. It is also possible that T_2 is placed such that G is further from the origin than B but closer than F. In either case, the area of the polygon OEGAcannot be used directly. The third problem in computing the variance of the number of visible trees is thus to find the area of $A_{fl}(T_1, \alpha_1) \cup A_{fr}(T_2, \alpha_2) \cup$ {the unit disk}.

3 Further questions

Some further questions to look at might be:

- Is it possible to find the variance?
- What happens when the radii are random?
 - If all of the radii are the equal, say $R \sim F$, we can simply make the intensity random, with some proper scaling to match the radii, and integrate over the density multiplied with $e^{-\lambda \pi} \left(\frac{\pi}{\lambda} + \frac{\pi^2}{2}\right)$ from Equation (21). The mean thus becomes

$$\int_{\Lambda} f_{\Lambda}(\lambda)^{-\lambda \pi} \left(\frac{\pi}{\lambda} + \frac{\pi^2}{2}\right) d\lambda,$$

where $f_{\Lambda}(\lambda)$ is the density of the intensity. The distribution of the intensity should of course be non-negative.



Figure 4: A case that might happen if the radii are random and iid. The tree T_2 is visible from origin both to the left and to the right of the tree T_1 .

- If the radii for each tree are different, it will not be possible to use the method, that has been used up to now. This since the forbidden areas rely on all trees having the same radius. A case that might occur, is that a tree with a large radius is visible on both sides of a tree with smaller radius closer to the origin. How this might look is drawn up in Figure 4.
- What would happen when conditioning on no trees overlapping?

4 Acknowledgements

I would like to thank my supervisors Pieter Trapman and Abhishek Pal Majumder for guiding me through the process of writing this paper. I would also like to thank my friends Magnus and Dasha being supportive, good company when writing and for proof reading this paper.

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