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Bayesian Portfolio Optimization: Out-of-sample Performance of the Market Portfolio

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Abstract

The main focus of this thesis lies on Bayesian approaches to portfolio optimization. By using such statistical methods, it is possible to account for parameter uncertainty in the optimization procedure. Two different Bayesian approaches are utilized within the framework of this thesis, alongside with the conventional approach. We derive the weights of the market portfolio, which maximizes the Sharpe ratio, in the Bayesian settings. The market portfolio is then applied out-of-sample on empirical data consisting of asset returns through years 2002-2023 from Swedish stocks included in the OMXS30 index, as well as simulated data. The out-of-sample Sharpe ratio of the market portfolio is tested against the equally-weighted portfolio and we conclude that, in general, the market portfolio does not outperform the equally-weighted approach.

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1 Introduction

The theory of portfolio optimization was pioneered by Harry Markowitz in the 1950s [1], with his original research focusing on constructing an optimal portfolio using the mean-variance definition of return and risk. For his research into the area, Markowitz received the Nobel Memorial Prize in Economic Sciences in 1990.

In the 70 years that have passed, the theory of portfolio optimization has undergone substantial development, with a range of different extensions and approaches. DeMiguel et al. [2] provides an overview of several such approaches; one example is shortsale-constrained portfolio optimization.

Furthermore, different concepts of risks have also been introduced within the theory framework. Instead of variance as a risk measure, it is possible to use value at risk (VaR) and conditional value at risk (CVaR) in the optimization procedure, as in for example Bodnar et al. [3].

Like most models, the theory of portfolio optimization has its limitations and weaknesses. As shown in the aforementioned and widely cited paper by DeMiguel et al. [2], many portfolio optimization techniques struggle with diffuculties in performing optimally when applied out-of-sample, on "unseen" data, owing to estimation error and parameter uncertainty.

One way to deal with these issues is to apply a Bayesian perspective to portfolio optimization, which takes parameter uncertainty into account in the optimization process. Such a perspective was introduced within portfolio theory in the 1970s by, among others, Winkler [4] and Barry [5].

Bayesian mean-variance portfolio optimization approaches are the main focus of this thesis, in which the methods and results derived by Bauder et al. [6] and Bodnar et al. [3] are utilized. We contribute to to the field by deriving the weights of the market portfolio, which maximizes the Sharpe ratio, in a similar Bayesian setting as the aforementioned authors. The properties of this portfolio are then analyzed over several time periods, using empirical data consisting of returns from Swedish stocks as well as and simulated data. Specifically, we analyze the out-of-sample performance of the market portfolio and test it against the naive equally-weighted portfolio, showing that it in general does not outperform the naive approach in terms of the Sharpe ratio.

2 Methods

2.1 General concepts in portfolio theory

2.1.1 Risk and return of a portfolio

Given the choice between two risky assets, a rational investor would naturally prefer to hold the the asset with the highest expected return, given that it has less than or equal risk compared to the asset with lower return. Following the definition of of Capinski & Zastawniak [7], the asset with higher return in this case *dominates* the one with less return. However, this also begs the question: What do we mean by *return* and *risk*?

For the remainder of this thesis, the return will be defined as the percentage change in price between two time periods. Furthermore, the risk is defined as either the variance V or as the standard deviation SD of said returns, depending on the context. In finance the latter is sometimes referred to as the *volatility* of the asset.

A portfolio P is a set of different assets, each with a specific weight w_i in relation to the other assets. A portfolio which satisfies the condition

$$\sum_{i=1}^{k} w_i = \mathbf{w}^\top \mathbf{1} = 1,$$

where $\mathbf{1}$ denotes a k-dimensional vector of ones, is called a *feasible portfolio*, following the definition of Capinski & Zastawniak [7]. The weights, in general, can be both positive and negative with the latter corresponding to short selling.

The mean return of a portfolio P is calculated as

$$R_P = \mathbf{w}^\top \boldsymbol{\mu},$$

with $\boldsymbol{\mu} \in \mathbb{R}^k$ denoting the mean vector of the asset returns. Moreover, the variance of the portfolio is calculated as

$$V_P = \mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w},$$

where $\boldsymbol{\Sigma}$ is the covariance matrix of the asset returns.

2.1.2 Sharpe ratio and risk-free rate

As a measure of performance for a portfolio, one may also consider the *Sharpe ratio*, named after Willam Sharpe who introduced the measure in 1966 [8].

Definition 2.1 (Sharpe ratio)**.** For a portfolio P with expected return R_P and standard deviation SD_P, the *ex-ante* Sharpe ratio is defined as

$$SR = \frac{R_P - r_f}{SD_P},$$

where r_f is the *risk-free rate.*

The risk-free rate is defined as the rate of return an investor can achieve without taking any risk in terms of variance. Government bonds are usually utilized as a proxy for the risk-free rate; more information on how the concept is treated within this thesis is available in section 3.3.

An advantage of the Sharpe ratio is that it considers not only the return yielded by an investor, but also the risk taken to achieve such a return. A larger Sharpe ratio is generally considered to be better. Since the standard deviation is bounded below by zero, a negative Sharpe ratio implies that the return of the portfolio is less than that of the risk-free asset.

2.1.3 Efficient frontier, market portfolio and capital market line (CML)

A portfolio is called *efficient* if there is no other portfolio that dominates it, and all efficient portfolios among the feasible portfolios together form the *efficient frontier*, again following the definitions of Capinski & Zastawniak [7].

The relationship between the risk-free rate, the Sharpe ratio and the efficient frontier are visualised in Figure 1.

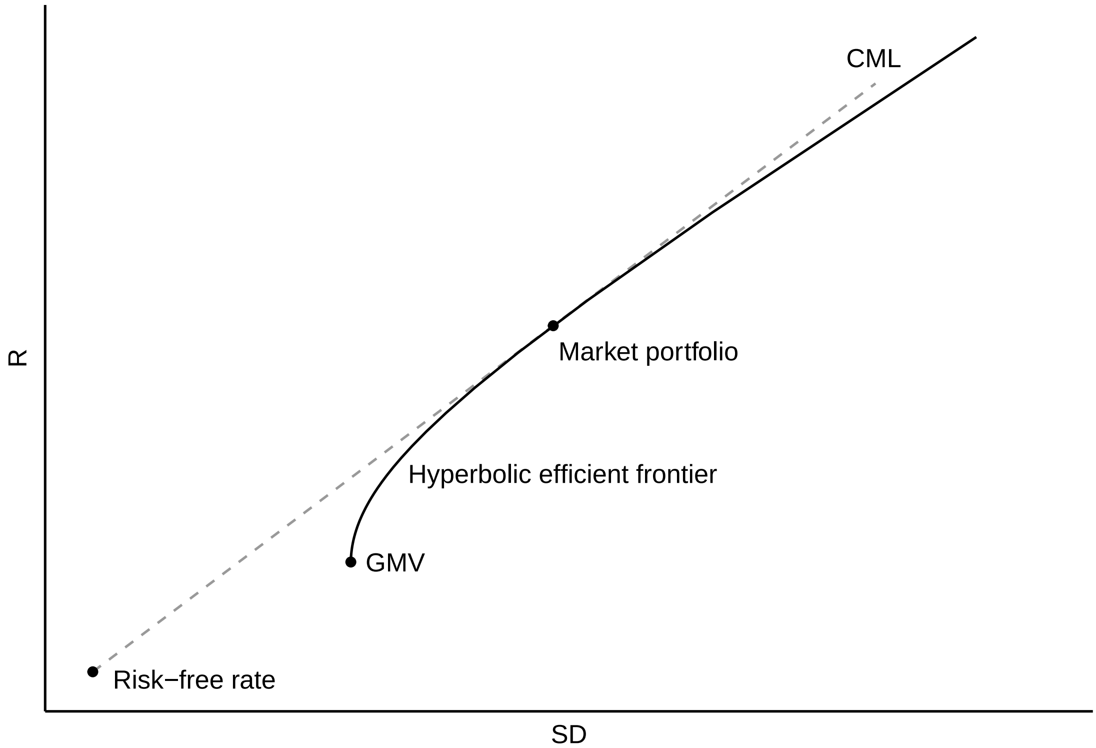

Figure 1: Hyperbolic efficient frontier and CML

In Figure 1, all feasible portfolios consisting of risky assets only form the hyperbolic efficient frontier.

The risk-free rate is here assumed to carry some level of return that is lower than the return of the *minimum variance portfolio*, R_{GMV}. This is the feasible portfolio consisting of risky assets only with the lowest possible variance of all such portfolios.

Between the hyperbolic efficient frontier and the risk-free rate lies the *capital market line* (CML). As can be noted from the figure, the CML does not lie in the feasible set of risky-only assets except at the tangency point with the hyperbolic efficient frontier, a special point called the *market portfolio.* As will be shown later on, the market portfolio maximizes the Sharpe ratio for an investor.

The portfolios that lie along the CML are, however, also feasible. The implication of this is that one allocates capital not only in the market portfolio, but also in the risk-free asset. The CML to the left of the market portfolio corresponds to placing some part of the capital also in the risk-free asset, while the CML to the right of the market portfolio corresponds to borrowing at the risk-free rate and investing the funds in the market portfolio. Another implication of this is that in the presence of a risk-free asset with a lower return than the minimum variance portfolio, the CML is the efficient frontier.

2.2 Portfolio optimization

Considering a portfolio of risky assets and, perhaps, also a risk-free asset, a natural question arises for an investor: How should one invest in these assets in order to maximize the return for some given level of risk? Or equivalently, for some given level of return, how should one invest in the assets in order to minimize the risk?

Through the theory of portfolio optimization, we aim to answer exactly these sort of questions.

In the following section a Bayesian approach to portfolio optimization is introduced first, using the methods and following the same structure and notations as presented by Bauder et al. [6], with some results derived from the development made by Bodnar et al. [3]. It is then followed by the conventional or classical way of portfolio optimization.

2.2.1 Bayesian approach

As in Bauder et al. [6], let $\mathbf{X}_t$ be the k-dimensional vector of asset returns at time t and $\mathbf{x}_{(t-1)} = (\mathbf{x}_{t-n}, ..., \mathbf{x}_{t-1})$ be the observation matrix of the asset returns, which are realizations of the former up to time $t-1$. Now, the distribution of $\mathbf{x}_{(t-1)}$ depends on the parameter vector $\boldsymbol{\theta} = (\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where $\boldsymbol{\mu}$ is the mean vector and $\boldsymbol{\Sigma}$ the covariance matrix of the asset returns. In applying Bayes' theorem, the posterior distribution of $\boldsymbol{\theta}$ is given by

$$\pi(\boldsymbol{\theta}|\mathbf{x}_{(t-1)}) \propto f(\mathbf{x}_{(t-1)}|\boldsymbol{\theta})\pi(\boldsymbol{\theta}),$$

where $\pi(\cdot)$ is the prior and $f(\mathbf{x}_{(t-1)}|\boldsymbol{\theta})$ is the likelihood function of $\mathbf{X}_{(t-1)}$. The return of a portfolio P at time t is expressed as

$$X_{P,t} = \mathbf{w}^\top \mathbf{X}_t,$$

with $\mathbf{w} = (w_1, ..., w_k)^\top$ as the k-dimensional vector of weights. The posterior *predictive* distribution of these returns is now given by

$$f(x_{P,t}|\mathbf{x}_{t-1}) = \int_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} f(x_{P,t}|\boldsymbol{\theta})\pi(\boldsymbol{\theta}|\mathbf{x}_{(t-1)})\, d\boldsymbol{\theta}. \tag{1}$$

Bauder et al. [6] have, under some assumptions and circumstances, derived a stochastic representation for (1), with further developments made by Bodnar et al. [3]. These will now be summarized.

Jeffreys prior

An investor with little or no prior knowledge about the asset returns may consider a non-informative prior on their distribution. One example of such an approach is the Jeffreys prior, named after Sir Harold Jeffreys. For multivariate distributions, it is defined as the square root of the determinant of the Fisher information matrix. Under this prior, the stochastic representation of (1) is given as in the following proposition, derived from Bauder et al. [6] and Bodnar et al. [3].

Proposition 2.1. *Let* $\mathbf{X}_1, \mathbf{X}_2, \ldots$ *be infinitely exchangeable and multivariate centred spherically symmetric. Let* $\pi(\boldsymbol{\theta}) = |\mathbf{F}|^{1/2}$ *be Jeffreys prior where* $|\mathbf{A}|$ *denotes the determinant of a square matrix* $\mathbf{A}$ *and* $\mathbf{F} = -\mathbb{E}[\partial^2 \log(f(\mathbf{x}_{(t-1)}|\boldsymbol{\theta}))/\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^\top]$ *is the Fisher information matrix. Assume* $n > k$, *and let* $t(q, a, b^2)$ *denote the univariate t-distribution with* q *degrees of freedom, location parameter* a *and scale parameter* b. *Then the stochastic representation of the random variable* $\widehat{\mathbf{X}}_{p,t}$ *whose density is the posterior predictive distribution* (1) *is given by* $t(d_{k,n,J}, \mathbf{w}^\top \bar{\mathbf{x}}_{t-1}, r_{k,n,J}\mathbf{w}^\top \mathbf{S}_{t-1}\mathbf{w})$ *with* $d_{k,n,J} = n - k$ *and*

$$r_{k,n,J} = \frac{n+1}{n(n-k)}, \quad \bar{\mathbf{x}}_{t-1} = \frac{1}{n}\sum_{i=t-n}^{t-1} \mathbf{x}_i \quad \text{and} \quad \mathbf{S}_{t-1} = \sum_{i=t-n}^{t-1} (\mathbf{x}_i - \bar{\mathbf{x}}_{t-1})(\mathbf{x}_i - \bar{\mathbf{x}}_{t-1})^\top. \tag{2}$$

The condition $n > k$ ensures that $\mathbf{S}_{t-1}$ is positive definite and thus invertible. Regarding the condition that the returns are multivariate centered spherically symmetric, this neither implies that the unconditional distribution is normal nor that the returns are independently distributed. A definition of the concept and its properties is available in section 4.4 of Bernardo and Smith [9].

As noted by Bauder et al. [6], the results of Proposition 4.6 in Bernardo and Smith [9] ensures that the conditional multivariate normal distribution satisfies both the assumption of infinite exchangeability and multivariate centered spherically symmetry. The multivariate t-distribution, however, does not not fulfill these assumptions.

Proposition 2.1 can be utilized to calculate important characteristics of the posterior predictive distribution, such as credible intervals. As for the expected return and variance, the aforementioned papers provide analytical expressions for these quantities.

Corollary 2.1. *Under the conditions of Proposition 2.1, let $n-k>2$. Then,*

$$\mathbb{E}[\mathbf{w}^\top \mathbf{X}_t|\mathbf{x}_{(t-1)}] = \mathbf{w}^\top \bar{\mathbf{x}}_{t-1}$$

and

$$Var(\mathbf{w}^\top \mathbf{X}_t|\mathbf{x}_{(t-1)}) = c_{k,n}\mathbf{w}^\top \mathbf{S}_{t-1}\mathbf{w} \quad \textit{with} \quad c_{k,n} = \frac{d_{k,n,J}\, r_{k,n,J}}{d_{k,n,J}-2}.$$

Proofs of Proposition 2.1 and Corollary 2.1 are given in the appendix of Bodnar et al. [3] and are based on the results of Bauder et al. [6].

Now, considering risky assets only, an investor aiming to construct an optimal portfolio at time $t-1$ for the next period maximizes the utility function

$$\begin{aligned} U(\mathbf{w}) &= \mathbb{E}[\mathrm{X}_{p,t}|\mathbf{x}_{(t-1)}] - \frac{\gamma}{2} Var(\mathrm{X}_{p,t}|\mathbf{x}_{(t-1)}) \\ &= \mathbf{w}^\top \bar{\mathbf{x}}_{t-1} - \frac{c_{k,n}\gamma}{2}\mathbf{w}^\top \mathbf{S}_{t-1}\mathbf{w}, \end{aligned} \tag{3}$$

under the constraint that $\mathbf{w}^\top \mathbf{1} = 1$ and where $\gamma > 0$ is the risk-aversion coefficient. The solution to this optimization problem is given by

$$\mathbf{w}_{MV,\gamma} = \frac{\mathbf{S}_{t-1}^{-1}\mathbf{1}}{\mathbf{1}^\top \mathbf{S}_{t-1}^{-1}\mathbf{1}} + \gamma^{-1}c_{k,n}^{-1}\mathbf{Q}_{t-1}\bar{\mathbf{x}}_{t-1}, \quad \text{where} \quad \mathbf{Q}_{t-1} = \mathbf{S}_{t-1}^{-1} - \frac{\mathbf{S}_{t-1}^{-1}\mathbf{1}\mathbf{1}^\top \mathbf{S}_{t-1}^{-1}}{\mathbf{1}^\top \mathbf{S}_{t-1}^{-1}\mathbf{1}}.$$

The expected return and variance is then expressed as

$$R_{MV,\gamma} = \frac{\mathbf{1}^\top \mathbf{S}_{t-1}^{-1}\bar{\mathbf{x}}_{t-1}}{\mathbf{1}^\top \mathbf{S}_{t-1}^{-1}\mathbf{1}} + \gamma^{-1}c_{k,n}^{-1}\bar{\mathbf{x}}_{t-1}^\top \mathbf{Q}_{t-1}\bar{\mathbf{x}}_{t-1},$$

and

$$V_{MV,\gamma} = \frac{c_{k,n}}{\mathbf{1}^\top \mathbf{S}_{t-1}^{-1}\mathbf{1}} + \gamma^{-2}c_{k,n}^{-1}\bar{\mathbf{x}}_{t-1}^\top \mathbf{Q}_{t-1}\bar{\mathbf{x}}_{t-1},$$

where it is used that $\mathbf{Q}_{t-1}\mathbf{1} = \mathbf{0}$ and $\mathbf{Q}_{t-1}\mathbf{S}_{t-1}\mathbf{Q}_{t-1} = \mathbf{Q}_{t-1}$ in the expression for $V_{MV,\gamma}$. The objective Bayesian efficient frontier, derived by Bauder et al. [6], is then expressed as

$$(R - R_{GMV})^2 = \frac{\bar{\mathbf{x}}_{t-1}^\top \mathbf{Q}_{t-1}\bar{\mathbf{x}}_{t-1}}{c_{k,n}}\left(V - V_{GMV}\right),$$

where

$$R_{GMV} = \frac{\mathbf{1}^\top \mathbf{S}_{t-1}^{-1}\bar{\mathbf{x}}_{t-1}}{\mathbf{1}^\top \mathbf{S}_{t-1}^{-1}\mathbf{1}} \quad \text{and} \quad V_{GMV} = \frac{c_{k,n}}{\mathbf{1}^\top \mathbf{S}_{t-1}^{-1}\mathbf{1}}$$

refers to the return and variance of the global minimum variance portfolio, respectively, whose weights are given by

$$\mathbf{w}_{GMV} = \frac{\mathbf{S}_{t-1}^{-1}\mathbf{1}}{\mathbf{1}^\top \mathbf{S}_{t-1}^{-1}\mathbf{1}}.$$

In the presence of a risk-free asset with return lower than that of the minimum variance portfolio, the efficient frontier is equal to the CML, given by

$$(R - r_f) = SD \cdot \left(\frac{R_M - r_f}{SD_{M,J}}\right). \tag{4}$$

Here, R_M and $SD_{M,J}$ corresponds to the return and standard deviation of the (Jeffreys) market portfolio, respectively. The CML and market portfolio weights are derived in the following proposition and corollary.

Proposition 2.2. *Under the conditions of Proposition 2.1 and the condition that $r_f < R_{GMV}$, let $n - k > 2$. The weights along the Jeffreys CML are then given by*

$$\mathbf{w}_{CML,J} = \gamma^{-1} c_{k,n}^{-1} \mathbf{S}_{t-1}^{-1}(\bar{\mathbf{x}}_{t-1} - r_f \mathbf{1}).$$

Now, the market portfolio is a special point along the CML, namely where the CML intersects with the hyperbolic efficient frontier.

Corollary 2.2. *Under the conditions of Proposition 2.1 and the condition that $r_f < R_{GMV}$, let $n - k > 2$. The weights of the market portfolio are then given by*

$$\mathbf{w}_M = \frac{\mathbf{S}_{t-1}^{-1}(\bar{\mathbf{x}}_{t-1} - r_f \mathbf{1})}{\mathbf{1}^\top \mathbf{S}_{t-1}^{-1}(\bar{\mathbf{x}}_{t-1} - r_f \mathbf{1})}.$$

Proofs of Proposition 2.2 and Corollary 2.2, from which Equation (4) follows, are given in the appendix.

The return of a portfolio P along the CML can be expressed as

$$R_{CML,P} = (\mathbf{w}_{CML,P}^\top \mathbf{1}) \mathbf{w}_M^\top \bar{\mathbf{x}}_{t-1} + (1 - \mathbf{w}_{CML,P}^\top \mathbf{1}) r_f,$$

which may be interpreted as investing some part of the available funds in the market portfolio, and some part in the risk-free asset. Since the (co)variance between the risky assets and risk-free asset is assumed to be zero, the variance of the portfolio can be found using Corollary 2.1.

Conjugate prior

As opposed to the vague Jeffreys prior, an investor may also want to consider a more informed approach. One such technique is using a conjugate prior, namely the Black-Litterman model devised by Fisher Black and Robert Litterman [10]. In this setting, it is possible for an investor to incorporate expert information into the portfolio optimization.

As in Bauder et al. [6], an extended variant of the Black-Litterman model is considered here, with a prior not only on $\boldsymbol{\mu}$ but also on $\boldsymbol{\Sigma}$.

The conjugate prior for $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ is given by

$$\boldsymbol{\mu}|\boldsymbol{\Sigma} \sim N_k\left(\mathbf{m}_0, \frac{1}{r_0}\boldsymbol{\Sigma}\right),$$
$$\boldsymbol{\Sigma} \sim IW_k(d_0, \mathbf{S}_0).$$

Here, N_k denotes the k-dimensional normal distribution, and IW_k is the inverse Wishart distribution with d_0 degrees of freedom. The parameter $\mathbf{m}_0$ reflects the investor's prior beliefs about $\boldsymbol{\mu}$, whereas the parameter $\mathbf{S}_0$ reflects the beliefs about $\boldsymbol{\Sigma}$. The parameters r_0 and d_0 are known as precision parameters for the two aforementioned parameters.

For the remainder of this thesis, we will set $r_0 = d_0 = 100$, following the same approach as Bauder et al. [6] in this regard. Furthermore, we will consider two different priors for $\mathbf{m}_0$ and $\mathbf{S}_0$.

In the first case, known as *Conjugate Bear*, the investor is negative towards the market and believes that $\mathbf{m}_0 = \frac{1}{2}\left(\bar{\mathbf{x}}_{t-1} + \boldsymbol{\varepsilon}\right)$, where $\varepsilon_i \sim N(-0.01, 0.01)$ are randomly simulated, and $\mathbf{S}_0 = 2 \cdot \mathbf{S}_{t-1}$.

In the second case, known as *Conjugate Bull*, the investor is positive towards the market and believes that $\mathbf{m}_0 = 2\left(\bar{\mathbf{x}}_{t-1} + \boldsymbol{\varepsilon}\right)$ where $\varepsilon_i \sim N(0.01, 0.01)$ and $\mathbf{S}_0 = \frac{1}{2} \cdot \mathbf{S}_{t-1}$.

Now, likewise to the Jeffreys prior, Bauder et al. [6] and Bodnar et al. [3] provides a stochastic representation of the posterior predictive distribution under the conjugate prior.

Proposition 2.3. *Let $\mathbf{X}_1, \mathbf{X}_2, \ldots$ be infinitely exchangeable and multivariate centred spherically symmetric. Assume $n + d_0 - 2k > 0$. Then, under the application of the conjugate prior the stochastic representation of the random variable $\widehat{\mathbf{X}}_{p,t}$ whose density is the posterior predictive distribution (1) is given by $t(d_{k,n,C}, \mathbf{w}^\top \bar{\mathbf{x}}_{t-1,C}, r_{k,n,C}\mathbf{w}^\top \mathbf{S}_{t-1,C}\mathbf{w})$ with $d_{k,n,C} = n + d_0 - 2k$ where*

$$r_{k,n,C} = \frac{n + r_0 + 1}{(n + r_0)(n + d_0 - 2k)}, \quad \bar{\mathbf{x}}_{t-1,I} = \frac{n\bar{\mathbf{x}}_{t-1} + r_0\mathbf{m}_0}{n + r_0}$$

and

$$\mathbf{S}_{t-1,I} = \mathbf{S}_{t-1} + \mathbf{S}_0 + nr_0 \frac{(\mathbf{m}_0 - \bar{\mathbf{x}}_{t-1,I})(\mathbf{m}_0 - \bar{\mathbf{x}}_{t-1,I})^\top}{n + r_0}.$$

The expected return and variance is now given in Corollary 2.3, that together with Proposition 2.3 is proven in the appendix of Bauder et al. [6] and Bodnar et al. [3].

Corollary 2.3. *Under the conditions of Proposition 2.3, let $n+d_0-2k > 2$. Then,*

$$\mathbb{E}[\mathbf{w}^\top \mathbf{X}_t | \mathbf{x}_{(t-1)}] = \mathbf{w}^\top \bar{\mathbf{x}}_{t-1,I}$$

and

$$Var(\mathbf{w}^\top \mathbf{X}_t | \mathbf{x}_{(t-1)}) = q_{k,n} \mathbf{w}^\top \mathbf{S}_{t-1,I} \mathbf{w} \quad \textit{with} \quad q_{k,n} = \frac{d_{k,n,C}\; r_{k,n,C}}{d_{k,n,C} - 2}.$$

Using the results of Corollary 2.3 in the optimization procecure in Equation (3) provides us with the weights for the optimal portfolios, expressed as

$$\mathbf{w}_{MV,\gamma} = \frac{\mathbf{S}_{t-1,I}^{-1} \mathbf{1}}{\mathbf{1}^\top \mathbf{S}_{t-1,I}^{-1} \mathbf{1}} + \gamma^{-1} q_{k,n}^{-1} \mathbf{Q}_{t-1,I} \bar{\mathbf{x}}_{t-1,I}, \quad \text{where}$$

$$\mathbf{Q}_{t-1,I} = \mathbf{S}_{t-1,I}^{-1} - \frac{\mathbf{S}_{t-1,I}^{-1} \mathbf{1} \mathbf{1}^\top \mathbf{S}_{t-1,I}^{-1}}{\mathbf{1}^\top \mathbf{S}_{t-1,I}^{-1} \mathbf{1}}.$$

It follows that the expected return and variance is then given by

$$R_{MV,\gamma} = \frac{\mathbf{1}^\top \mathbf{S}_{t-1,I}^{-1} \bar{\mathbf{x}}_{t-1,I}}{\mathbf{1}^\top \mathbf{S}_{t-1,I}^{-1} \mathbf{1}} + \gamma^{-1} q_{k,n}^{-1} \bar{\mathbf{x}}_{t-1,I}^\top \mathbf{Q}_{t-1,I} \bar{\mathbf{x}}_{t-1,I},$$

and

$$V_{MV,\gamma} = \frac{q_{k,n}}{\mathbf{1}^\top \mathbf{S}_{t-1,I}^{-1} \mathbf{1}} + \gamma^{-2} q_{k,n}^{-1} \bar{\mathbf{x}}_{t-1,I}^\top \mathbf{Q}_{t-1,I} \bar{\mathbf{x}}_{t-1,I}.$$

The Black-Litterman Bayesian efficient frontier, derived by Bauder et al. [6] is now expressed as

$$(R - R_{GMV,I})^2 = \frac{\bar{\mathbf{x}}_{t-1,I}^\top \mathbf{Q}_{t-1,I} \bar{\mathbf{x}}_{t-1,I}}{q_{k,n}} (V - V_{GMV,I}),$$

where

$$R_{GMV,I} = \frac{\mathbf{1}^\top \mathbf{S}_{t-1,I}^{-1} \bar{\mathbf{x}}_{t-1,I}}{\mathbf{1}^\top \mathbf{S}_{t-1,I}^{-1} \mathbf{1}} \quad \text{and} \quad V_{GMV,I} = \frac{q_{k,n}}{\mathbf{1}^\top \mathbf{S}_{t-1,I}^{-1} \mathbf{1}}$$

refers to the expected return and variance of the Black-Litterman global minimum variance portfolio, whose weights are given by

$$\mathbf{w}_{GMV,I} = \frac{\mathbf{S}_{t-1,I}^{-1}\mathbf{1}}{\mathbf{1}^\top \mathbf{S}_{t-1,I}^{-1}\mathbf{1}}.$$

As noted in the former sections, in the prescence of a risk-free asset the efficient frontier is equal to the CML expressed as

$$(R - r_f) = SD \cdot \left(\frac{R_{M,I} - r_f}{SD_{M,I}}\right). \tag{5}$$

Here, $R_{M,I}$, $SD_{M,I}$ corresponds to the return and standard deviation of the Black-Litterman market portfolio, respectively. The CML and market portfolio weights under the informative prior are now derived.

Proposition 2.4. *Under the conditions of Proposition 2.3 and the condition that $r_f < R_{GMV,I}$, let $n + d_0 - 2k > 2$. The weights along the Black-Litterman CML are then given by*

$$\mathbf{w}_{CML,I} = \gamma^{-1} q_{k,n}^{-1} \mathbf{S}_{t-1,I}^{-1}(\bar{\mathbf{x}}_{t-1,I} - r_f \mathbf{1}).$$

The proof of Proposition 2.4 and the upcoming Corollary 2.4 follows analogously from the proof in the appendix.

Corollary 2.4. *Under the conditions of Proposition 2.3 and the condition that $r_f < R_{GMV,I}$, let $n + d_0 - 2k > 2$. The weights of the Black-Litterman market portfolio are then given by*

$$\mathbf{w}_{M,I} = \frac{\mathbf{S}_{t-1,I}^{-1}(\bar{\mathbf{x}}_{t-1,I} - r_f \mathbf{1})}{\mathbf{1}^\top \mathbf{S}_{t-1,I}^{-1}(\bar{\mathbf{x}}_{t-1,I} - r_f \mathbf{1})}.$$

2.2.2 Conventional approach

An investor wishing to maximize the portfolio return $\mathbf{w}^\top \boldsymbol{\mu}$ for some given level of risk $\mathbf{w}^\top \mathbf{C} \mathbf{w}$, considering a set of risky assets only, aims to solve the optimization problem

$$\max \mathbf{w}^\top \boldsymbol{\mu} - \frac{\gamma}{2} \mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w} \quad \text{subject to } \mathbf{w}^\top \mathbf{1} = 1,$$

where $\gamma > 0$ corresponds to the risk-aversion coefficient, describing the investor's risk appetite, $\boldsymbol{\mu}$ corresponds to the mean vector and $\boldsymbol{\Sigma}$ is the covariance matrix.

The solution to this optimization problem, as given in Okhrin and Schmid [11], is attained using Lagrange multipliers and yields

$$\mathbf{w}_{P,\gamma} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\mathbf{1}} + \gamma^{-1}\mathbf{R}\boldsymbol{\mu}, \quad \text{where} \quad \mathbf{R} = \boldsymbol{\Sigma}^{-1} - \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}}{\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\mathbf{1}}.$$

Then, the expected return and variance are given by

$$R_{P,\gamma} = \frac{\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\mathbf{1}} + \gamma^{-1}\boldsymbol{\mu}^\top\mathbf{R}\boldsymbol{\mu},$$

$$V_{P,\gamma} = \frac{1}{\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\mathbf{1}} + \gamma^{-2}\boldsymbol{\mu}^\top\mathbf{R}\boldsymbol{\mu}.$$

Since $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are unknown quantities, they need to be estimated from the return samples.

With $\bar{\mathbf{x}}_{t-1}$ and $\mathbf{S}_{t-1}$ as in (2) we then get

$$\hat{\boldsymbol{\mu}} = \bar{\mathbf{x}}_{t-1} \quad \text{and} \quad \hat{\boldsymbol{\Sigma}} = d_n\mathbf{S}_{t-1} \quad \text{with} \quad d_n = \frac{1}{n-1}.$$

Then, the weights for the sample optimal portfolio are given by

$$\mathbf{w}_{S,\gamma} = \frac{\mathbf{S}_{t-1}^{-1}\mathbf{1}}{\mathbf{1}^\top\mathbf{S}_{t-1}^{-1}\mathbf{1}} + \gamma^{-1}d_n^{-1}\mathbf{Q}_{t-1}\bar{\mathbf{x}}_{t-1}, \quad \text{where} \quad \mathbf{Q}_{t-1} = \mathbf{S}_{t-1}^{-1} - \frac{\mathbf{S}_{t-1}^{-1}\mathbf{1}\mathbf{1}^\top\mathbf{S}_{t-1}^{-1}}{\mathbf{1}^\top\mathbf{S}_{t-1}^{-1}\mathbf{1}}.$$

The sample estimators for expected return and variance are now obtained by

$$R_{S,\gamma} = \frac{\mathbf{1}^\top\mathbf{S}_{t-1}^{-1}\bar{\mathbf{x}}_{t-1}}{\mathbf{1}^\top\mathbf{S}_{t-1}^{-1}\mathbf{1}} + \gamma^{-1}d_n^{-1}\bar{\mathbf{x}}_{t-1}^\top\mathbf{Q}_{t-1}\bar{\mathbf{x}}_{t-1},$$

and

$$V_{S,\gamma} = \frac{d_n}{\mathbf{1}^\top\mathbf{S}_{t-1}^{-1}\mathbf{1}} + \gamma^{-2}d_n^{-1}\bar{\mathbf{x}}_{t-1}^\top\mathbf{Q}_{t-1}\bar{\mathbf{x}}_{t-1}.$$

The sample efficient frontier is then, as in Bodnar and Schmid [12], expressed by

$$(R - R_{GMV,S})^2 = \frac{\bar{\mathbf{x}}_{t-1}^\top\mathbf{Q}_{t-1}\bar{\mathbf{x}}_{t-1}}{d_n}\left(V - V_{GMV,S}\right),$$

where

$$R_{GMV,S} = \frac{\mathbf{1}^\top\mathbf{S}_{t-1}^{-1}\bar{\mathbf{x}}_{t-1}}{\mathbf{1}^\top\mathbf{S}_{t-1}^{-1}\mathbf{1}} \quad \text{and} \quad V_{GMV,S} = \frac{d_n}{\mathbf{1}^\top\mathbf{S}_{t-1}^{-1}\mathbf{1}}.$$

The sample CML is given by

$$(R - r_f) = SD \cdot \left(\frac{R_M - r_f}{SD_{M,S}}\right),$$

with R_M and $SD_{M,S}$ corresponding to the return and standard deviation of the (sample) market portfolio. The weights along the sample CML are derived in Proposition 2.5.

Proposition 2.5. *Under the conditions that* $r_f < R_{GMV}$ *and* $\det \mathbf{S}_{t-1} \neq 0$, *the weights along the sample CML are given by*

$$\mathbf{w}_{CML,S} = \gamma^{-1} d_n^{-1} \mathbf{S}_{t-1}^{-1} (\bar{\mathbf{x}}_{t-1} - r_f \mathbf{1}).$$

The above proposition can be derived analogously from the proof in the appendix, together with the fact that the weights of the sample market portfolio are the same as in Corollary 2.2.

2.3 Performance evaluation

2.3.1 Out-of-sample analysis

To measure the performance of the portfolio optimization methods presented in this thesis, a similar but modified out-of-sample analysis as the one implemented by DeMiguel et al. [2] is considered.

That is, at time $t = 0$, the historical returns of the preceeding $n \in \{260, 520\}$ weeks (corresponding to five and ten years) are used in the optimization procedure to find the weights, expected return and variance of the market portfolio. The weights are then applied to the succeeding 104 weeks (two years), in order to generate a set of out-of-sample returns. From these, the *ex-post* out-of-sample Sharpe ratio of the period is computed using the sample mean and standard deviation. All portfolio returns are calculated under the assumption that an investor is trading without transaction costs.

The sample window is then shifted forward, dropping the first 104 returns and now including the 104 returns that were out-of-sample in the last period, thus keeping n fixed during the optimization process. New out-of-sample returns are then generated for the succeeding 104 weeks. This procedure is repeated until the end of the dataset.

For each time period, the resulting out-of-sample Sharpe ratio will be compared to the estimators yielded in the portfolio optimization approaches. Furthermore, the out-of-sample Sharpe ratio will be tested against the Sharpe ratio of a more naive approach to portfolio allocation, namely the equally-weighted $(1/k)$ portfolio. In this setting, for a portfolio with $k \in \{10, 20\}$ assets, each asset has weight $1/k$. Finally, the cumulative *ex-post* out-of-sample Sharpe ratio is computed by concatenating the out-of-sample returns from each period, yieleded in the iterative process described above.

2.3.2 Test statistic

In order to analyze whether there is any statistically significant difference between the Sharpe ratio of the optimal portfolios and the equally-weighted portfolio, the same hypothesis test as the one used by DeMiguel et al. [2] is considered.

Following the results of Jobson and Korkie [13], with the corrections made by Memmel [14], the hypothesis test $H_0: \ SR_i - SR_j = 0$ between two different portfolios i, j can be evaluated through the test statistic

$$\frac{SR_i - SR_j}{\sqrt{\hat{\vartheta}}}, \quad \text{where} \quad \vartheta = \frac{1}{W}(2 - 2\rho_{i,j} + \frac{1}{2}(SR_i^2 + SR_j^2 - 2 \cdot SR_i \cdot SR_j \cdot \rho_{i,j}^2)) \tag{6}$$

which is asymptotically normally distributed. In Equation (6), $\rho_{i,j}$ corresponds to the correlation between the returns of the two portfolios and W is the sample size. The p-value for the two-tailed test is then calculated from the above statistic using the cumulative distributive function of the standard normal distribution as $2\Phi(-|z|)$. The test relies on the assumptions that the portfolio returns are serially independent, normally distributed and that their distribution does not change through time.

Furthermore, we also consider a more robust test, which does not rely on assumptions of normality, serial independence and identical distributions. Such a test has been developed by Ledoit and Wolf [15] in 2008, by using a studentized bootstrap method. The mathematics of this test is a lot more involved than in the former case; the test has been deployed within this thesis using the function **sharpeTesting** from the R-package **PeerPerformance** [16], using 1000 bootstrap samples and optimal block length. We refer to [15] for more technical details.

As noted by, for example, Auer and Schumacher [17] and Kazak and Pohlmeier [18], it is worth mentioning that both of these tests exhibit weaknesses in terms of power and other properties.

3 Data

3.1 Empirical returns

For the empirical study, adjusted weekly returns for 20 of the stocks included in the Swedish OMXS30 index as of late 2023 are considered, with data ranging from April 2002 to March 2023. The reason for not including all 30 stocks from the index stocks is that not all have been publicly traded for a satisfactory length of time.

Furthermore, a smaller portfolio consisting of 10 stocks only are also considered. These stocks have been randomly selected from the larger data set. A full disclosure of the assets included in the study are available in Table 1, where bold rows correspond to the stocks in the 10-asset portfolio.

All data has been collected from Yahoo Finance [19].

Table 1: Included stocks

Ticker	Full name
ALIV-SDB	Autoliv SDB
ASSA-B	**Assa Abloy B**
ATCO-A	**Atlas Copco A**
AZN	Astra Zeneca
ELUX-B	**Electrolux B**
ERIC-B	**Ericsson B**
GETI-B	**Getinge B**
HEXA-B	Hexagon B
HM-B	**Hennes & Mauritz B**
INVE-B	**Investor B**
KINV-B	**Kinnevik B**
NDA-ABP	Nordea Abp
NIBE-B	Nibe B
SAND	Sandvik
SEB-A	SEB A
SHB-A	**Handelsbanken A**
SKF-B	SKF B
SWED-A	Swedbank A
TEL2-B	Tele2 B
VOLV-B	**Volvo B**

3.2 Simulated returns

For the simulation study, the sample mean vector and sample covariance matrix are computed using the empirical returns as data, evaluated over the same time periods as in the the empirical study. The sample mean vector and sample covariance matrix are then used as parameters in a multivariate normal distribution, from which new returns are simulated.

In the robustness analysis, a multivariate t-distribution with five degrees of freedom is considered, using the same parameters as above.

For each of the considered time periods, $n + 104$ with $n \in \{260, 520\}$ samples are simulated. The first n returns are then used as data in the optimization procedure, while the last 104 samples are used to generate out-of-sample returns. This ensures that all returns considered are coming from the same distribution, which is one of the assumptions of the Jobson and Korkie [13] test statistic, described in section 2.3.2.

Simulated returns are also used to construct the efficient frontiers in section 4.1. In order to more clearly visualize the difference of the considered approaches, a shorter estimation period is used here than in the out-of-sample analysis. The returns in this section are simulated using the sample mean vector and sample covariance matrix from the empirical time period 02-04-08 - 03-03-31 (52 weeks). With these parameters, $n = 104$ samples are simulated from a multivariate normal distribution. Those samples are then used in both estimation periods $n \in \{52, 104\}$.

3.3 Risk-free rate

As a proxy for the risk-free rate, the yield on Swedish 2-year government bonds are used. The yield has been adjusted to weekly, continuously compounded return rates. All data has been collected from the Swedish Riksbank [20].

4 Results

This section begins with a review of the efficient frontiers of the various portfolio approaches in section 4.1.

The subsections 4.2-4.4 account for the out-of-sample analysis of the market portfolio, executed by using the techniques described in section 3.2. The tables in these subsections accounts for the results of five-year (n = 260 weeks) sample periods and two-year (104 weeks) out-of-sample periods. Results for longer sample periods are available in the appendix, and will be referenced when relevant to the text.

4.1 Efficient frontiers

Figure 2 displays the efficient frontiers of all considered portfolio optimization approaches, with two different portfolio sizes and two different lengths of return samples. The data is drawn from the same distribution in all cases, as described in section 3.2.

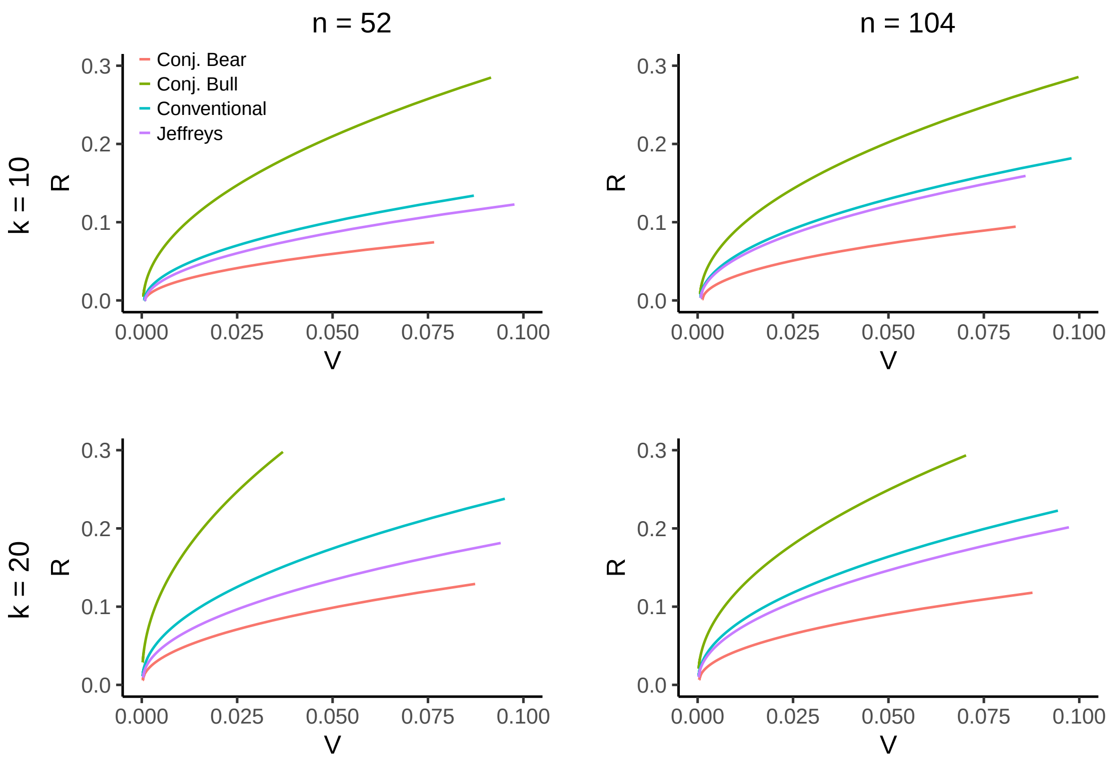

Figure 2: Efficient frontiers

The plots exhibits the varying effect that the different priors have on the efficient frontier, together with the influence of the portfolio size and sample length. In all cases, the Conjugate Bull prior displays quite exaggerated behavior as compared to the other priors, especially in the case when $n = 52$. The Jeffreys prior and the Conjugate Bear prior lead to more conservative estimates.

Bauder et al. [6] notes that the conventional approach is prone to overoptimism, and through simulations show that the non-informative Bayesian prior is preferable in this regard. However, while the Jeffreys prior exhibit a slightly more tentative efficient frontier in comparison to the conventional approach, these discrepancies are dampened when the sample length is enlarged.

4.2 Empirical study

This section focuses on the out-of-sample results yielded when applying the market portfolio weights to empirical asset returns, which are accounted for in section 3.1. It should be mentioned that it is by no means possible to guarantee that the various assumptions regarding the asset returns made in Proposition 2.1 and Proposition 2.3 will hold for the empirical data. The results should be read with this in mind.

4.2.1 Results for 20 stocks

Table 2 contains the estimators for the market portfolio Sharpe ratio in each of the employed portfolio approaches, over all of the considered time periods. In each time period, $n = 260$ weeks (five years) of return samples are used in the optimization procedure, while 104 (two years) returns are kept out-of-sample.

As was indicated in Figure 2, the Jeffreys prior and Conjugate Bear prior both show more conservative esimates of the Sharpe ratio as compared to both the conventional approach and, especially, the Conjugate Bull prior.

Table 2: Sharpe ratio estimators, empirical (20 stocks)

Sample period (out-of-sample period)	Conventional	Jeffreys	Conjugate Bear	Conjugate Bull
1. 02-04-08 - 07-03-26 (07-04-02 - 09-03-23)	0.353	0.338	0.191	0.443
2. 04-04-05 - 09-03-23 (09-03-30 - 11-03-21)	0.220	0.211	0.121	0.268
3. 06-04-03 - 11-03-21 (11-03-28 - 13-03-18)	0.141	0.135	0.075	0.181
4. 08-03-31 - 13-03-18 (13-03-25 - 15-03-16)	0.174	0.166	0.094	0.229
5. 10-03-29 - 15-03-16 (15-03-23 - 17-03-13)	0.304	0.291	0.168	0.390
6. 12-03-26 - 17-03-13 (17-03-20 - 19-03-11)	0.252	0.241	0.139	0.317
7. 14-03-24 - 19-03-11 (19-03-18 - 21-03-08)	0.285	0.273	0.156	0.343
8. 16-03-21 - 21-03-08 (21-03-15 - 23-03-06)	0.279	0.267	0.152	0.338

The third and fourth period display notably lower Sharpe ratio estimates in comparison to the other periods. This is not very surprising, given the fact that both of these periods contain empirical samples from the 2007–2008 financial crisis, which saw catastrophic drops in asset prices that influence the estimators.

Table 3 accounts for the Sharpe ratio of the equally-weighted ($1/k$) portfolio, as well as the out-of-sample Sharpe ratio yielded when applying the weights of the various market portfolios to the out-of-sample returns, using the technique described in section 2.3.1. Since the weights for the Jeffreys and conventional portfolios coincide, they have the exact same result and as such they are listed together in the table.

Displayed within parenthesis is the p-value of the out-of-sample market portfolio tested against the equally-weighted portfolio, using the tests accounted for in section 2.3.2. The first p-value correspond to the Jobson and Korkie [13] test, while the second correspond to the Ledoit and Wolf [15] method. For the cumulative Sharpe ratio, where the returns in each period have been concatenated together, only the Ledoit and Wolf test is displayed since these returns explicitly violate the assumption of non-changing distributions in the Jobson and Korkie test. As will become apparent throughout this section, the difference in p-values between these two tests are small.

Table 3: Out-of-sample Sharpe ratios and p-values, empirical (20 stocks)

Sample period (out-of-sample period)	1/k	Jeffreys / Conventional	Conjugate Bear	Conjugate Bull
1. 02-04-08 - 07-03-26 (07-04-02 - 09-03-23)	-0.114	-0.157 (0.53 / 0.52)	-0.161 (0.51 / 0.50)	-0.173 (0.39 / 0.39)
2. 04-04-05 - 09-03-23 (09-03-30 - 11-03-21)	0.277	-0.191 (0.00 / 0.01)	-0.190 (0.00 / 0.01)	-0.150 (0.01 / 0.01)
3. 06-04-03 - 11-03-21 (11-03-28 - 13-03-18)	0.056	-0.003 (0.49 / 0.48)	-0.011 (0.45 / 0.44)	-0.001 (0.50 / 0.49)
4. 08-03-31 - 13-03-18 (13-03-25 - 15-03-16)	0.250	0.316 (0.50 / 0.49)	0.333 (0.39 / 0.38)	0.349 (0.27 / 0.26)
5. 10-03-29 - 15-03-16 (15-03-23 - 17-03-13)	0.028	-0.003 (0.75 / 0.75)	-0.004 (0.75 / 0.75)	-0.013 (0.67 / 0.67)
6. 12-03-26 - 17-03-13 (17-03-20 - 19-03-11)	0.081	0.131 (0.69 / 0.70)	0.121 (0.75 / 0.76)	0.108 (0.82 / 0.82)
7. 14-03-24 - 19-03-11 (19-03-18 - 21-03-08)	0.137	0.119 (0.90 / 0.90)	0.122 (0.92 / 0.92)	0.139 (0.98 / 0.98)
8. 16-03-21 - 21-03-08 (21-03-15 - 23-03-06)	0.047	0.107 (0.59 / 0.60)	0.114 (0.55 / 0.55)	0.126 (0.47 / 0.48)
Cumulative Sharpe ratio	0.073	0.005 (0.12)	0.003 (0.12)	0.027 (0.27)

It is immediately recognizable that the out-of-sample Sharpe ratios in Table 3 are generally much lower than their estimated counterparts in Table 2 and are, in several cases, negative. They are in fact lower than their estimates, for all portfolios, in all periods except one: period 4, which happens to contain returns from the financial crisis in-sample that has caused an underestimation of the means for the upcoming out-of-sample period. Moreover, although the considered optimization approaches displayed quite different estimators in Table 2, their out-of-sample Sharpe ratios are all very similar.

Clearly, the optimal portfolios do not show any general tendencies to consistently outperform the equally-weighted approach. For $\alpha \leq 0.05$ where α denotes the probability of rejecting the null hypothesis, both of the test statistics employed only recognizes a significant difference in period 2. However, in this period, the optimal portfolios significantly *underperforms* the equally-weighted approach. This displays the catastrophic consequences of the risk exposure that may come with the market portfolio during periods of financial turmoil.

The results in Table 3 are, as we shall see, quite general (with some exceptions of course) to all the considered portfolio sizes, estimation periods, and data types within this thesis. They underline the negative effects that estimation error has on the optimal portfolios, and the results are in line with what DeMiguel et al. [2] and others have shown before. Namely, that the out-of-sample Sharpe ratio of optimal portfolios seldomly outperform the equally-weighted portfolio.

Although there is lack of statistical evidence to any general difference in terms of the Sharpe ratio, it is however evident from Figure 3 that there is nonetheless a quite large discrepancy between the equally-weighted portfolio and the optimal portfolios.

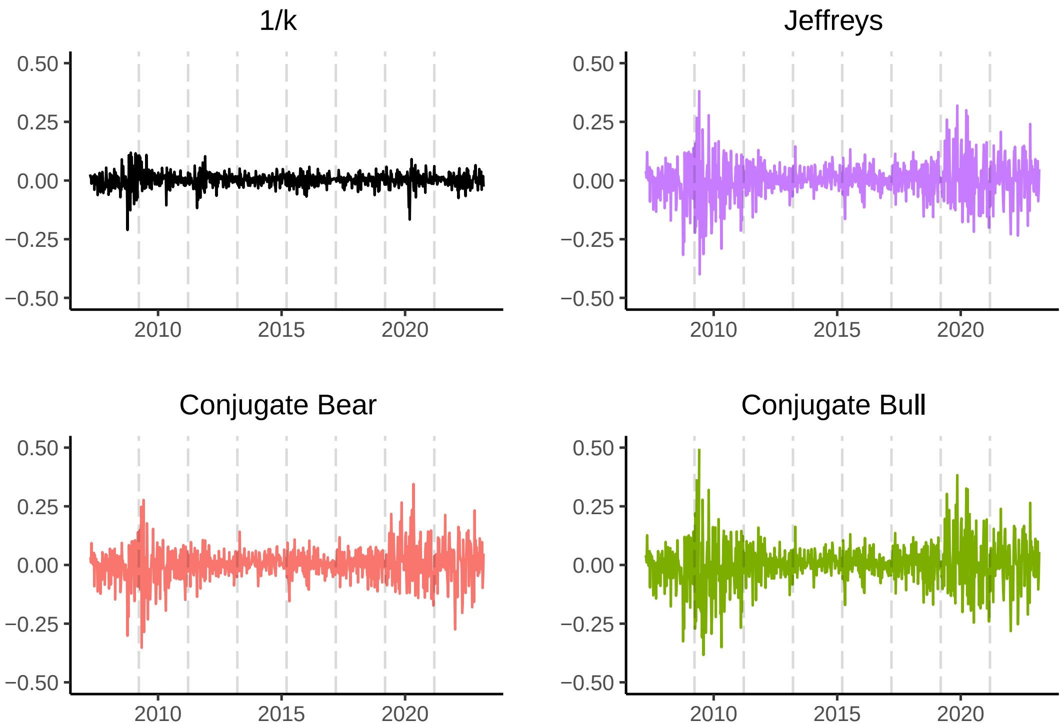

Figure 3: Weekly returns, empirical (20 stocks)

In the above figure, the out-of-sample returns of the portfolios in each time period are concatenated together. The dashed vertical lines represent re-optimization instances, where old returns are dropped from the optimization sample and new returns are included, as described in section 2.3.1.

All of the optimal portfolio approaches display much larger movements in terms of return in comparison to the equally-weighted portfolio, which again illustrates the significant risks an investor may expose herself to in applying the market portfolio in practice. The consequences of this risk is also apparent in Figure 4, which shows the cumulative return for all of the considered portfolios, with an initial value of 100. Here, each of the distinct periods has again been concatenated together to form a series.

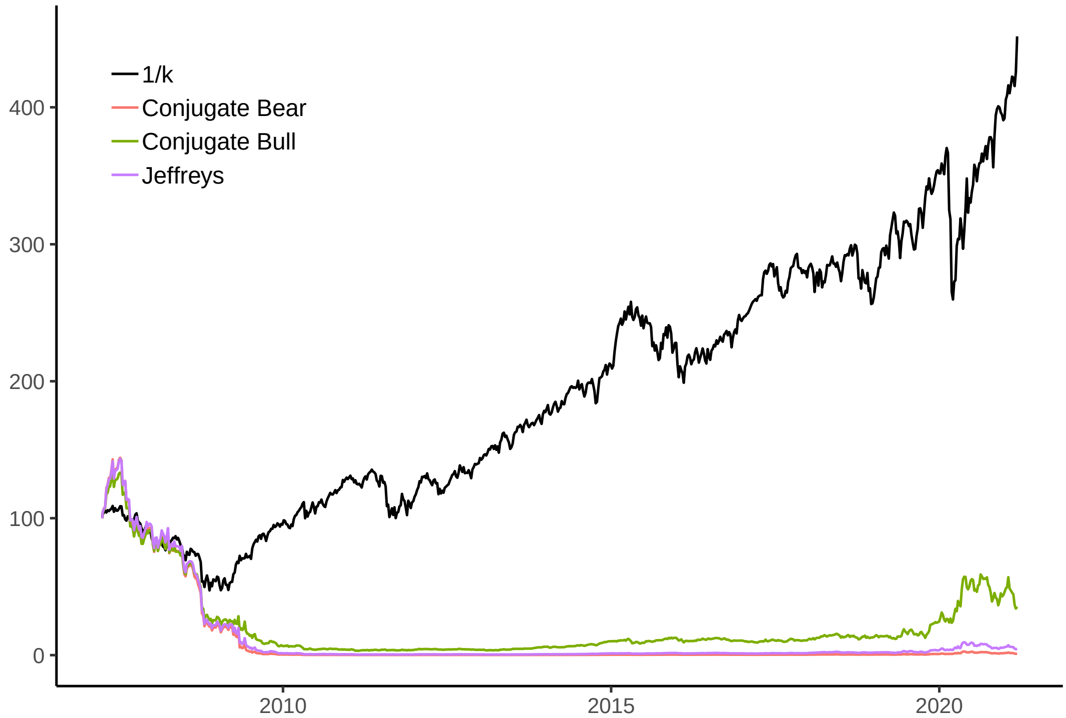

Figure 4: Cumulative returns, empirical (20 stocks)

In applying any one of the optimal portfolios continuously into the market, the investor early on loses almost all wealth during the financial crisis. This wealth is not recaptured in the proceeding years to any significant degree, although the portfolio with the Conjugate Bull prior display somewhat promising tendencies at the end. This is in stark contrast to the equally-weighted portfolio. Heavy losses are incurred in this portfolio as well, however not even nearly to the same degree as in the optimal portfolios.

The picture becomes somewhat different when looking at the cumulative return *within* each period, as in Figure 5.

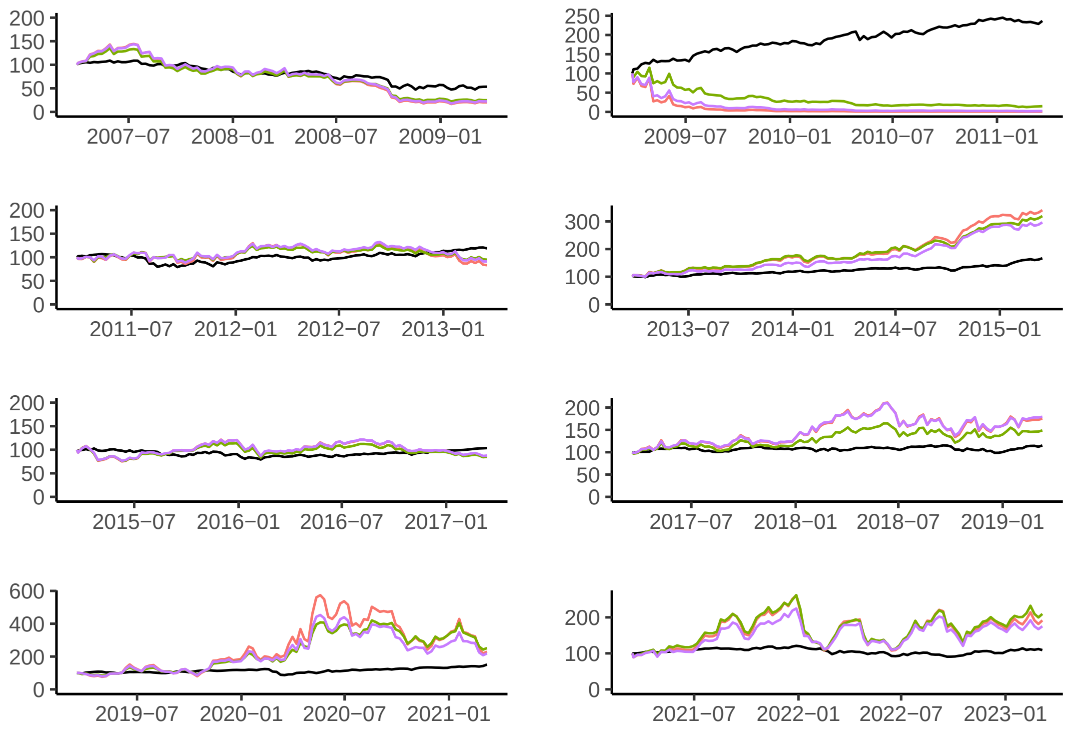

Figure 5: Cumulative returns in each period, empirical (20 stocks)

The first period, which contain returns from the financial crisis out-of-sample, lead to significant losses for the optimal investor as well as the equally-weighted investor. In the second period, a large part of the negative returns of the financial crisis are included in-sample, leading to a skewed estimation of the returns within the optimization procedure. In applying the weights of these optimal portfolios to the proceeding period, large losses are again incurred, which however is not the case for the equally-weighted portfolio.

The returns in the other periods are slightly less negative, although the within-period variance is high in several cases. It is again obvious that the three different optimization approaches lead to quite similar results.

As for the longer sample period of $n = 520$ weeks, where tables are available in the appendix, the conclusion that the optimal portfolios fail to outperform the equally-weighted approach is also valid. The estimators for the Sharpe ratio in these cases are generally lower than in Table 2, and the estimators for the Conjugate Bull prior are much closer to those of the conventional and Jeffreys prior. The out-of-sample Sharpe ratios generally display more positive tendencies than in Table 3.

4.2.2 Results for 10 stocks

This section considers a set of smaller portfolios consisting of 10 stocks only, which are listed in bold in Table 1. For the sake of parsimonity, an in-depth analysis of the kind made in the last section will not be done here.

In Table 4, the estimators of the Sharpe ratios are noticeably smaller as compared to the 20 stock-portfolio in Table 2.

Table 4: Sharpe ratio estimators, empirical (10 stocks)

Sample period (out-of-sample period)	Conventional	Jeffreys	Conjugate Bear	Conjugate Bull
1. 02-04-08 - 07-03-26 (07-04-02 - 09-03-23)	0.266	0.260	0.146	0.340
2. 04-04-05 - 09-03-23 (09-03-30 - 11-03-21)	0.161	0.157	0.087	0.175
3. 06-04-03 - 11-03-21 (11-03-28 - 13-03-18)	0.094	0.092	0.051	0.118
4. 08-03-31 - 13-03-18 (13-03-25 - 15-03-16)	0.132	0.129	0.072	0.167
5. 10-03-29 - 15-03-16 (15-03-23 - 17-03-13)	0.264	0.258	0.147	0.328
6. 12-03-26 - 17-03-13 (17-03-20 - 19-03-11)	0.200	0.195	0.111	0.252
7. 14-03-24 - 19-03-11 (19-03-18 - 21-03-08)	0.179	0.175	0.097	0.210
8. 16-03-21 - 21-03-08 (21-03-15 - 23-03-06)	0.186	0.181	0.099	0.217

As in the last section, turning to the out-of-sample Sharpe ratios in Table 5, it is again recognizable that the optimal portfolios do not outperform the equally-weighted approach.

Table 5: Out-of-sample Sharpe ratios and p-values, empirical (10 stocks)

Sample period (out-of-sample period)	1/k	Jeffreys / Conventional	Conjugate Bear	Conjugate Bull
1. 02-04-08 - 07-03-26 (07-04-02 - 09-03-23)	-0.122	-0.107 (0.80 / 0.81)	-0.111 (0.85 / 0.85)	-0.127 (0.94 / 0.94)
2. 04-04-05 - 09-03-23 (09-03-30 - 11-03-21)	0.289	-0.080 (0.02 / 0.04)	-0.086 (0.02 / 0.04)	-0.058 (0.03 / 0.04)
3. 06-04-03 - 11-03-21 (11-03-28 - 13-03-18)	0.059	0.026 (0.47 / 0.46)	0.028 (0.52 / 0.52)	0.049 (0.81 / 0.81)
4. 08-03-31 - 13-03-18 (13-03-25 - 15-03-16)	0.221	0.337 (0.21 / 0.19)	0.323 (0.29 / 0.27)	0.316 (0.30 / 0.29)
5. 10-03-29 - 15-03-16 (15-03-23 - 17-03-13)	0.043	-0.017 (0.51 / 0.52)	-0.016 (0.53 / 0.53)	-0.017 (0.52 / 0.52)
6. 12-03-26 - 17-03-13 (17-03-20 - 19-03-11)	0.067	0.126 (0.63 / 0.63)	0.116 (0.70 / 0.70)	0.103 (0.77 / 0.77)
7. 14-03-24 - 19-03-11 (19-03-18 - 21-03-08)	0.158	0.001 (0.11 / 0.10)	-0.012 (0.09 / 0.09)	0.002 (0.06 / 0.05)
8. 16-03-21 - 21-03-08 (21-03-15 - 23-03-06)	0.033	0.128 (0.21 / 0.23)	0.129 (0.21 / 0.22)	0.131 (0.15 / 0.16)
Cumulative Sharpe ratio	0.075	-0.003 (0.09)	-0.014 (0.07)	0.007 (0.09)

Like in the case with 20 assets, period 2 suffers from significant underperformance, while period 7 and the cumulative Sharpe ratio are close to significantly underperforming. For the longer sample periods, with data available in the appendix, a significant underperformance is noted during period 3.

It is a somewhat surprising result that a smaller portfolio would perform worse or equally bad as compared to a larger portfolio, given that there are fewer parameters present in a smaller portfolio which should thus lead to smaller amplification of estimation error. This result, in fact, conflicts with the conclusions of DeMiguel et al. [2], who claim that a smaller portfolio has a greater chance to outperform the equally-weighted approach.

4.3 Simulation study

This section considers normally distributed data, simulated using the techniques described in section 3.2. The multivariate normal distribution satisfies the assumptions of multivariate centered spherical symmetry and infinite exchangeability, which are the main assumptions in Proposition 2.1 and Proposition 2.3. Furthermore, the setup of the simulation also implies that the out-of-sample returns have the same distribution as the in-sample returns.

4.3.1 Results for 20 stocks

Table 6 shows the Sharpe ratio estimators for the market portfolio in all of the considered approaches. Likewise to the last section, the Jeffreys prior and the Conjugate Bear prior both show more conservative estimates – especially the latter. The estimated Sharpe ratios are generally higher than in the empirical study.

Table 6: Sharpe ratio estimators, normal distribution (20 stocks)

Sample period (out-of-sample period)	Conventional	Jeffreys	Conjugate Bear	Conjugate Bull
1. 02-04-08 - 07-03-26 (07-04-02 - 09-03-23)	0.540	0.517	0.295	0.660
2. 04-04-05 - 09-03-23 (09-03-30 - 11-03-21)	0.422	0.403	0.229	0.490
3. 06-04-03 - 11-03-21 (11-03-28 - 13-03-18)	0.402	0.385	0.219	0.485
4. 08-03-31 - 13-03-18 (13-03-25 - 15-03-16)	0.289	0.277	0.155	0.347
5. 10-03-29 - 15-03-16 (15-03-23 - 17-03-13)	0.409	0.392	0.226	0.506
6. 12-03-26 - 17-03-13 (17-03-20 - 19-03-11)	0.418	0.400	0.234	0.516
7. 14-03-24 - 19-03-11 (19-03-18 - 21-03-08)	0.396	0.379	0.215	0.465
8. 16-03-21 - 21-03-08 (21-03-15 - 23-03-06)	0.375	0.359	0.204	0.450

In Table 7, the resulting out-of-sample Sharpe ratios yielded when applying the weights of the market portfolios are listed, together with the Sharpe ratio of the equally-weighted portfolio and the p-values of the two tests within parenthesis.

Table 7: Out-of-sample Sharpe ratios and p-values, normal distribution (20 stocks)

Sample period (out-of-sample period)	1/k	Jeffreys / Conventional	Conjugate Bear	Conjugate Bull
1. 02-04-08 - 07-03-26 (07-04-02 - 09-03-23)	0.092	0.301 (0.09 / 0.09)	0.303 (0.09 / 0.09)	0.310 (0.07 / 0.07)
2. 04-04-05 - 09-03-23 (09-03-30 - 11-03-21)	0.045	0.123 (0.55 / 0.56)	0.131 (0.52 / 0.53)	0.149 (0.43 / 0.43)
3. 06-04-03 - 11-03-21 (11-03-28 - 13-03-18)	0.134	0.172 (0.74 / 0.73)	0.168 (0.77 / 0.76)	0.164 (0.79 / 0.79)
4. 08-03-31 - 13-03-18 (13-03-25 - 15-03-16)	0.046	0.144 (0.30 / 0.29)	0.149 (0.28 / 0.26)	0.180 (0.15 / 0.13)
5. 10-03-29 - 15-03-16 (15-03-23 - 17-03-13)	0.004	0.112 (0.32 / 0.31)	0.109 (0.34 / 0.33)	0.096 (0.39 / 0.38)
6. 12-03-26 - 17-03-13 (17-03-20 - 19-03-11)	0.169	0.278 (0.30 / 0.31)	0.289 (0.28 / 0.28)	0.310 (0.20 / 0.21)
7. 14-03-24 - 19-03-11 (19-03-18 - 21-03-08)	0.137	0.294 (0.23 / 0.23)	0.295 (0.23 / 0.22)	0.301 (0.20 / 0.20)
8. 16-03-21 - 21-03-08 (21-03-15 - 23-03-06)	0.204	0.114 (0.43 / 0.45)	0.119 (0.45 / 0.47)	0.126 (0.48 / 0.49)
Cumulative Sharpe ratio	0.096	0.170 (0.08)	0.167 (0.10)	0.191 (0.02)

As in the case for the empirical data, the out-of-sample Sharpe ratios are in many cases lower than their estimator counterparts. However, in comparison to the empirical study, they are in general higher and none of the periods have negative Sharpe ratios. Nevertheless, only the cumulative Sharpe ratio for the Conjugate Bull prior displays a significant outperformance at level $\alpha \leq 0.05$, albeit that the p-values in the other periods are often smaller than in the empirical study. These results are in line with the conclusions of DeMiguel et al. [2], who notes that even for normally distributed data there is a need for extremely long estimation periods if the optimal portfolios are to outperform the equally-weighted portfolio.

Although there is no consistent outperformance in terms of the Sharpe ratio within the periods, and the cumulative Sharpe ratio at first glance may seem only slightly better, the difference between the optimal portfolios and the equally-weighted portfolio is striking when looking at the cumulative returns.

In the empirical setting, the optimal portfolios led to catastrophic losses. In this simulation setting, the results are the opposite.

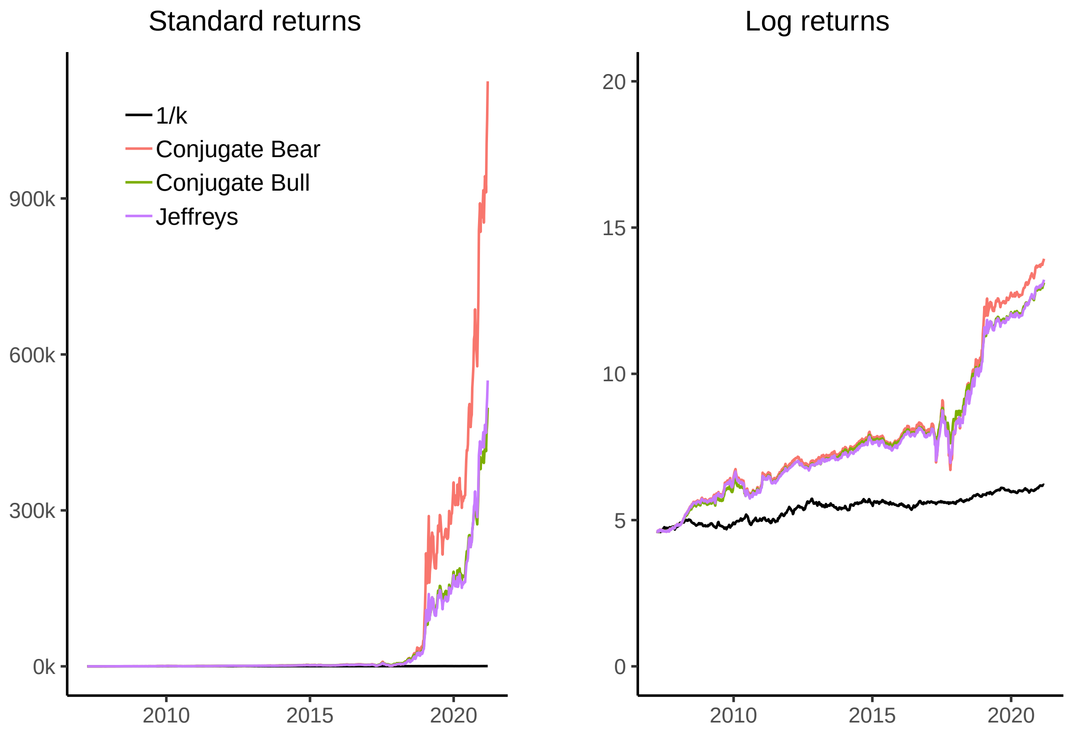

Figure 6: Cumulative returns, normal distribution (20 stocks)

Figure 6 displays extreme returns for the optimal portfolios, especially for the Conjugate Bear prior. This also, again, illustrates the issue with looking only at Sharpe ratios as a performance measure. Looking more closely within each period, as in Figure 7, it is apparent that period 6 contributes a lot to the accumulated returns.

Figure 7: Cumulative returns in each period, normal distribution (20 stocks)

It should be noted that period 6 also, at instances, contain significant losses. This exhibits the high risk of the various market portfolios within this period, and accounts for the fact that the Sharpe ratio is not significantly different from that of the equally-weighted portfolio. It also demonstrates another important fact of the market portfolio: When the return of the minumum variance portfolio and the risk-free rate are close to eachother, which is the case in period 6, the tangency point between the CML and the hyperbolic efficient frontier (see Figure 1 for a visual reference) will lie in a section with higher risk. In this specific example, the market portfolio for the Conjugate Bear prior corresponds to a risk-aversion coefficient γ of less than one, implying that it is associated with very high risk. Clearly, however, this extreme risk has also led to extreme rewards in this specific setting.

As for the longer sample periods accounted for in the appendix, it is interesting to note that all periods generally show smaller Sharpe ratios out-of-sample, many which are lesser than that of the equally-weighted portfolio. However, no significant difference is noted here.

4.3.2 Results for 10 stocks

Looking at the smaller portfolios consisting of 10 stocks only, the estimators for the Sharpe ratios in Table 8 are, as in the empirical case, lower.

Table 8: Sharpe ratio estimators, normal distribution (10 stocks)

Sample period (out-of-sample period)	Conventional	Jeffreys	Conjugate Bear	Conjugate Bull
1. 02-04-08 - 07-03-26 (07-04-02 - 09-03-23)	0.380	0.371	0.211	0.471
2. 04-04-05 - 09-03-23 (09-03-30 - 11-03-21)	0.280	0.273	0.156	0.327
3. 06-04-03 - 11-03-21 (11-03-28 - 13-03-18)	0.247	0.241	0.138	0.315
4. 08-03-31 - 13-03-18 (13-03-25 - 15-03-16)	0.209	0.205	0.113	0.258
5. 10-03-29 - 15-03-16 (15-03-23 - 17-03-13)	0.281	0.274	0.156	0.343
6. 12-03-26 - 17-03-13 (17-03-20 - 19-03-11)	0.365	0.357	0.209	0.453
7. 14-03-24 - 19-03-11 (19-03-18 - 21-03-08)	0.209	0.204	0.116	0.257
8. 16-03-21 - 21-03-08 (21-03-15 - 23-03-06)	0.294	0.287	0.161	0.350

The out-of-sample Sharpe ratios in 9 again exhibit the curious fact that the smaller portfolios seems to perform worse than the larger ones, a fact which is also confirmed by the longer sample periods available in the appendix.

Table 9: Out-of-sample Sharpe ratios and p-values, normal distribution (10 stocks)

Sample period (out-of-sample period)	1/k	Jeffreys / Conventional	Conjugate Bear	Conjugate Bull
1. 02-04-08 - 07-03-26 (07-04-02 - 09-03-23)	0.125	0.121 (0.97 / 0.97)	0.116 (0.93 / 0.92)	0.123 (0.98 / 0.98)
2. 04-04-05 - 09-03-23 (09-03-30 - 11-03-21)	-0.020	0.085 (0.43 / 0.44)	0.081 (0.46 / 0.46)	0.073 (0.48 / 0.48)
3. 06-04-03 - 11-03-21 (11-03-28 - 13-03-18)	0.156	0.112 (0.60 / 0.61)	0.110 (0.59 / 0.60)	0.105 (0.55 / 0.55)
4. 08-03-31 - 13-03-18 (13-03-25 - 15-03-16)	0.058	0.150 (0.22 / 0.20)	0.147 (0.23 / 0.21)	0.151 (0.23 / 0.21)
5. 10-03-29 - 15-03-16 (15-03-23 - 17-03-13)	0.010	0.114 (0.23 / 0.22)	0.111 (0.24 / 0.23)	0.099 (0.28 / 0.28)
6. 12-03-26 - 17-03-13 (17-03-20 - 19-03-11)	0.176	0.176 (1.00 / 1.00)	0.172 (0.96 / 0.96)	0.175 (0.99 / 0.99)
7. 14-03-24 - 19-03-11 (19-03-18 - 21-03-08)	0.105	0.199 (0.43 / 0.43)	0.213 (0.38 / 0.38)	0.240 (0.25 / 0.25)
8. 16-03-21 - 21-03-08 (21-03-15 - 23-03-06)	0.156	0.113 (0.69 / 0.69)	0.117 (0.71 / 0.72)	0.126 (0.77 / 0.78)
Cumulative Sharpe ratio	0.088	0.102 (0.74)	0.093 (0.91)	0.106 (0.66)

Studying the cumulative returns in Figure 8, it is evident that the smaller portfolios generates remarkably lower returns over time as compared to the larger portfolios in Figure 6. However, the accumulated return of the optimal portfolios are still a lot larger than that of the equally-weighted portfolio.

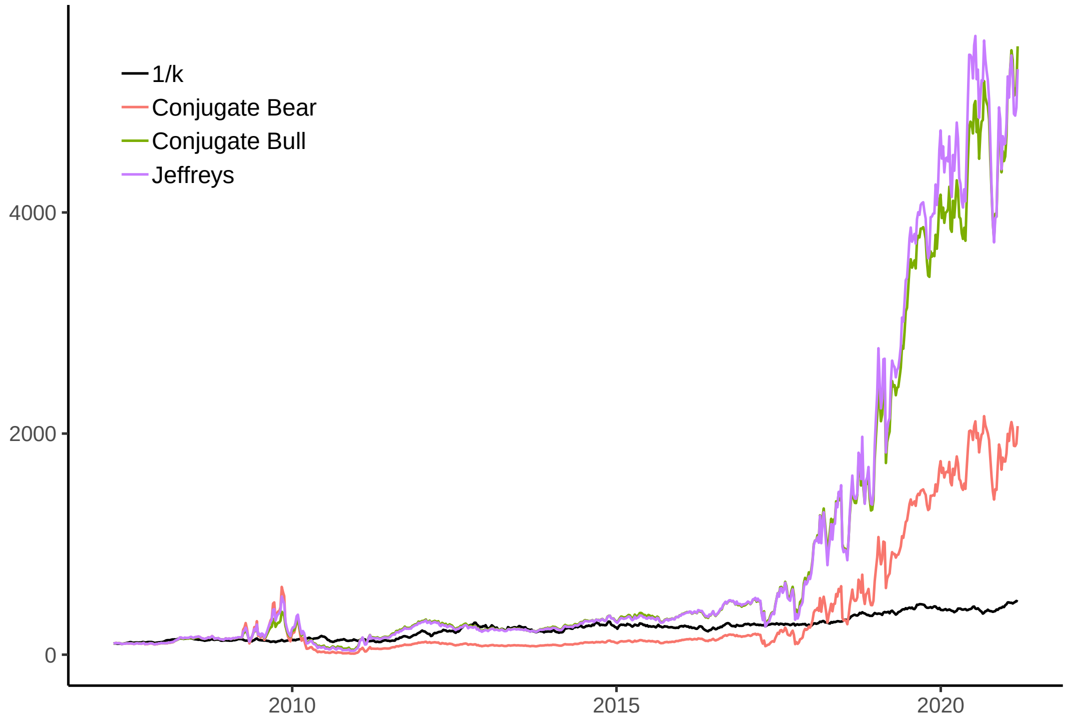

Figure 8: Cumulative returns, normal distribution (10 stocks)

4.4 Robustness analysis

This section considers data simulated from a multivariate t-distribution with five degrees of freedom, as described in section 3.2. The setup of the simulation study is such that the out-of-sample returns come from the same distribution as the in-sample returns. However, the multivariate t-distribution does not satisfy the assumptions of multivariate centered spherical symmetry and infinite exchangeability.

4.4.1 Results for 20 stocks

As in the preceding sections, the analysis begins by looking at the Sharpe ratio estimators in Table 10. In the first period, all portfolios except the Conjugate Bull prior are omitted. This is due to the fact that the return of the minimum variance portfolio in these cases are smaller than that of the risk-free rate, which implies that the market portfolio does not exist.

Table 10: Sharpe ratio estimators, t-distribution (20 stocks)

Sample period (out-of-sample period)	Conventional	Jeffreys	Conjugate Bear	Conjugate Bull
1. 02-04-08 - 07-03-26 (07-04-02 - 09-03-23)	-	-	-	0.306
2. 04-04-05 - 09-03-23 (09-03-30 - 11-03-21)	0.392	0.375	0.218	0.480
3. 06-04-03 - 11-03-21 (11-03-28 - 13-03-18)	0.248	0.237	0.134	0.292
4. 08-03-31 - 13-03-18 (13-03-25 - 15-03-16)	0.307	0.294	0.170	0.369
5. 10-03-29 - 15-03-16 (15-03-23 - 17-03-13)	0.336	0.322	0.185	0.417
6. 12-03-26 - 17-03-13 (17-03-20 - 19-03-11)	0.282	0.270	0.158	0.367
7. 14-03-24 - 19-03-11 (19-03-18 - 21-03-08)	0.306	0.293	0.167	0.356
8. 16-03-21 - 21-03-08 (21-03-15 - 23-03-06)	0.500	0.479	0.270	0.565

It is notable that the estimators in Table 10 are in many cases smaller than in the normally distributed simulation setting, as accounted for in Table 6. Turning to Table 11, it is once more apparent that the out-of-sample Sharpe ratios are in general much lower than their estimator counterparts.

Table 11: Out-of-sample Sharpe ratios and p-values, t-distribution (20 stocks)

Sample period (out-of-sample period)	1/k	Jeffreys / Conventional	Conjugate Bear	Conjugate Bull
1. 02-04-08 - 07-03-26 (07-04-02 - 09-03-23)	0.035	-	-	0.186 (0.28 / 0.30)
2. 04-04-05 - 09-03-23 (09-03-30 - 11-03-21)	0.051	0.084 (0.75 / 0.74)	0.087 (0.73 / 0.73)	0.083 (0.75 / 0.75)
3. 06-04-03 - 11-03-21 (11-03-28 - 13-03-18)	0.222	0.051 (0.10 / 0.10)	0.052 (0.11 / 0.10)	0.054 (0.10 / 0.10)
4. 08-03-31 - 13-03-18 (13-03-25 - 15-03-16)	-0.006	-0.142 (0.24 / 0.26)	-0.130 (0.29 / 0.30)	-0.100 (0.39 / 0.41)
5. 10-03-29 - 15-03-16 (15-03-23 - 17-03-13)	0.173	0.316 (0.16 / 0.13)	0.314 (0.18 / 0.15)	0.307 (0.20 / 0.17)
6. 12-03-26 - 17-03-13 (17-03-20 - 19-03-11)	-0.108	0.024 (0.14 / 0.14)	0.020 (0.16 / 0.16)	0.006 (0.20 / 0.20)
7. 14-03-24 - 19-03-11 (19-03-18 - 21-03-08)	0.104	0.036 (0.58 / 0.59)	0.040 (0.60 / 0.60)	0.057 (0.67 / 0.67)
8. 16-03-21 - 21-03-08 (21-03-15 - 23-03-06)	0.099	0.141 (0.71 / 0.71)	0.144 (0.69 / 0.69)	0.155 (0.61 / 0.61)
Cumulative Sharpe ratio	0.076	0.038 (0.31)	0.037 (0.302)	0.078 (0.95)

No significant differences are noted in comparison to the equally-weighted portfolio, and the same thing can be said about the longer sample periods in the appendix.

Figure 9 accounts for the accumulated returns of the portfolios, where the first period in which the Jeffreys and Conjugate Bear market portfolio did not exist has been dropped from the data.

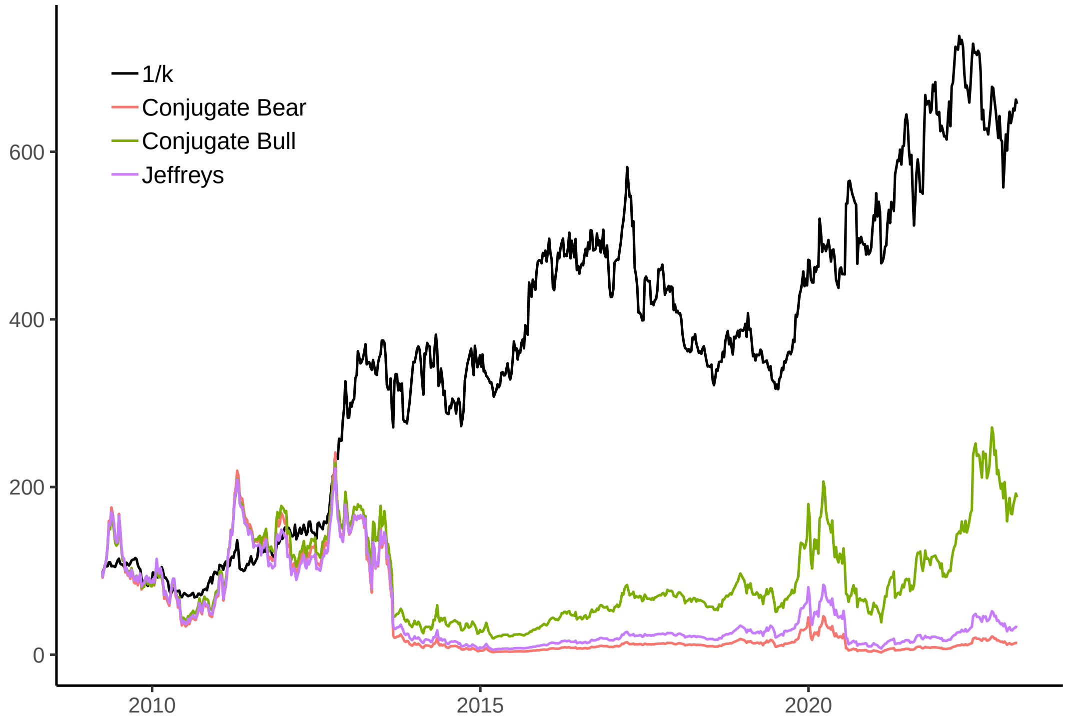

Figure 9: Cumulative returns, t-distribution (20 stocks)

The accumulated returns in the above figure exhibit a behavior which is more reminiscent of the empirical returns in Figure 4 than in the normally distributed data. The change of distribution in the simulated setting has, apparently, had a profound effect on the resulting cumulative returns.

4.4.2 Results for 10 stocks

As a final analysis, we consider the smaller 10-stock portfolios under the t-distribution setting. In Table 12, all of the considered approaches are omitted in the first period due to the market portfolio not existing.

Table 12: Sharpe ratio estimators, t-distribution (10 stocks)

Sample period (out-of-sample period)	Conventional	Jeffreys	Conjugate Bear	Conjugate Bull
1. 02-04-08 - 07-03-26 (07-04-02 - 09-03-23)	-	-	-	-
2. 04-04-05 - 09-03-23 (09-03-30 - 11-03-21)	0.234	0.228	0.129	0.260
3. 06-04-03 - 11-03-21 (11-03-28 - 13-03-18)	0.213	0.208	0.120	0.259
4. 08-03-31 - 13-03-18 (13-03-25 - 15-03-16)	0.237	0.232	0.135	0.293
5. 10-03-29 - 15-03-16 (15-03-23 - 17-03-13)	0.252	0.246	0.140	0.310
6. 12-03-26 - 17-03-13 (17-03-20 - 19-03-11)	0.156	0.152	0.085	0.199
7. 14-03-24 - 19-03-11 (19-03-18 - 21-03-08)	0.254	0.248	0.143	0.307
8. 16-03-21 - 21-03-08 (21-03-15 - 23-03-06)	0.296	0.289	0.165	0.353

As before, we note lower Sharpe ratio estimators in the smaller portfolios. Regarding the out-of-sample returns in Table 13, a significant underperformance against the equally-weighted portfolio is noted for all optimal portfolios in period 3. No significant differences are noted in the longer sample periods, as shown in the appendix.

Table 13: Out-of-sample Sharpe ratios and p-values, t-distribution (10 stocks)

Sample period (out-of-sample period)	1/k	Jeffreys / Conventional	Conjugate Bear	Conjugate Bull
1. 02-04-08 - 07-03-26 (07-04-02 - 09-03-23)	-	-	-	-
2. 04-04-05 - 09-03-23 (09-03-30 - 11-03-21)	0.037	0.069 (0.75 / 0.75)	0.067 (0.76 / 0.76)	0.064 (0.78 / 0.78)
3. 06-04-03 - 11-03-21 (11-03-28 - 13-03-18)	0.234	0.031 (0.02 / 0.02)	0.030 (0.03 / 0.02)	0.026 (0.02 / 0.02)
4. 08-03-31 - 13-03-18 (13-03-25 - 15-03-16)	-0.026	-0.135 (0.31 / 0.32)	-0.132 (0.35 / 0.35)	-0.131 (0.33 / 0.33)
5. 10-03-29 - 15-03-16 (15-03-23 - 17-03-13)	0.161	0.256 (0.34 / 0.29)	0.262 (0.30 / 0.25)	0.263 (0.27 / 0.23)
6. 12-03-26 - 17-03-13 (17-03-20 - 19-03-11)	-0.110	-0.109 (0.98 / 0.98)	-0.102 (0.91 / 0.91)	-0.091 (0.79 / 0.79)
7. 14-03-24 - 19-03-11 (19-03-18 - 21-03-08)	0.135	0.151 (0.87 / 0.86)	0.143 (0.93 / 0.93)	0.138 (0.98 / 0.97)
8. 16-03-21 - 21-03-08 (21-03-15 - 23-03-06)	0.101	0.206 (0.31 / 0.31)	0.197 (0.36 / 0.36)	0.188 (0.39 / 0.39)
Cumulative Sharpe ratio	0.078	0.041 (0.353)	0.034 (0.3)	0.04 (0.304)

5 Conclusions and discussion

As mentioned in the introduction, many portfolio optimization techniques have been shown to struggle with difficulties when applied out-of-sample. By turning to Bayesian statistics it is possible to account for the parameter uncertainty when constructing an optimal portfolio which, at least in theory, takes into account the estimation error.

Within the framework of this thesis, we have introduced both the conventional theory of portfolio optimization and a Bayesian approach with several different prior distributions. The latter have been shown by Bauder et al. [6] to, under some circumstances, improve the estimation of the expected return, variance and thereby the efficient frontier.

The aim of this thesis was to test the methods introduced by Bauder et al. [6] and developed by Bodnar et al. [3] out-of-sample, and to see if they could outperform the equally-weighted portfolio in terms of the Sharpe ratio. To this end, we have mathematically derived the weights of the market portfolio, which maximizes the Sharpe ratio, in a Bayesian setting with two different priors. Using both empirical and simulated historical data, the market portfolio has then been applied out-of-sample in a series of periods.

The overall conclusions are in line with earlier research, namely that the various optimal portfolios generally fail to outperform the equally-weighted approach in terms of the Sharpe ratio. This conclusion comes, however, with a word of caution given that the power of the tests applied in the thesis have been questioned, as mentioned in section 2.3.2. Furthermore, the choice of variance as a risk measure and the Sharpe ratio as a performance measure is by no means arbitrary and may very well be put into question. In utilizing the variance as risk, large deviations from the mean return inflates the perceived risk – even though such deviations may be positive. This can be illustrated by an example.

Consider the following sequences of returns:

(a) 10%, 10%, 0.5%, -1%

(b) 2%, 2%, 2%, -1%.

The cumulative return of sequence (a) is equal to 20.39%, which is notably higher than that of sequence (b) with 5.06%. However, the standard deviation in (a) is also higher. Assuming a risk-free rate of zero, this leads to to the fact that (b) will have a better Sharpe ratio of 0.833 than that of (a) with a Sharpe ratio of 0.819.

It is no easy task to explain why a rational investor would prefer a portfolio or an asset with returns as in (b) rather than (a). In utilizing variance as a risk measure, we are stuck in this counterintuitive and illogical corner. As an alternative, one may consider other risk measures, such as value at risk (VaR) and conditional value at risk (CVaR). These are, however, beyond the scope of this thesis.

Even though the optimal market portfolios fail to outperform the equally-weighted portfolio in terms of the Sharpe ratio, we have illustrated that there is still a large difference between these approaches in terms of pure return. Part of this difference may be attributed to the risk that comes with the market portfolio which, as have been shown, can lead to both ruin and extreme riches. When applying the market portfolio in practice, the results depend on a range of different factors: portfolio size, sample size, choice of prior, relationship between the risk-free rate and the return of the minimum variance portfolio and – of course – the properties of the returns themselves, to mention some of them.

The results of the thesis may seem disheartening. Nevertheless, applying a Bayesian perspective to portfolio optimization still leads to several interesting contributions. There are many aspects which have not been touched within the scope of this thesis, such as computing credible intervals, studying more closely the distributional properties of the weights in a Bayesian optimal portfolio, or utilizing different risk measures.

6 Appendix

6.1 Proof: CML and market portfolio

This proof is written with respect to the Jeffreys prior; the proof for the conjugate prior and sample approach follows analogously.

Consider as in Figure 1 a line drawn from the point $(0, r_f)$, where r_f is the return of the risk-free asset, up to the hyperbolic efficient frontier. The slope of such a line, drawn up to the expected return and variance of a feasible portfolio P, can be written as

$$\frac{R_P - r_f}{SD_P} = \frac{\mathbf{w}^\top \bar{\mathbf{x}}_{t-1} - r_f}{\sqrt{c_{k,n}\mathbf{w}^\top \mathbf{S}_{t-1}\mathbf{w}}}, \tag{7}$$

where $\mathbf{w}^\top \bar{\mathbf{x}}_{t-1}$ and $\mathbf{S}_{t-1}$ are as in (2) and the results from Corollary 2.1 are used in the right hand side of the equation.

Maximizing the slope of this line is equivalent to maximizing the Sharpe ratio, since the left hand side of (7) is equal to the Sharpe ratio. In order to find the weights of the CML and market portfolio we thus maximize the utility function

$$\max \frac{\mathbf{w}^\top \bar{\mathbf{x}}_{t-1} - r_f}{\sqrt{c_{k,n}\mathbf{w}^\top \mathbf{S}_{t-1}\mathbf{w}}}$$

over vectors $\mathbf{w} \in \mathbb{R}^k$ subject to the constraint $\mathbf{w}^\top \mathbf{1} = 1$.

Forming the Lagrangian

$$F(\boldsymbol{w}, \lambda) = \frac{\mathbf{w}^\top \bar{\mathbf{x}}_{t-1} - r_f}{\sqrt{c_{k,n}\mathbf{w}^\top \mathbf{S}_{t-1}\mathbf{w}}} - \lambda(\mathbf{w}^\top \mathbf{1} - 1),$$

and taking the gradient and equating it to zero yields

$$\nabla_{\mathbf{w}} F(\mathbf{w}, \lambda) = \frac{\sqrt{c_{k,n}\mathbf{w}^\top \mathbf{S}_{t-1}\mathbf{w}}\, \bar{\mathbf{x}}_{t-1} - \frac{\mathbf{w}^\top \bar{\mathbf{x}}_{t-1} - r_f}{\sqrt{c_{k,n}\mathbf{w}^\top \mathbf{S}_{t-1}\mathbf{w}}} c_{k,n}\mathbf{S}_{t-1}\mathbf{w}}{c_{k,n}\mathbf{w}^\top \mathbf{S}_{t-1}\mathbf{w}} - \lambda \mathbf{1} = 0.$$

After simplification and using the identities for variance/standard deviation and expected return of the portfolio, this yields

$$\frac{R_P - r_f}{V_P} c_{k,n}\mathbf{S}_{t-1}\mathbf{w} = \bar{\mathbf{x}}_{t-1} - \lambda SD_P \mathbf{1}. \tag{8}$$

By multiplying from the left with $\mathbf{w}^\top$ we get

$$\frac{R_P - r_f}{V_P} c_{k,n} \mathbf{w}^\top \mathbf{S}_{t-1} \mathbf{w} = R_P - \lambda SD_P,$$

so that

$$\lambda = \frac{r_f}{SD_P}.$$

Setting $\gamma = \frac{R_P - r_f}{V_P}$ in (8) now yields the equation

$$\gamma c_{k,n} \mathbf{S}_{t-1} \mathbf{w} = (\bar{\mathbf{x}}_{t-1} - r_f \mathbf{1}). \tag{9}$$

Noting that γ depends on $\boldsymbol{w}$, if we now let the weights in γ be parametrized, and let $n - k > 2$ so that $\det \mathbf{S}_{t-1} \neq 0$, we can solve for the CML weights as

$$\boldsymbol{w}_{CML} = \gamma^{-1} c_{k,n}^{-1} \mathbf{S}_{t-1}^{-1} (\bar{\mathbf{x}}_{t-1} - r_f \mathbf{1}).$$

Assume now instead that we wish to solve (9) for the market portfolio weights, which should sum to one. Using the constraint $\mathbf{w}^\top \mathbf{1} = 1$ again yields

$$\gamma = \mathbf{1}^\top c_{k,n}^{-1} \mathbf{S}_{t-1}^{-1} (\bar{\mathbf{x}}_{t-1} - r_f \mathbf{1}).$$

It then follows that the weights of the market portfolio are given by

$$\mathbf{w}_M = \frac{\mathbf{S}_{t-1}^{-1} (\bar{\mathbf{x}}_{t-1} - r_f \mathbf{1})}{\mathbf{1}^\top \mathbf{S}_{t-1}^{-1} (\bar{\mathbf{x}}_{t-1} - r_f \mathbf{1})},$$

due to the fact that the constant $c_{k,n}$ is cancelled out in the numerator and denominator. This also implies that the weights for the market portfolio is the same for the conventional sample approach and the Jeffreys prior.

It is now necessary to confirm that $\gamma \neq 0$ in the case of the market portfolio. This can be shown by using the expected return of the minimum variance portfolio. Now

$$R_{GMV} = \mathbf{w}_{GMV}^\top \bar{\mathbf{x}}_{t-1} = \frac{\mathbf{S}_{t-1}^{-1} \mathbf{1}}{\mathbf{1}^\top \mathbf{S}_{t-1}^{-1} \mathbf{1}} \bar{\mathbf{x}}_{t-1}.$$

Following the assumption of $r_f < R_{GMV}$ and $\mathbf{1}^\top \mathbf{S}_{t-1}^{-1} \mathbf{1} > 0$, we now have

$$\gamma = \mathbf{1}^\top \mathbf{S}_{t-1}^{-1} \bar{\mathbf{x}}_{t-1} - r_f \mathbf{1}^\top \mathbf{S}_{t-1}^{-1} \mathbf{1} > \mathbf{0},$$

where

$$\mathbf{1}^\top \mathbf{S}_{t-1}^{-1} \bar{\mathbf{x}}_{t-1} = \mathbf{S}_{t-1}^{-1} \mathbf{1} \bar{\mathbf{x}}_{t-1}$$

due to the symmetry of $\mathbf{S}_{t-1}^{-1}$.

The return of a CML portfolio can be expressed as

$$R_{CML,P} = (\mathbf{w}_{CML,P}^\top \mathbf{1})\mathbf{w}_M^\top \bar{\mathbf{x}}_{t-1} + (1 - \mathbf{w}_{CML,P}^\top \mathbf{1})r_f.$$

Since the (co)variance of the risk-free return is assumed to be zero, the variance of this portfolio can be attained using Corollary 2.3 and analogously for the conjugate and sample case.

Interpreting again Equation (7) as the slope parameter of the CML, we can now parametrize the line using SD_P for some portfolio P so that the line is given by

$$(R_P - r_f) = SD_P \left(\frac{R_M - r_f}{SD_M} \right).$$

This also implies that the Sharpe ratio is equal for all portfolios along the CML.

□

6.2 Results for longer sample periods

6.2.1 Empirical returns - 20 stocks

Table 14: Sharpe ratios, empirical (20 stocks, long)

Sample period (out-of-sample period)	Conventional	Jeffreys	Conjugate Bear	Conjugate Bull
1. 02-04-08 - 12-03-19 (12-03-26 - 14-03-17)	0.171	0.168	0.095	0.184
2. 04-04-05 - 14-03-17 (14-03-24 - 16-03-14)	0.164	0.160	0.091	0.180
3. 06-04-03 - 16-03-14 (16-03-21 - 18-03-12)	0.153	0.150	0.085	0.166
4. 08-03-31 - 18-03-12 (18-03-19 - 20-03-09)	0.186	0.182	0.104	0.200
5. 10-03-29 - 20-03-09 (20-03-16 - 22-03-07)	0.218	0.213	0.121	0.228

Table 15: Out-of-sample Sharpe ratios and p-values, empirical (20 stocks, long)

Sample period (out-of-sample period)	1/k	Jeffreys / Conventional	Conjugate Bear	Conjugate Bull
1. 02-04-08 - 12-03-19 (12-03-26 - 14-03-17)	0.161	0.029 (0.12 / 0.12)	0.038 (0.15 / 0.15)	0.060 (0.23 / 0.24)
2. 04-04-05 - 14-03-17 (14-03-24 - 16-03-14)	0.077	0.122 (0.56 / 0.56)	0.122 (0.57 / 0.57)	0.117 (0.59 / 0.59)
3. 06-04-03 - 16-03-14 (16-03-21 - 18-03-12)	0.148	0.008 (0.25 / 0.25)	0.007 (0.25 / 0.25)	-0.001 (0.21 / 0.20)
4. 08-03-31 - 18-03-12 (18-03-19 - 20-03-09)	0.001	0.026 (0.83 / 0.83)	0.018 (0.89 / 0.89)	0.016 (0.89 / 0.89)
5. 10-03-29 - 20-03-09 (20-03-16 - 22-03-07)	0.214	0.135 (0.57 / 0.56)	0.137 (0.58 / 0.57)	0.149 (0.64 / 0.63)
Cumulative Sharpe ratio	0.112	0.071 (0.44)	0.071 (0.45)	0.076 (0.49)

6.2.2 Empirical returns - 10 stocks

Table 16: Sharpe ratio estimators, empirical (10 stocks, long)

Sample period (out-of-sample period)	Conventional	Jeffreys	Conjugate Bear	Conjugate Bull
1. 02-04-08 - 12-03-19 (12-03-26 - 14-03-17)	0.137	0.136	0.076	0.141
2. 04-04-05 - 14-03-17 (14-03-24 - 16-03-14)	0.136	0.134	0.075	0.140
3. 06-04-03 - 16-03-14 (16-03-21 - 18-03-12)	0.116	0.114	0.064	0.123
4. 08-03-31 - 18-03-12 (18-03-19 - 20-03-09)	0.147	0.145	0.082	0.150
5. 10-03-29 - 20-03-09 (20-03-16 - 22-03-07)	0.152	0.151	0.085	0.157

Table 17: Out-of-sample Sharpe ratios and p-values, empirical (10 stocks, long)

Sample period (out-of-sample period)	1/k	Jeffreys / Conventional	Conjugate Bear	Conjugate Bull
1. 02-04-08 - 12-03-19 (12-03-26 - 14-03-17)	0.133	0.051 (0.26 / 0.28)	0.047 (0.25 / 0.27)	0.053 (0.28 / 0.31)
2. 04-04-05 - 14-03-17 (14-03-24 - 16-03-14)	0.082	0.086 (0.95 / 0.96)	0.088 (0.94 / 0.94)	0.091 (0.90 / 0.90)
3. 06-04-03 - 16-03-14 (16-03-21 - 18-03-12)	0.159	-0.106 (0.01 / 0.01)	-0.113 (0.01 / 0.01)	-0.125 (0.01 / 0.01)
4. 08-03-31 - 18-03-12 (18-03-19 - 20-03-09)	0.017	-0.009 (0.82 / 0.82)	-0.015 (0.78 / 0.78)	-0.015 (0.77 / 0.77)
5. 10-03-29 - 20-03-09 (20-03-16 - 22-03-07)	0.212	0.151 (0.63 / 0.63)	0.143 (0.59 / 0.59)	0.148 (0.61 / 0.61)
Cumulative Sharpe ratio	0.113	0.041 (0.13)	0.036 (0.11)	0.037 (0.10)

6.2.3 Simulated returns - 20 stocks

Table 18: Sharpe ratio estimators, normal distribution (20 stocks, long)

Sample period (out-of-sample period)	Conventional	Jeffreys	Conjugate Bear	Conjugate Bull
1. 02-04-08 - 12-03-19 (12-03-26 - 14-03-17)	0.265	0.259	0.147	0.264
2. 04-04-05 - 14-03-17 (14-03-24 - 16-03-14)	0.289	0.283	0.161	0.291
3. 06-04-03 - 16-03-14 (16-03-21 - 18-03-12)	0.244	0.239	0.136	0.245
4. 08-03-31 - 18-03-12 (18-03-19 - 20-03-09)	0.263	0.258	0.147	0.272
5. 10-03-29 - 20-03-09 (20-03-16 - 22-03-07)	0.189	0.185	0.105	0.202

Table 19: Out-of-sample Sharpe ratios and p-values, normal distribution (20 stocks, long)

Sample period (out-of-sample period)	1/k	Jeffreys / Conventional	Conjugate Bear	Conjugate Bull
1. 02-04-08 - 12-03-19 (12-03-26 - 14-03-17)	0.197	0.022 (0.24 / 0.24)	0.020 (0.24 / 0.24)	0.023 (0.23 / 0.24)
2. 04-04-05 - 14-03-17 (14-03-24 - 16-03-14)	0.001	0.009 (0.95 / 0.95)	0.018 (0.89 / 0.89)	0.032 (0.79 / 0.79)
3. 06-04-03 - 16-03-14 (16-03-21 - 18-03-12)	-0.058	-0.022 (0.70 / 0.70)	-0.017 (0.67 / 0.67)	-0.001 (0.53 / 0.53)
4. 08-03-31 - 18-03-12 (18-03-19 - 20-03-09)	-0.051	0.010 (0.54 / 0.53)	0.015 (0.51 / 0.51)	0.016 (0.51 / 0.50)
5. 10-03-29 - 20-03-09 (20-03-16 - 22-03-07)	0.075	-0.095 (0.05 / 0.05)	-0.098 (0.05 / 0.05)	-0.082 (0.07 / 0.06)
Cumulative Sharpe ratio	0.035	0.002 (0.58)	0.004 (0.61)	0.008 (0.63)

6.2.4 Simulated returns - 10 stocks

Table 20: Sharpe ratio estimators, normal distribution (10 stocks, long)

Sample period (out-of-sample period)	Conventional	Jeffreys	Conjugate Bear	Conjugate Bull
1. 02-04-08 - 12-03-19 (12-03-26 - 14-03-17)	0.199	0.196	0.113	0.204
2. 04-04-05 - 14-03-17 (14-03-24 - 16-03-14)	0.253	0.250	0.144	0.263
3. 06-04-03 - 16-03-14 (16-03-21 - 18-03-12)	0.165	0.163	0.093	0.168
4. 08-03-31 - 18-03-12 (18-03-19 - 20-03-09)	0.200	0.197	0.111	0.200
5. 10-03-29 - 20-03-09 (20-03-16 - 22-03-07)	0.149	0.147	0.083	0.155

Table 21: Out-of-sample Sharpe ratios and p-values, normal distribution (10 stocks, long)

Sample period (out-of-sample period)	1/k	Jeffreys / Conventional	Conjugate Bear	Conjugate Bull
1. 02-04-08 - 12-03-19 (12-03-26 - 14-03-17)	0.184	0.025 (0.24 / 0.24)	0.026 (0.25 / 0.25)	0.035 (0.27 / 0.27)
2. 04-04-05 - 14-03-17 (14-03-24 - 16-03-14)	-0.023	-0.049 (0.81 / 0.81)	-0.045 (0.84 / 0.84)	-0.039 (0.88 / 0.88)
3. 06-04-03 - 16-03-14 (16-03-21 - 18-03-12)	-0.078	0.011 (0.34 / 0.34)	0.018 (0.31 / 0.31)	0.037 (0.20 / 0.20)
4. 08-03-31 - 18-03-12 (18-03-19 - 20-03-09)	-0.059	0.012 (0.46 / 0.46)	0.021 (0.41 / 0.41)	0.028 (0.36 / 0.35)
5. 10-03-29 - 20-03-09 (20-03-16 - 22-03-07)	0.106	0.010 (0.26 / 0.25)	-0.004 (0.20 / 0.20)	-0.015 (0.14 / 0.14)
Cumulative Sharpe ratio	0.026	0.004 (0.69)	0.007 (0.74)	0.013 (0.81)

6.2.5 Robustness analysis - 20 stocks

Table 22: Sharpe ratio estimators, t-distribution (20 stocks, long)

Sample period (out-of-sample period)	Conventional	Jeffreys	Conjugate Bear	Conjugate Bull
1. 02-04-08 - 12-03-19 (12-03-26 - 14-03-17)	0.275	0.269	0.152	0.274
2. 04-04-05 - 14-03-17 (14-03-24 - 16-03-14)	0.278	0.272	0.155	0.283
3. 06-04-03 - 16-03-14 (16-03-21 - 18-03-12)	0.235	0.230	0.134	0.255
4. 08-03-31 - 18-03-12 (18-03-19 - 20-03-09)	0.235	0.230	0.134	0.251
5. 10-03-29 - 20-03-09 (20-03-16 - 22-03-07)	0.282	0.276	0.158	0.292

Table 23: Out-of-sample Sharpe ratios and p-values, t-distribution (20 stocks, long)

Sample period (out-of-sample period)	1/k	Jeffreys / Conventional	Conjugate Bear	Conjugate Bull
1. 02-04-08 - 12-03-19 (12-03-26 - 14-03-17)	0.041	0.055 (0.89 / 0.89)	0.052 (0.91 / 0.91)	0.039 (0.98 / 0.98)
2. 04-04-05 - 14-03-17 (14-03-24 - 16-03-14)	-0.033	0.058 (0.43 / 0.43)	0.062 (0.41 / 0.42)	0.067 (0.38 / 0.38)
3. 06-04-03 - 16-03-14 (16-03-21 - 18-03-12)	0.083	-0.088 (0.24 / 0.25)	-0.085 (0.25 / 0.26)	-0.077 (0.26 / 0.27)
4. 08-03-31 - 18-03-12 (18-03-19 - 20-03-09)	0.090	0.075 (0.89 / 0.89)	0.074 (0.89 / 0.88)	0.079 (0.92 / 0.92)
5. 10-03-29 - 20-03-09 (20-03-16 - 22-03-07)	0.104	0.014 (0.46 / 0.45)	0.011 (0.45 / 0.44)	0.008 (0.43 / 0.42)
Cumulative Sharpe ratio	0.055	0.026 (0.59)	0.026 (0.59)	0.025 (0.58)

6.2.6 Robustness analysis - 10 stocks

Table 24: Sharpe ratio estimators, t-distribution (10 stocks, long)

Sample period (out-of-sample period)	Conventional	Jeffreys	Conjugate Bear	Conjugate Bull
1. 02-04-08 - 12-03-19 (12-03-26 - 14-03-17)	0.230	0.228	0.132	0.241
2. 04-04-05 - 14-03-17 (14-03-24 - 16-03-14)	0.190	0.187	0.107	0.194
3. 06-04-03 - 16-03-14 (16-03-21 - 18-03-12)	0.144	0.142	0.080	0.142
4. 08-03-31 - 18-03-12 (18-03-19 - 20-03-09)	0.181	0.179	0.102	0.183
5. 10-03-29 - 20-03-09 (20-03-16 - 22-03-07)	0.164	0.162	0.091	0.168

Table 25: Out-of-sample Sharpe ratios and p-values, t-distribution (10 stocks, long)

Sample period (out-of-sample period)	1/k	Jeffreys / Conventional	Conjugate Bear	Conjugate Bull
1. 02-04-08 - 12-03-19 (12-03-26 - 14-03-17)	0.011	0.121 (0.22 / 0.22)	0.123 (0.22 / 0.22)	0.120 (0.22 / 0.23)
2. 04-04-05 - 14-03-17 (14-03-24 - 16-03-14)	-0.041	-0.001 (0.72 / 0.72)	0.000 (0.71 / 0.71)	-0.005 (0.74 / 0.74)
3. 06-04-03 - 16-03-14 (16-03-21 - 18-03-12)	0.060	-0.100 (0.27 / 0.27)	-0.101 (0.27 / 0.27)	-0.102 (0.26 / 0.26)
4. 08-03-31 - 18-03-12 (18-03-19 - 20-03-09)	0.074	0.071 (0.98 / 0.98)	0.069 (0.97 / 0.97)	0.069 (0.96 / 0.96)
5. 10-03-29 - 20-03-09 (20-03-16 - 22-03-07)	0.115	0.158 (0.66 / 0.67)	0.164 (0.62 / 0.63)	0.180 (0.50 / 0.51)
Cumulative Sharpe ratio	0.040	0.056 (0.75)	0.058 (0.73)	0.056 (0.75)

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