

## A computation of the cost-of-capital margin from loss triangles and a comparison to the risk margin in Solvency II

Hanna Magnusson

Masteruppsats i matematisk statistik Master Thesis in Mathematical Statistics

Masteruppsats 2017:9 Försäkringsmatematik Juni 2017

www.math.su.se

Matematisk statistik Matematiska institutionen Stockholms universitet 106 91 Stockholm

# Matematiska institutionen



Mathematical Statistics Stockholm University Master Thesis **2017:9** http://www.math.su.se

## A computation of the cost-of-capital margin from loss triangles and a comparison to the risk margin in Solvency II

Hanna Magnusson\*

June 2017

## Abstract

The upcoming IFRS 17 Insurance entails new starting points for accounting and valuation of insurance contracts. The goal of this thesis is to look further into assigning a value to the aggregate outstanding liability for an insurance company, including a risk margin. We will apply a market-consistent two-stage valuation procedure of an insur- ance liability based on data from loss triangles. The first step is to find a portfolio of zero-coupon bonds that generates cash flows matching the liability cash flow in expected values. Residual cash flow will arise due to the imperfect replication in the first step, which is managed by repeated one-period replication using only cash funds. The value of the residual cash flow is what we call the cost-ofcapital margin. This is compared to the risk margin in the Solvency II framework, calculated according to a proposed approximation technique by EIOPA that is commonly used in the industry. Moreover, we consider two stochastic models for the comparison of the risk margin objects. Our results are both based on data used in the literature and on data from a Swedish insurance company. We will find that the risk margin in Solvency II may overestimate as well as underestimate the risk margin compared to the more correct valuation procedure. We will also see that the ap- proximation technique is performing well for insurance products that is less volatile and furthermore that the value of total outstanding insurance liability is not very different among the approaches.

<sup>\*</sup>Postal address: Mathematical Statistics, Stockholm University, SE-106 91, Sweden. E-mail: hannamag91@gmail.com. Supervisor: Filip Lindskog.

## Acknowledgements

I would like to thank my supervisor professor Filip Lindskog for his guidance and support throughout this project. He has helped me to realize ideas and also made me find this subject very interesting. Moreover, he has given me good advice that I will keep in mind in the future working as an Actuary. I would also like to thank my other supervisor Totte Pikanen for discussing problems and for explaining mathematical problems in an easy way.

## Contents

1	Introduction									
<b>2</b>	The	eory	6							
	2.1	Risk margin according to Solvency II	6							
	2.2	The cost-of-capital margin	7							
		2.2.1 Gaussian cashflows	9							
	2.3	Our model	9							
		2.3.1 Value of the outstanding liability	10							
		2.3.2 Estimators and predictors	12							
	2.4	Chain-ladder	14							
	2.5	Chain-ladder time series model	15							
	2.6	Stochastic model inspired by chain-ladder	15							
		2.6.1 Value of the outstanding liability	15							
		2.6.2 Estimators and predictors	17							
	2.7	Prediction of future solvency capital requirements	17							
	2.8	Balance sheet for the reference undertaking under Solvency								
		II and under the cost-of-capital margin approach	19							
3	Met	thod	<b>21</b>							
	3.1	Data	21							
	3.2	Model check	21							
	3.3	Simulation study	21							
4	Res	sults	23							
	4.1	Simulation study for data used in the literature	23							
	4.2	Results using data from a Swedish insurance company	29							
	4.3	Change in runoff pattern	36							
5	Die	cussion	40							
0	5.1	Future work	41							
р	. <b>f</b>		40							
п	eiere	nces	42							
$\mathbf{A}$	App	pendix	<b>43</b>							
	A.1	Residuals	43							
	A.2	Empirical distributions of risk margin objects	44							
	A.3	Distribution of parameter estimates	46							
	A.4	Values of risk margin objects for different cost-of-capital rates	47							
	A.5	"True" weights vs. best estimate ratio in Solvency II	48							
	A.6	Set up for changing the runoff pattern	49							

## 1 Introduction

The current International Financial Reporting Standards (IFRS 4) is aimed at insurance companies and is a guide to the accounting of insurance contracts. In the future, significant changes will occur when IFRS 4 is replaced by IFRS 17 Insurance. The International Accounting Standards Board (IASB) published an exposure draft in June 2013, called Exposure Draft ED/2013/7 *insurance contracts*, with proposals on the new standard, see [4]. This entails new starting points for accounting and valuation of insurance contracts with aim to increase transparency and reduce the differences in the accounting of insurance contracts. The final standard is expected to be released in May 2017 and will probably take effect on January 1, 2021.

On account of the upcoming IFRS 17, we will in this thesis look further into assigning a value to the aggregate outstanding liability for an insurance company, including a risk margin. In the current solvency regulatory framework Solvency II the market consistent valuation of the insurance liability consists of the so-called technical provisions. This corresponds to the sum of a discounted best estimate and a risk margin with the aim of capturing capital costs. The Solvency II framework defines the risk margin (RM) according to the following formula

$$RM = CoC \sum_{t \ge 0} \frac{SCR(t)}{(1 + r(t+1))^{t+1}},$$
(1)

where CoC denotes the cost-of-capital rate, SCR(t) denotes the required solvency capital after t years and r(t+1) is the risk free rate for t+1 years. This is prescribed by the European Insurance and Occupational Pensions Authority (EIOPA), see Article 37 in [2], where it is also assumed that the cost-of-capital rate is deterministic and set to 6%. Thus, the risk margin in the light of Solvency II corresponds to the present value of what one has to set aside today to yield 6% in return on the capital that is required for each year until runoff. The delegated regulation [2] also prescribes that the solvency capital requirement (SCR) should be calculated either by a standard formula, given in the framework, or by an internal model. This standard formula is calibrated by the Value-at-Risk (VaR) of the basic own funds with a confidence level of 99.5% over a time period of one year. The latter implies that the company will be insufficient with 0.5% probability. Note here the following, that for future years when t > 0, SCR(t) depends on the state at the beginning of year t, which is currently not known. Hence, the regulation allows insurance companies to use simplified methods when calculating the risk margin, which is something that can be questioned. Criticism and suggestions for better notations of risk margin objects are found in [6], [12] and [10].

In contrast to the current solvency regulatory framework, where one explicitly states how the risk margin should be calculated, one does not state any particular formula for the risk margin in the exposure draft for IFRS 17, there called risk adjustments. Instead one can find principles of how to approach the risk margin. Insurance companies will then be able to interpret their own computation of the risk margin in line with these principles. The new accounting standard, together with the criticism against the costof-capital formula (1), make a new derivation of the risk margin of great interest. We will in thesis look at another approach for calculating the risk margin consistent with the new regulation for insurance contracts. One alternative to calculate the risk margin under IFRS 17 is still according to the cost-of-capital formula in Solvency II.

The delagated regulation under Solvency II states that "...assets and liabilities are valued at the amount for which they could be exchanged in the case of assets or transferred or settled in the case of liabilities between knowledgeable and willing parties in an arm's length transaction". In the exposure draft for the new accounting standard for insurance contracts the risk adjustment is defined as "the compensation that an entity requires for bearing the uncertainty about the amount and timing of the cash flows that arise as the entity fulfils the insurance contract". It appears that the framework under Solvency II and IFRS 17 coincide under these two point of views. Therefore, to determine the aggregate liability cash flow we will consider a so-called reference undertaking situation where the insurance liability is hypothetically transferred to a separate entity. This entity is empty before the transfer and will contain assets with the purpose of matching the future liability cash flow as well as possible. The reference undertaking situation is also considered in the Solvency II framework.

Our procedure to valuation of the aggregate insurance liability and to determine the cost-of-capital margin is based on [1] and consists of two parts. Since liability cash flows are not typically replicated by financial instruments our first step will be to find a replicating portfolio that generates cash flows matching the liability cash flow with expected values. Secondly we want to asses the residual cash flow that will arise from the imperfect replication in the first step. The value of the residual liability cash flow corresponds to the compensation the reference undertaking would require for taking over the liability. These non-hedgeable risks give rise to capital requirements that the reference undertaking needs to meet. The initial capital that the company is given to compensate for the risk is not enough to cover all future capital requirements. Capital providers, such as share holders, are then asked to provide buffer capital throughout the runoff which in turn requires compensation for providing the capital. In line with this valuation procedure, we will apply the work in [5] to a valuation of an insurance liability cash flow based on data from loss triangles. This approach is presented when using an autoregressive model on incremental payments. We will further extend the approach and derive an expression for the value of the liability cash flow using a stochastic model inspired by the well-known claims reserving method chain-ladder. This is performed in the same setting as for the autoregressive model. In this way we can assign a value to the liability cash flow with different underlying models and analyze the difference. Moreover, we will compare and discuss the difference between the cost-of-capital margin and the risk margin in Solvency II. Since the latter most often is simplified by approximating the future solvency capital requirement by the ratio of best estimate, see (2), we will use the same technique here.

Theory underlying the valuation procedure is described in Section 2, together with other necessary notation for the data analysis. Methods for producing results underlying our analysis are described in Section 3 and the corresponding results are found in Section 4. Our data analysis is first based on loss triangle data used in the literature, e.g by Mack in [7] and by Verrall in [11]. The analysis is then extended using triangle data from a Swedish insurance company. Finally, in Section 5, we compare and discuss the results for the different approaches in assigning a value to the outstanding insurance liability. Here, we will also mention what can be done in future work.

## 2 Theory

Throughout the thesis we will consider the case when, for all times and maturities interest rates are zero. Although this assumption is unrealistic it allows us compare notions of risk margin and interpret differences.

### 2.1 Risk margin according to Solvency II

The risk margin according to Solvency II is explicitly defined by the cost-ofcapital formula (1). As mentioned in the introduction, the solvency capital requirement for future year t depends on the state at the beginning of year t, which is currently not known. Therefore, various proxies are used in practice. One common technique is by using the ratio of the best estimate at future year t and the best estimate at the valuation date. This simplified method is proposed by EIPOA, see Guideline 61 in [3]. Thus, when disregarding discounting, the cost-of-capital formula reduces to

$$RM = CoC \sum_{t=0}^{T} \frac{BE(t)}{BE(0)} SCR(0), \qquad (2)$$

where BE(t) equals the expected, seen from time 0, remaining liability cash flow from time t and CoC the cost-of-capital rate. Note that SCR(0) is the solvency capital requirement for the reference undertaking and does only contain non-hedgeable insurance risks.

## 2.2 The cost-of-capital margin

In this section we introduce the recursion formula for the cost-of-capital margin based on economic arguments that is done in [1]. This is introduced without going into any deeper mathematical details, however, these can be found in [1], section 3.

We consider an aggregate insurance liability cash flow  $X^o = (X_t^o)_{t=1}^T$  that corresponds to a stochastic process and where t represent time-periods of one year, i.e. [t, t+1). As is prescribed by EIOPA in [2] and in the exposure draft for IFRS 17 [4] we consider a hypothetical transfer of the insurance liabilities to a separate entity, a so called reference undertaking. This entity is empty before the transfer and is supposed to have assets matching the aggregate liability as well as possible. The first step in the valuation procedure is then to find a portfolio generating a cash flow replicating  $X^o$  until runoff at year T. This static replicating portfolio is purchased at time 0 and has a market price  $\pi$  generating the cash flow  $X^s = (X_t^s)_{t=1}^T$ . In this thesis we will consider the market price of a portfolio of zero-coupon bonds generating the cash flow  $X^s = \mathbb{E}[X^o] = (\mathbb{E}[X^o_t])_{t=1}^T$ . Here, under the assumption of zero interest rate,  $\pi = (\mathbb{E}[X_t^o])_{t=1}^T$ . The sum of the market price  $\pi$  of the replicating portfolio and the value of  $V_0(X)$  of the residual cash flow  $X := X^o - X^s$  defines the value of the original liability.  $V_0(X)$  is what we call the cost-of-capital margin and is calculated from repeated one-period replication. We will also refer to  $V_t(X)$  as the cost-of-capital margin for all  $t = \{0, ..., T\}$ , which furthermore will be defined in terms of  $X_{t+1}$  and  $V_{t+1}(X)$ .

Economic arguments lead us to a recursion defining the cost-of-capital margin  $V_0(X)$ . The one-period replication is using only cash funds where the capital provider requires compensation for capital costs and has limited liability. At time t, the capital provider is asked to provide the difference between the capital requirement and the value of the residual liability cash flow, i.e.

$$C_t := \operatorname{VaR}_{t,0.005}(-X_{t+1} - V_{t+1}(X)) - V_t(X), \tag{3}$$

where  $VaR_{t,0.005}$  is a risk measure (see Definition 2.1) motivated by the Solvency II regulations at a 99.5% confidence level.

**Definition 2.1.** In a one-year setting, the Value-at-Risk at time t of a random loss variable L at time t+1 at level  $u \in (0,1)$  is defined as

$$VaR_{t,u}(X) := \min\{m \in \mathbb{R} : \mathbb{P}(m+X < 0) \le u\}$$
$$= \min\{m \in \mathbb{R} : \mathbb{P}(-X \le m) \ge 1 - u\}$$
$$= \min\{l \in \mathbb{R} : \mathbb{P}(L \le l) \ge 1 - u\},$$

where the loss variable L = -X.

If buffer capital is provided at time t, the available capital at time t + 1 equals

$$\operatorname{VaR}_{t,0.005}(-X_{t+1} - V_{t+1}(X)) - X_{t+1},$$

where the capital provider collects the excess capital as compensation if this amount exceeds  $V_{t+1}(X)$ . If no excess capital is available at time t + 1 the capital provider has no obligation to offset any deficit. Thus, the investor has limited liability if

$$\operatorname{VaR}_{t,0.005}(-X_{t+1} - V_{t+1}(X)) - X_{t+1} - V_{t+1}(X) < 0.$$

The acceptability condition for the capital provider is expressed by the following relation

$$\mathbb{E}_{t}[(\operatorname{VaR}_{t,0.005}(-X_{t+1} - V_{t+1}(X)) - X_{t+1} - V_{t+1}(X))_{+}] \ge (1 + \eta_{t})C_{t}, \quad (4)$$

where  $\eta_t > 0$  corresponds to a return-on-capital rate and  $x_+ = \max(x, 0)$ .

Now, if we combine (3) and (4) and let  $Y_{t+1} := X_{t+1} + V_{t+1}(X)$ , we get

$$V_t(X) \ge \operatorname{VaR}_{t,0.005}(-Y_{t+1}) - \frac{1}{1+\eta_t} \mathbb{E}_t[(\operatorname{VaR}_{t,0.005}(-Y_{t+1}) - Y_{t+1})_+].$$

Thus, if we have a strict inequality the capital provider obtains an opportunity to more compensation than required. If this would be the case, the policy holders would have to pay a higher premium than what would be reasonable. Therefore, if we replace the inequality with an equality we define the value of the cash flow as the minimum value for which the capital provider finds acceptable and get

$$V_t(X) := \operatorname{VaR}_{t,0.005}(-Y_{t+1}) - \frac{1}{1+\eta_t} \mathbb{E}_t[(\operatorname{VaR}_{t,0.005}(-Y_{t+1}) - Y_{t+1})_+],$$

with the initial condition  $V_T(X) = 0$ .

#### 2.2.1 Gaussian cashflows

In [1], the cost-of-capital margin is further derived when considering Gaussian cash flows and a fixed cost-of-capital rate  $\eta_t = \eta_0$ . This assumption will provide a reasonable approximation since the cost-of-capital margin is mainly intended for aggregate cash flows. Below, we state the definition of a Gaussian model and the final proposition in the derivation of the cost-of-capital margin, since the latter is applied for the models we consider.

**Definition 2.2.** Let  $\Gamma$  be a finite set of Gaussian vectors in  $\mathbb{R}^T$  that are jointly Gaussian. Let

$$\mathcal{G}_0 := \{\emptyset, \Omega\}, \quad \mathcal{G}_t := \left( \lor_{Z \in \Gamma} \sigma(Z_t) \right) \lor \mathcal{G}_{t-1} \quad for \ t = 1, ..., T.$$

 $\mathbb{G} := (\mathcal{G}_t)_{t=0}^T$  is called a Gaussian filtration, and, if  $X \in \Gamma$ , then  $(X, \mathbb{G})$  is called a Gaussian model.

**Proposition 2.1.** Let  $(X, \mathbb{G})$ , be a zero mean Gaussian model, then for  $t \in \{0, ..., T-1\}$ ,

$$V_{t,\mathbb{G}}(X) = \mathbb{E}\Big[\sum_{s=t+1}^{T} X_s | \mathcal{G}_t\Big] + \sum_{s=t+1}^{T} \operatorname{Var}\Big(\mathbb{E}\Big[\sum_{u=s}^{T} X_u | \mathcal{G}_s\Big] | \mathcal{G}_{s-1}\Big)^{1/2} c.$$

Moreover,

$$V_{0,\mathbb{G}}(X) = \sum_{s=1}^{T} \left( \operatorname{Var}\left(\sum_{u=s}^{T} X_u | \mathcal{G}_{s-1}\right) - \operatorname{Var}\left(\sum_{u=s}^{T} X_u | \mathcal{G}_s\right) \right)^{1/2} c,$$
  
where  $c := \Phi^{-1}(0.995) - \frac{1}{1+\eta_0} \left( 0.995 \Phi^{-1}(0.995) + \varphi(\Phi^{-1}(0.995)) \right)$  and  $\eta_0 > 0.$ 

#### 2.3 Our model

Consider K lines of business where  $k \in \{1, ..., K\} = \mathcal{K}$ . Let  $Z_{ij}^{(k)}$  for development year  $j \in \{1, ..., J^{(k)}\} := \mathcal{J}$  and accident year  $i \in \{i_0^{(k)}, ..., J^{(k)}+1\} := \mathcal{I}$ , where  $i_0^{(k)} \leq 1$ , denote the incremental claims payment for the k:th line of business. Negative values for  $i_0^{(k)}$  correspond to fully developed accident years. Let  $I_{ij}^{(k)} := Z_{ij}^{(k)}/v_i^{(k)}$  denote the normalized incremental payments where  $v_i^{(k)}$  is equal to the premium volume for accident year *i*. We consider the following stochastic model for incremental payments

$$I_{ij}^{(k)} = \alpha_j^{(k)} + \beta_j^{(k)} I_{i,j-1}^{(k)} + \frac{\sigma_j^{(k)}}{\sqrt{v_i^{(k)}}} \epsilon_{ij}^{(k)},$$
(5)

where for every  $k \ \epsilon_{ij}^{(k)}$ ,  $(i, j) \in \mathcal{I} \times \mathcal{J}$  are mutually independent and standard normally distributed and  $I_{i0} := 0$ . We assume that the upper triangle

$$D_0 := \{ I_{ij}^{(k)} : (i, j, k) \in \mathcal{I} \times \mathcal{J} \times \mathcal{K}, i + j \le J^{(k)} + 1 \}$$

contains observed data and that all  $v_i^{(k)}$ ,  $(i, k) \in \mathcal{I} \times \mathcal{K}$  are known constants.

#### 2.3.1 Value of the outstanding liability

We want to assign a value to the future liability cash flow from contracts that may still generate claims, i.e. Incurred But Not Reported (IBNR) and Reported But Not Settled (RBNS) and from active contracts, i.e. Non Incurred claim (NI) claims. The lower triangle

$$\{Z_{ij}^{(k)}: (i,j,k) \in \mathcal{I} \times \mathcal{J} \times \mathcal{K}, i \ge 2, i+j > J^{(k)}+1\}$$

corresponds to the outstanding liability cash flow. Note that NI claims from still active contracts corresponds to accident year  $J^{(k)} + 1$ . Let  $T = \max\{J^{(k)} : k \in \mathcal{K}\}$  and

$$X_t^{o,(k)} := \begin{cases} \sum_{u=t+1}^{J^{(k)}+1} Z_{u,J^{(k)}-u+t+1}^{(k)}, & t = 1, ..., J^{(k)}, \\ 0 & t = J^{(k)}+1, ..., T. \end{cases}$$

Also let

$$X_t^o := \sum_{k=1}^K X_t^{o,(k)}, \quad t = 1, ..., T$$
$$X^o := (X_1^o, ..., X_T^o).$$

Given that available zero-coupon bond prices for the maturity times 1, ..., T exists and that the relevant conditional expectations and conditional variances can be determined, the value of the outstanding liability cash flow  $X^{o,(k)}$  is given by

$$\sum_{t=1}^{T} \mathbb{E}[X_t^o | \mathcal{F}_0] + c \sum_{t=1}^{T} \left( \operatorname{Var}(X_T^c | \mathcal{F}_{t-1}) - \operatorname{Var}(X_T^c | \mathcal{F}_t) \right)^{1/2}, \tag{6}$$

where the market price  $\pi$  of the bond portfolio giving the holder the cash

flow  $\mathbb{E}[X^o|\mathcal{F}_0]$  is given by  $\sum_{t=1}^T \mathbb{E}[X_t^o|\mathcal{F}_0]$ , and

$$\begin{split} X &:= X^o - \mathbb{E}[X^o | \mathcal{F}_0], \\ X_T^c &:= \sum_{s=1}^T X_s, \\ \mathcal{F}_t &:= \sigma(D_t), \\ D_t &:= \{ I_{ij}^{(k)} : (i, j, k) \in \mathcal{I} \times \mathcal{J} \times \mathcal{K}, i+j \leq J^{(k)} + 1 + t \}, \\ c &:= \Phi^{-1}(0.995) - \frac{1}{1+\eta_0} \Big( 0.995 \Phi^{-1}(0.995) + \varphi(\Phi^{-1}(0.995)) \Big). \end{split}$$

Moreover,

$$\operatorname{Var}(X_T^c | \mathcal{F}_s) = \operatorname{Var}\left(\sum_{u=1}^T (X_u^o - \mathbb{E}[X_u^o | \mathcal{F}_0] | \mathcal{F}_s)\right) = \operatorname{Var}\left(\sum_{u=1}^T X_u^o | \mathcal{F}_s\right)$$
$$= \operatorname{Var}(X_T^U | \mathcal{F}_s),$$

i.e. the variance of all the ultimate claims amount for all accident years given the information at times, for  $s \ge 0$ , where  $X_T^U := \sum_{u=1}^T X_u^o$ .

Notice that

$$\mathbb{E}[X^{o}|\mathcal{F}_{0}] = \sum_{k=1}^{K} \mathbb{E}[X^{o,(k)}|\mathcal{F}_{0}],$$

where

$$\mathbb{E}[X^{o,(k)}|\mathcal{F}_0] = \sum_{u=t+1}^{J^{(k)}+1} v_u^{(k)} \mathbb{E}[I_{u,J^{(k)}-u+t+1}^{(k)}|I_{u,J^{(k)}-u+1}^{(k)}],$$

for  $t = 1, ..., J^{(k)}$ . Furthermore,

$$\begin{split} \mathbb{E}[I_{u,J^{(k)}-u+t+1}^{(k)}|I_{u,J^{(k)}-u+1}^{(k)}] \\ &= I_{u,J^{(k)}-u+1}\prod_{l=J^{(k)}-u+2}^{J^{(k)}-u+t+1}\beta_l^{(k)} + \sum_{l=J^{(k)}-u+2}^{J^{(k)}-u+t+1}\alpha_l^{(k)}\prod_{m=l+1}^{J^{(k)}-u+t+1}\beta_m^{(k)}. \end{split}$$

It remains to derive the conditional variances in the latter part of equation (6). We have that

$$\operatorname{Var}(X_T^U | \mathcal{F}_t) = \sum_{k=1}^K \operatorname{Var}(X_T^{U,(k)} | \mathcal{F}_t) + 2 \sum_{k < l} \operatorname{Cov}(X_T^{U,(k)}, X_T^{U,(l)} | \mathcal{F}_t).$$

Since accident years are considered independent

$$\begin{aligned} \operatorname{Var}(X_T^{U,(k)}|\mathcal{F}_t) &= \sum_{i=t+2}^{J^{(k)}+1} (v_i^{(k)})^2 \operatorname{Var}\left(\sum_{j=J^{(k)}-i+2+t}^{J^{(k)}} I_{i,j}^{(k)} |\mathcal{F}_t\right) \\ &= \sum_{i=t+2}^{J^{(k)}+1} (v_i^{(k)})^2 \operatorname{Var}\left(\sum_{j=J^{(k)}-i+2+t}^{J^{(k)}} \sum_{l=J^{(k)}-i+2+t}^{j} \frac{\sigma_l^{(k)}}{\sqrt{v_i^{(k)}}} \epsilon_{i,l}^{(k)} \prod_{m=l+1}^{j} \beta_m^{(k)}\right) \\ &= \sum_{i=t+2}^{J^{(k)}+1} (v_i^{(k)})^2 \operatorname{Var}\left(\sum_{l=J^{(k)}-i+2+t}^{J^{(k)}} \sum_{j=l}^{J^{(k)}} \frac{\sigma_l^{(k)}}{\sqrt{v_i^{(k)}}} \epsilon_{i,l}^{(k)} \prod_{m=l+1}^{j} \beta_m^{(k)}\right) \\ &= \sum_{i=t+2}^{J^{(k)}+1} v_i^{(k)} \sum_{l=J^{(k)}-i+2+t}^{J^{(k)}} (\sigma_l^{(k)})^2 \left(1 + \sum_{j=l+1}^{J^{(k)}} \prod_{m=l+1}^{j} \beta_m^{(k)}\right)^2. \end{aligned}$$

Under the assumption that the ultimate claims amounts for the different lines of business are uncorrelated, i.e. that  $\text{Cov}(X_T^{U,(k)}, X_T^{U,(l)} | \mathcal{F}_t) = 0$  for all k < l and all t,

$$\sum_{t=1}^{T} \left( \operatorname{Var}(X_T^U | \mathcal{F}_{t-1}) - \operatorname{Var}(X_T^U | \mathcal{F}_t) \right)^{1/2}$$
  
=  $\sum_{t=1}^{T} \left( \sum_{k=1}^{K} \sum_{i=t+1}^{J^{(k)}} v_i^{(k)} (\sigma_{J^{(k)}-i+1+t}^{(k)})^2 \left( 1 + \sum_{j=J^{(k)}-i+2+t}^{J^{(k)}} \prod_{m=J^{(k)}-i+2+t}^{j} \beta_m^{(k)} \right)^2 \right)^{1/2}.$ 

Given suitable parameter estimators we may estimate  $\mathbb{E}[X_t^o|\mathcal{F}_0]$  by

$$\sum_{k=1}^{K} \sum_{u=t+1}^{J^{(k)}+1} v_{u}^{(k)} \Big( I_{u,J^{(k)}-u+1}^{(k)} \prod_{l=J^{(k)}-u+2}^{J^{(k)}-u+t+1} \hat{\beta}_{l}^{(k)} + \sum_{l=J^{(k)}-u+2}^{J^{(k)}-u+t+1} \hat{\alpha}_{l}^{(k)} \prod_{m=l+1}^{J^{(k)}-u+t+1} \hat{\beta}_{m}^{(k)} \Big)$$

and the sum of the conditional variances in (6) by

$$\sum_{t=1}^{T} \Big( \sum_{k=1}^{K} \sum_{i=t+1}^{J^{(k)}} v_i^{(k)} (\hat{\sigma}_{J^{(k)}-i+1+t}^{(k)})^2 \Big( 1 + \sum_{j=J^{(k)}-i+2+t}^{J^{(k)}} \prod_{m=J^{(k)}-i+2+t}^{j} \hat{\beta}_m^{(k)} \Big)^2 \Big)^{1/2}.$$

## 2.3.2 Estimators and predictors

Unbiased estimators for  $\alpha^{(k)}$  and  $\beta^{(k)}$  are found by weighted least squares based on the observed data. Write

$$\begin{bmatrix} I_{i_0^{(k)},j}^{(k)} \\ \vdots \\ I_{J^{(k)}-j+1,j}^{(k)} \end{bmatrix} = \begin{bmatrix} 1 & I_{i_0^{(k)},j-1}^{(k)} \\ \vdots & \vdots \\ 1 & I_{J^{(k)}-j+1,j-1}^{(k)} \end{bmatrix} \begin{bmatrix} \alpha_j^{(k)} \\ \beta_j^{(k)} \end{bmatrix} + \begin{bmatrix} e_{i_0^{(k)},j}^{(k)} \\ \vdots \\ e_{J^{(k)}-j+1,j}^{(k)} \end{bmatrix},$$

where  $\mathbf{e}_{j}^{(k)}$  has zero mean and covariance matrix  $(\sigma_{j}^{(k)})^{2} \mathbf{\Sigma}_{j}^{(k)}$  with  $\mathbf{\Sigma}_{j}^{(k)} = \operatorname{diag}(1/v_{i_{0}}^{(k)}, ..., 1/v_{J^{(k)}}^{(k)})$ . Compactly we can write the matrix equation above as  $\mathbf{I}_{j}^{(k)} = \mathbf{A}_{j}^{(k)} \boldsymbol{\theta}_{j}^{(k)} + \mathbf{e}_{j}^{(k)}$ . The weighted least squares estimator  $\mathbf{e}_{j}^{(k)} = ((\mathbf{f}_{j}^{(k)})^{T} (\mathbf{\Sigma}_{j}^{(k)})^{-1} (\mathbf{f}_{j}^{(k)})^{-1} (\mathbf$ 

$$\boldsymbol{\theta}_{j}^{(k)} = ((\mathbf{A}_{j}^{(k)})^{T} (\boldsymbol{\Sigma}_{j}^{(k)})^{-1} \mathbf{A}_{j}^{(k)})^{-1} (\mathbf{A}_{j}^{(k)})^{T} (\boldsymbol{\Sigma}_{j}^{(k)})^{-1} \mathbf{I}_{j}^{(k)}$$

is given by general theory of linear models. With  $v^{(k)} = \sum_{i=i_0^{(k)}}^{J^{(k)}-j+1} v_i^{(k)},$ 

$$\hat{\beta}_{j}^{(k)} = \frac{\sum_{i=i_{0}^{(k)}}^{J^{(k)}-j+1} \frac{v_{i}^{(k)}}{v^{(k)}} \left(I_{ij}^{(k)} - \sum_{i=i_{0}^{(k)}}^{J^{(k)}-j+1} \frac{v_{i}^{(k)}}{v^{(k)}} I_{ij}^{(k)}\right) I_{i,j-1}^{(k)}}{\sum_{i=i_{0}^{(k)}}^{J^{(k)}-j+1} \frac{v_{i}^{(k)}}{v^{(k)}} \left(I_{i,j-1}^{(k)} - \sum_{i=i_{0}^{(k)}}^{J^{(k)}-j+1} \frac{v_{i}^{(k)}}{v^{(k)}} I_{i,j-1}^{(k)}\right) I_{i,j-1}^{(k)}}\right.$$
(7)
$$= \beta_{j}^{(k)} + \sigma_{j}^{(k)} \frac{\sum_{i=i_{0}^{(k)}}^{J^{(k)}-j+1} \frac{v_{i}^{(k)}}{v^{(k)}} \left(\frac{\epsilon_{ij}^{(k)}}{\sqrt{v_{i}^{(k)}}} - \sum_{i=i_{0}^{(k)}}^{J^{(k)}-j+1} \frac{v_{i}^{(k)}}{v^{(k)}} \frac{\epsilon_{ij}^{(k)}}{\sqrt{v_{i}^{(k)}}}\right) I_{i,j-1}^{(k)}}{\sum_{i=i_{0}^{(k)}}^{J^{(k)}-j+1} \frac{v_{i}^{(k)}}{v^{(k)}} \left(I_{i,j-1}^{(k)} - \sum_{i=i_{0}^{(k)}}^{J^{(k)}-j+1} \frac{v_{i}^{(k)}}{v^{(k)}} I_{i,j-1}^{(k)}\right) I_{i,j-1}^{(k)}},$$
$$\hat{\alpha}_{j}^{(k)} = \sum_{i=i_{0}^{(k)}}^{J^{(k)}-j+1} \frac{v_{i}^{(k)}}{v^{(k)}} \left(I_{ij}^{(k)} - \hat{\beta}_{j}^{(k)} I_{i,j-1}^{(k)}\right).$$
(8)

Moreover, if  $J^{(k)} - i_0^{(k)} - j \ge 1$ , then

$$(\hat{\sigma}_{j}^{(k)})^{2} = \frac{1}{J^{(k)} - i_{0}^{(k)} - j} (\mathbf{I}_{j}^{(k)} - \mathbf{A}_{j}^{(k)} \hat{\boldsymbol{\theta}}_{j}^{(k)})^{T} (\boldsymbol{\Sigma}_{j}^{(k)})^{-1} (\mathbf{I}_{j}^{(k)} - \mathbf{A}_{j}^{(k)} \hat{\boldsymbol{\theta}}_{j}^{(k)})$$
$$= \frac{1}{J^{(k)} - i_{0}^{(k)} - j} \sum_{i=i_{0}^{(k)}}^{J^{(k)} - j+1} v_{i}^{(k)} (I_{ij}^{(k)} - \hat{\alpha}_{j}^{(k)} - \hat{\beta}^{(k)} I_{i,j-1}^{(k)})^{2}$$
(9)

is an unbiased estimator of  $(\sigma_j^{(k)})^2$ .

From equation (5) we can construct predictors of  $\hat{I}_{ij}^{(k)}$  based on the observed data  $D_0$  for  $i + j > J^{(k)} + 1$ 

$$\hat{I}_{ij}^{(k)} = I_{i,J^{(k)}-i+1} \prod_{l=J^{(k)}-i+2}^{j} \hat{\beta}_{l}^{(k)} + \sum_{l=J^{(k)}-i+2}^{j} \hat{\alpha}_{l}^{(k)} \prod_{m=l+1}^{j} \hat{\beta}_{m}^{(k)}$$

Moreover, set

$$\hat{X}_{t}^{o,(k)} := \begin{cases} \sum_{u=t+1}^{J^{(k)}+1} v_{u}^{(k)} \hat{I}_{u,J^{(k)}-u+t+1}, & t = 1, ..., J^{(k)}, \\ 0 & t = J^{(k)} + 1, ..., T. \end{cases}$$

and  $\hat{X}^{o} := \sum_{k=1}^{K} \hat{X}^{o,(k)}$ .

#### 2.4 Chain-ladder

The chain-ladder method is one of the most popular actuarial loss reserving technique. The aim of the method is to estimate IBNR claims and to project ultimate loss amounts. The fact that the chain-ladder method is distribution-free and seems to work with almost no assumptions is the reason for its simplicity. We will now present some basic notation and results for the chain-ladder method by Mack, see [7].

Let  $C_{ij}$  denote the accumulated paid claims amount for accident year i,  $i_0 \leq i \leq J$ , and development year j,  $1 \leq j \leq J$ . We assume that we have an observation in the triangle if  $i + k \leq J + 1$ , which corresponds to the upper triangle. The idea behind chain-ladder is that there exists development factors  $f_1, \dots f_{I-1} > 0$  with

$$\mathbb{E}[C_{i,j+1}|C_{i1},...,C_{ij}] = C_{ij}f_j.$$
(10)

The chain-ladder method consists of estimating development factors  $f_{j}$  by

$$\hat{f}_j = \frac{\sum_{i=i_0}^{J-j} C_{i,j+1}}{\sum_{i=i_0}^{J-j} C_{ij}}, \qquad 1 \le j \le J-1,$$
(11)

and the ultimate claims amount  $C_{iJ}$  by

$$\hat{C}_{iJ} = C_{i,J+1-i}\hat{f}_{J+1-i}...\hat{f}_{J-1}.$$

Moreover, chain-ladder does not account for dependencies between accident years so one assumes that accident years are independent, i.e.

$$\{C_{i1}, ..., C_{iJ}\}, \{C_{k1}, ..., C_{kJ}\}, i \neq k, \text{are independent.}$$
 (12)

This assumption makes the estimator  $f_j$  an unbiased estimator of  $f_j$  which is a desirable property of a good estimator. The expressions (10) and (12) are two of the underlying assumptions for the chain-ladder method. The last assumption for Mack's chain-ladder is called the variance assumption, where the variance of  $C_{ij}$  is assumed to be equal to

$$Var(C_{i,j+1}|C_{i1},...,C_{ij}) = C_{ij}\lambda_j^2,$$
(13)

with unknown parameters  $\lambda_j^2$ . The estimate for  $\lambda_j^2$  is given by the following formula

$$\hat{\lambda}_{j}^{2} = \frac{1}{J-j-1} \sum_{i=1}^{J-j} C_{ij} \left( \frac{C_{i,j+1}}{C_{ij}} - \hat{f}_{j} \right)^{2}, \quad 1 \le j \le J-2,$$
(14)

where  $\hat{\lambda}_j^2$  is an unbiased estimator of  $\lambda_j^2$ . If the claims development is believed to be finished after J - 1 years we can put  $\hat{\lambda}_j^2 = 0$  otherwise  $\hat{\lambda}_j^2 = \min(\hat{\lambda}_{J-2}^4/\hat{\lambda}_{J-3}^2, \min(\hat{\lambda}_{J-3}^2, \hat{\sigma}_{J-2}^2)).$ 

#### 2.5 Chain-ladder time series model

Consider the following time series model for the accumulated payments

$$C_{i,j+1}^{(k)} = C_{ij}^{(k)} f_j^{(k)} + \lambda_j^{(k)} \sqrt{C_{ij}^{(k)}} \epsilon_{i,j+1}^{(k)},$$
(15)

where  $\epsilon_{i,j+1}^{(k)}$  are mutually independent and standard normally distributed. This model satisfies the underlying assumptions of Mack's chain-ladder model presented in the previous section. Thus, the conditional expectation and variance are equal to

$$\begin{split} \mathbb{E}[C_{i,j+1}^{(k)}|C_{i1}^{(k)},...,C_{ij}^{(k)}] &= C_{ij}^{(k)}f_j^{(k)}\\ \mathrm{Var}(C_{i,j+1}^{(k)}|C_{i1}^{(k)},...,C_{ij}^{(k)}) &= C_{ij}^{(k)}(\lambda_j^{(k)})^2, \end{split}$$

where the parameter estimates for  $f_j^{(k)}$  and  $\lambda_j^{(k)}$  are found by equation (11) respectively (14).

## 2.6 Stochastic model inspired by chain-ladder

We will in this section present a Gaussian stochastic model inspired by chain-ladder and furthermore derive the value of the aggregate outstanding liability. Note that we will use exactly the same notation as in section 2.3 but now for the following model

$$C_{i,j+1}^{(k)} = C_{ij}^{(k)} f_{j0}^{(k)} + \sigma_{j0}^{(k)} \epsilon_{i,j+1}^{(k)},$$
(16)

where the  $\epsilon_{i,j+1}^{(k)}$  are mutually independent and standard normally distributed. The conditional expectation and variance for this model equals

$$\mathbb{E}[C_{i,j+1}^{(k)}|C_{i1}^{(k)},...,C_{ij}^{(k)}] = C_{ij}^{(k)}f_{j0}^{(k)}$$
$$\operatorname{Var}(C_{i,j+1}^{(k)}|C_{i1}^{(k)},...,C_{ij}^{(k)}) = (\sigma_{j0}^{(k)})^{2}.$$

Hence, the variance assumption underlying the chain-ladder model introduced by Mack is not satisfied since we assume the same variance for all accident years. Note, that f and  $\sigma$  in (16) are not the same parameters as in the chain-ladder model and the autoregressive model on incremental payments. Therefore we have added an additional index '0'.

## 2.6.1 Value of the outstanding liability

The incremental payments  $Z_{ij}^{(k)}$  for this model is the difference between the accumulated payments for development period j and j-1 for accident year

i, i.e.  $C_{ij}^{(k)}-C_{i,j-1}^{(k)}.$  The expected value of the outstanding liability cash flow for future year t equals

$$\begin{split} \mathbb{E}[X_{t}^{o,(k)}|\mathcal{F}_{0}] &= \sum_{u=t+1}^{J^{(k)}} \mathbb{E}[Z_{u,J^{(k)}-u+t+1}^{(k)}|\mathcal{F}_{0}] \\ &= \sum_{u=t+1}^{J^{(k)}} \mathbb{E}[C_{u,J^{(k)}-u+t+1}^{(k)} - C_{u,J^{(k)}-u+t}^{(k)}|\mathcal{F}_{0}] \\ &= \sum_{u=t+1}^{J^{(k)}} \left( \mathbb{E}[C_{u,J^{(k)}-u+t+1}^{(k)}|C_{u,J^{(k)}-u+1}^{(k)}] \right) \\ &= \sum_{u=t+1}^{J^{(k)}} \left( C_{u,J^{(k)}-u+1}^{(k)} \prod_{j=J^{(k)}-u+1}^{J^{(k)}-u+1} f_{j0}^{(k)} \right) \\ &= C_{u,J^{(k)}-u+1}^{(k)} \prod_{j=J^{(k)}-u+1}^{J^{(k)}-u+1} f_{j0}^{(k)} \right) \end{split}$$

and the conditional variances in (6)

$$\operatorname{Var}(X_T^{U,(k)}|\mathcal{F}_t) = \sum_{i=t+2}^{J^{(k)}} \operatorname{Var}\left(\sum_{j=J^{(k)}-i+2+t}^{J^{(k)}} Z_{ij}^{(k)}|\mathcal{F}_t\right)$$
$$= \sum_{i=t+2}^{J^{(k)}} \operatorname{Var}\left(\sum_{j=J^{(k)}-i+2+t}^{J^{(k)}} (C_{ij}^{(k)} - C_{i,j-1}^{(k)})|\mathcal{F}_0\right)$$
$$= \sum_{i=t+2}^{J^{(k)}} \operatorname{Var}(C_{iJ^{(k)}}^{(k)}|\mathcal{F}_t)$$
$$= \sum_{i=t+2}^{J^{(k)}} \sum_{j=J^{(k)}-i+1+t}^{J^{(k)}-1} (\sigma_{j0}^{(k)})^2 \prod_{m=j+1}^{J^{(k)}-1} (f_{m0}^{(k)})^2.$$

Thus, under the assumption  $\text{Cov}(X_T^{U,(k)}, X_T^{U,(l)} | \mathcal{F}_t) = 0$  for all k < l and all t,

$$\sum_{t=1}^{T} \left( \operatorname{Var}(X_T^U | \mathcal{F}_{t-1}) - \operatorname{Var}(X_T^U | \mathcal{F}_t) \right)^{1/2}$$
$$= \sum_{t=1}^{T} \left( \sum_{k=1}^{K} \sum_{i=t}^{J^{(k)}-1} (\sigma_{i0}^{(k)})^2 \prod_{j=i+1}^{J^{(k)}} (f_{j0}^{(k)})^2 \right)^{1/2}.$$

Given suitable parameter estimators we may estimate  $\mathbb{E}[X_t^o|\mathcal{F}_0]$  by

$$\sum_{k=1}^{K} \sum_{u=t+1}^{J^{(k)}} \left( C_{u,J^{(k)}-u+1}^{(k)} \prod_{j=J^{(k)}-u+1}^{J^{(k)}-u+t} \hat{f}_{j0}^{(k)} - C_{u,J^{(k)}-u+1}^{(k)} \prod_{j=J^{(k)}-u+1}^{J^{(k)}-u+t-1} \hat{f}_{j0}^{(k)} \right)$$

and the sum of the conditional variances in (6) by

$$\sum_{t=1}^{T} \left( \sum_{k=1}^{K} \sum_{i=t}^{J^{(k)}-1} (\hat{\sigma}_{i0}^{(k)})^2 \prod_{j=i+1}^{J^{(k)}-1} (\hat{f}_{j0}^{(k)})^2 \right)^{1/2}.$$

#### 2.6.2 Estimators and predictors

The estimators for  $f_{j0}^{(k)}$  are found by weighted least squares based on the observed data. We then want to minimize the following expression for a fixed j by taking the derivative with respect to  $f_{j0}^{(k)}$ 

$$\sum_{i=i_0^{(k)}}^{J^{(k)}-i_0^{(k)}-j} \Big(\frac{C_{i,j+1}^{(k)}-C_{ij}^{(k)}f_{j0}^{(k)}}{\sigma_{j0}^{(k)}}\Big)^2.$$

By setting the derivative equal to zero and solving for  $f_{j0}^{(k)}$  we get that the minimizing parameter is

$$\hat{f}_{j0}^{(k)} = \frac{\sum_{i=i_0^{(k)}}^{J^{(k)}-i_0^{(k)}-j} C_{ij}^{(k)} C_{i,j+1}^{(k)}}{\sum_{i=i_0^{(k)}}^{J^{(k)}-i_0^{(k)}-j} (C_{ij}^{(k)})^2}$$

Moreover, if  $J^{(k)} - i_0^{(k)} - j \ge 1$ 

$$(\hat{\sigma}_{j0}^{(k)})^2 = \frac{1}{J^{(k)} - i_0^{(k)} - j} \sum_{i=i_0^{(k)}}^{J^{(k)} - i_0^{(k)} - j} (C_{i,j+1}^{(k)} - C_{ij}^{(k)} f_{j0}^{(k)})^2$$

is an unbiased estimator of  $(\sigma_{i0}^{(k)})^2$ .

### 2.7 Prediction of future solvency capital requirements

Using the work in [1], we will here derive an expression for the prediction of the solvency capital requirement at time t. This makes us able to compare the approximation of SCR(t) in the cost-of-capital formula against the prediction of SCR(t) according to the cost-of-capital margin approach. In contrast to the simplified risk margin (2), the solvency capital requirement for future year t is kept as a random variable in the derivation of the cost-of-capital margin  $V_0(X)$ . We have the following expression for the solvency capital requirement at time  $t \in \{0,...,T\}$ 

$$\begin{aligned} \operatorname{VaR}_{t,0.005}(-X_{t+1} - V_{t+1}(X)) &= \mathbb{E}\Big[\sum_{s=t+1}^{T} X_s | \mathcal{F}_t\Big] \\ &+ c \sum_{s=t+2}^{T} \Big(\operatorname{Var}\Big(\sum_{u=s}^{T} X_u | \mathcal{F}_{s-1}\Big) - \operatorname{Var}\Big(\sum_{u=s}^{T} X_u | \mathcal{F}_s\Big)\Big)^{1/2} \\ &+ \Phi^{-1}(0.995)\Big(\operatorname{Var}\Big(\sum_{s=t+1}^{T} X_s | \mathcal{F}_t\Big) - \operatorname{Var}\Big(\sum_{s=t+1}^{T} X_s | \mathcal{F}_{t+1}\Big)\Big)^{1/2}, \end{aligned}$$

where for our models we can simplify according to

$$\operatorname{VaR}_{t,0.005}(-X_{t+1} - V_{t+1}(X)) = \sum_{s=t+1}^{T} \left( \mathbb{E} \left[ X_s^o | \mathcal{F}_t \right] - \mathbb{E} \left[ X_s^o | \mathcal{F}_0 \right] \right) + c \sum_{s=t+2}^{T} \left( \operatorname{Var} \left( \sum_{u=s}^{T} X_u^o | \mathcal{F}_{s-1} \right) - \operatorname{Var} \left( \sum_{u=s}^{T} X_u^o | \mathcal{F}_s \right) \right)^{1/2} + \Phi^{-1}(0.995) \left( \operatorname{Var} \left( \sum_{s=t+1}^{T} X_s^o | \mathcal{F}_t \right) - \operatorname{Var} \left( \sum_{s=t+1}^{T} X_s^o | \mathcal{F}_{t+1} \right) \right)^{1/2}.$$

At time 0, we predict the solvency capital requirement at time t by its expected value given the information up to time 0, i.e. by

$$\begin{split} & \mathbb{E}[\operatorname{VaR}_{t,0.005}(-X_{t+1} - V_{t+1}(X))|\mathcal{F}_{0}] = \\ & + \mathbb{E}\Big[\sum_{s=t+1}^{T} \Big(\mathbb{E}\Big[X_{s}^{o}|\mathcal{F}_{t}\Big] - \mathbb{E}\Big[X_{s}^{o}|\mathcal{F}_{0}\Big]\Big)|\mathcal{F}_{0}\Big] \\ & + c\mathbb{E}\Big[\sum_{s=t+2}^{T} \Big(\operatorname{Var}\Big(\sum_{u=s}^{T} X_{u}^{o}|\mathcal{F}_{s-1}\Big) - \operatorname{Var}\Big(\sum_{u=s}^{T} X_{u}^{o}|\mathcal{F}_{s}\Big)\Big)^{1/2}|\mathcal{F}_{0}\Big] \\ & + \Phi^{-1}(0.995)\mathbb{E}\Big[\Big(\operatorname{Var}\Big(\sum_{s=t+1}^{T} X_{s}^{o}|\mathcal{F}_{t}\Big) - \operatorname{Var}\Big(\sum_{s=t+1}^{T} X_{s}^{o}|\mathcal{F}_{t+1}\Big)\Big)^{1/2}|\mathcal{F}_{0}\Big]. \end{split}$$

Note that the expected value of the first sum above, denoting the difference between the prediction of future cash flows at time t and at time 0, vanishes when we perform the prediction at time 0. The prediction of the future

solvency capital requirement at time t then reduces to

$$\begin{split} & \mathbb{E}[\operatorname{VaR}_{t,0.005}(-X_{t+1} - V_{t+1}(X))|\mathcal{F}_{0}] = \\ & c \mathbb{E}\Big[\sum_{s=t+2}^{T} \Big(\operatorname{Var}\Big(\sum_{u=s}^{T} X_{u}^{o}|\mathcal{F}_{s-1}\Big) - \operatorname{Var}\Big(\sum_{u=s}^{T} X_{u}^{o}|\mathcal{F}_{s}\Big)\Big)^{1/2}|\mathcal{F}_{0}\Big] \\ & + \Phi^{-1}(0.995) \mathbb{E}\Big[\Big(\operatorname{Var}\Big(\sum_{s=t+1}^{T} X_{s}^{o}|\mathcal{F}_{t}\Big) - \operatorname{Var}\Big(\sum_{s=t+1}^{T} X_{s}^{o}|\mathcal{F}_{t+1}\Big)\Big)^{1/2}|\mathcal{F}_{0}\Big]. \end{split}$$

For the stochastic model on incremental payments we get the following expression for the prediction

$$\mathbb{E}[\operatorname{VaR}_{t,0.005}(-X_{t+1} - V_{t+1}(X))] = c \sum_{s=t+2}^{T} \Big(\sum_{i=s+1}^{J^{(k)}} v_i^{(k)} (\sigma_{J^{(k)}-i+1+s}^{(k)})^2 \Big(1 + \sum_{j=J^{(k)}-i+2+s}^{J^{(k)}} \prod_{m=J^{(k)}-i+2+s}^{j} \beta_m^{(k)} \Big)^2 \Big)^{1/2} + \Phi^{-1}(0.995) \Big(\sum_{i=t+2}^{J^{(k)}} v_i^{(k)} (\sigma_{J^{-i+2+t}}^{(k)})^2 \Big(1 + \sum_{j=J^{(k)}-i+3+t}^{J^{(k)}} \prod_{m=J^{(k)}-i+3+t}^{j} \beta_m^{(k)} \Big)^2 \Big)^{1/2}$$

and for the stochastic model inspired by chain-ladder

$$\mathbb{E}[\operatorname{VaR}_{t,0.005}(-X_{t+1} - V_{t+1}(X))] = c \sum_{s=t+2}^{T} \left(\sum_{i=s}^{J^{(k)}-1} (\sigma_{i0}^{(k)})^2 \prod_{j=i+1}^{J^{(k)}-1} (f_{j0}^{(k)})^2 \right)^{1/2} + \Phi^{-1}(0.995) \left(\sum_{i=t+1}^{J^{(k)}-1} (\sigma_{i0}^{(k)})^2 \prod_{j=i+1}^{J^{(k)}-1} (f_{j0}^{(k)})^2 \right)^{1/2}.$$

The quantities are estimated by replacing the unknown parameter values by their estimates based on the upper triangle observed at time 0.

## 2.8 Balance sheet for the reference undertaking under Solvency II and under the cost-of-capital margin approach

The risk margin under Solvency II is calculated according to the cost-ofcapital formula (1), which equals the cost of holding capital meeting the solvency capital requirement for each year until runoff. Thus, the reference undertaking needs to have assets covering the best estimate, but it is also given additional capital, i.e. the risk margin. The risk margin then technically covers some of the capital requirement that the reference undertaking needs to meet at each future year t. This point of view is not considered in the Solvency II framework. Instead, one calculates the cost of holding the whole solvency capital requirement at each future year t without considering the additional capital. Consequently, there is a double counting. In the derivation of the cost-of-capital margin, one only considers capital costs for covering the difference between the capital requirement and the cost-of-capital margin at time t. Let

> $\mathcal{A}_t$  = assets at time t $\mathcal{L}_t$  = insurance liability at time t $RM_t$  = risk margin according to Solvency II at time t

and recall that  $C_t$  denotes the buffer capital that investors are asked to provide and  $V_t$  the cost-of-capital margin at time t. In figure 2.1, the balance sheet for the reference undertaking company according to each method for calculating the risk margin object is shown. There, the difference between the two approaches is clearly observable. Thus, if the two approaches is predicting the future solvency capital requirements equally, we can expect to get a higher value for the risk margin according to the simplified cost-ofcapital formula than when using the cost-of-capital margin approach. Since the simplified cost-of-capital formula is approximating SCR(t) we will in our data analysis study whether this approximation is good or not.



Figure 2.1: Balance sheet for the reference undertaking under the cost-ofcapital margin approach (left) and the cost-of-capital formula in Solvency II (right).

## 3 Method

In this section, we describe the underlying data and how the study was implemented in details. Here, the index k is dropped to avoid unnecessary heavy notation.

## 3.1 Data

Our study is based on loss triangle data which have the following structure,



where  $D_t$  denotes the known upper triangle at time t and consists of either incremental payments,  $I_{ij}$ , or accumulated payments,  $C_{ij}$ , depending on the underlying model.

### 3.2 Model check

When selecting a model to assign a value to the outstanding insurance liability, one should check whether the underlying model assumptions are fulfilled by the data. Methods to assure whether the underlying chain-ladder assumptions are considered to be met are described by Mack in [8]. For the other stochastic models presented in the theory section, the residuals should be distributed according to a standard normal distribution. If it appears that a model is not satisfied by the data, another model should be used to get the best prediction of the outstanding liability.

### 3.3 Simulation study

The overall aim in this thesis is to compare the different approaches for assigning a value to the risk margin in the balance sheet, i.e. when using the cost-of-capital margin formula and the approximation of the cost-ofcapital formula in Solvency II. We also want to say something about whether the choice of underlying model have any impact on the calculations of risk margin. To answer these questions, we will perform a simulation study where we consider the following three models

a) 
$$I_{ij} = \alpha_j + \beta_j I_{i,j-1} + \frac{\sigma_j}{\sqrt{v_i}} \epsilon_{ij}$$
  
b)  $C_{ij} = C_{i,j-1} f_{j-1} + \lambda_{j-1} \sqrt{C_{ij}} \epsilon_{ij}$   
c)  $C_{ij} = C_{i,j-1} f_{j-1,0} + \sigma_{j-1,0} \epsilon_{ij}$ .

From now on we will call these models by a), b) and c) for easier notation. The following algorithm describes how the simulation study is implemented step by step.

## Simulation algorithm

- 1) Estimate model parameters for the models a), b) and c) based on an observed loss triangle, i.e.  $\hat{\alpha}_j$ ,  $\hat{\beta}_j$ ,  $\hat{\sigma}_j$ ,  $\hat{f}_j$ ,  $\hat{\lambda}_j$ ,  $\hat{f}_{j0}$  and  $\hat{\sigma}_{j0}$ .
- 2) Simulate a fully developed runoff triangle by model a), using the parameter estimates calculated in the previous step and by simulating  $\epsilon_{ij}$  from a standard normal distribution according to the model.
- 3) Now, for the simulated runoff triangle in 2), consider the lower triangle as unknown and calculate new parameter estimates for each of the three models a), b) and c) based on the upper triangle.
- 4) Predict the lower triangle based on the new parameter estimates in3). Thereafter we are able to calculate:
  - The prediction error  $R \hat{R}$ , where R denotes the sum of the simulated lower triangle in 3) and  $\hat{R}$  denotes the sum of the predicted lower triangle.
  - The cost-of-capital margin,  $V_0$ , and the approximated risk margin, RM, in Solvency II for a chosen rate  $\eta_0 = CoC$ , based on the new parameter estimates in 3). We are also able to predict  $V_t$ , RM<sub>t</sub> and SCR<sub>t</sub> using these estimates. In the costof-capital margin approach, the latter is predicted according to the expressions derived in section 2.7 for model a) and c) respectively. For the approximated risk margin in Solvency II this is predicted by multiplying SCR(0) by the ratio of best estimate at each time point.
- 5) Repeat 2)-4) an arbitrary number of times. We then get the empirical distributions for the parameter estimators, the prediction error,

the cost-of-capital margin, the risk margin and the solvency capital requirements when simulating underlying data from model a).

6) Repeat 2)-5) but now for model b) and c) instead of a). To simulate a new fully developed runoff triangle in 2) for these two models, we will simulate the first column from N(â<sub>1</sub>, ô<sub>1</sub>) as for model a).

The outcome of this algorithm produce empirical distributions of the prediction error for each of the three models when varying the underlying model for simulating loss triangle data. This makes us able to say something about how well each model can predict the outstanding liability cash flow when the loss triangle is simulated by model a), b) and c). Here, it is desirable to look for a model that is not that much effected by how the underlying triangle data is distributed. Furthermore, the simulation study will give us the empirical distributions for the cost-of-capital margin and the risk margin in Solvency II. This makes us able to compare the two methods for assigning a value to the risk margin in the balance sheet. We are also able to compare the two approaches when simulating loss triangle data from different underlying models, i.e. model a) and c). Note that model b) is not a Gaussian model and we are therefore not able to calculate the cost-of-capital margin for it. Moreover, we can compare the two prediction methods for assigning a value to the future solvency capital requirement, i.e. according to the proposed approximation and when it is seen as a random variable.

## 4 Results

In this section, we present results that will lay the foundation for answering the main questions in this thesis. Our study is first be based on loss triangle data that is used in the literature, e.g. by Mack in [7] and by Verrall in [11]. Thereafter, we will present results based on data from six lines of businesses in a Swedish insurance company. Throughout this thesis, all results will be visualised according to a cost-of-capital rate of 6% if nothing else is stated.

### 4.1 Simulation study for data used in the literature

We will here present some results after running the simulation algorithm described in section 3.3 based on the runoff triangle used by Mack in [7]. This triangle data is similar from year to year and we can therefore set the weights  $v_i = 1$  for the stochastic model on incremental payments. The weights are mainly intended to adjust for a large variety of the claims amount from year to year, which primarily depends on the size of the business for each year. Table 4.1 shows the calculated parameter estimates for each of the three models a), b) and c) based on our loss triangle data. Before running the simulation algorithm we calculate the residuals,  $\epsilon_{ij}$ , for each

model with the parameter estimates below. These are found in figure A.1 in the Appendix and should be distributed according to a standard normal distribution.

After running the simulation algorithm 10.000 times we plot the empirical distributions of the prediction error,  $R - \hat{R}$ , for each model. These results are found in figure 4.1, where the first row shows the distribution of the prediction error for each model when simulating the underlying loss triangle data by model a). Similarly, the second row shows the corresponding distributions when simulating the underlying loss triangle data with model b) and the third row when simulating with model c). The empirical expected values and standard deviations of the prediction errors are found in table 4.2.

	Parameter estimates											
j	$\hat{lpha}_j$	$\hat{eta}_j$	$\hat{\sigma}_j$	$\hat{f}_j$	$\hat{\lambda}_j$	$\hat{f}_{j0}$	$\hat{\sigma}_{j0}$					
1	367140	0.0	47027	3.5	400	3.4	226558					
2	1564926	-1.7	96027	1.7	194	1.7	209652					
3	551694	0.4	225279	1.5	205	1.5	273774					
4	503533	0.5	307676	1.2	123	1.2	204716					
5	786802	-0.3	132968	1.1	117	1.1	187558					
6	555288	-0.4	133138	1.1	90	1.1	156684					
$\overline{7}$	648955	-0.9	105154	1.1	21	1.1	33401					
8	88360	0.4	10518	1.1	34	1.1	48285					

0.0

0.0

1.0

\_

21

\_

1.0

\_

0

\_

9

10

7897

67948

1.6

0.0

Table 4.1: Parameter estimates for each of the three models a), b) and c).

In each simulation we have also calculated the the cost-of-capital margin,  $V_0$ , and the approximated risk margin, RM, in Solvency II. Thus, we are able to plot the empirical distributions of these objects when simulating underlying loss triangle data from model a), b) and c), respectively. The results for the two prediction models a) and c) are found in figure A.2 and A.3 in the Appendix. Moreover, table A.1 summarize the empirical expected values and standard deviations of the parameter estimators that are used when calculating the risk margin objects. There, one can see the effect on the estimates when varying the underlying model for simulating loss triangle data. Note that the parameter estimates for model a) varies more than for model c) when the underlying data is simulated from another model than itself. Particularly, one can see that the standard deviation for both  $\hat{\beta}_j$  and  $\hat{\sigma}_j$  is larger when data is simulated from model b) and c) than for model a),



Figure 4.1: Empirical distributions of the prediction error for each model when varying the underlying model for simulating loss triangle data. The predicted values of the outstanding liability cash flow  $\hat{R}$  equals 16.6, 18.7 and 18.5 (in millions) for the models a), b) and c) respectively.

(Millions)										
Simulation	Expected value &		Prediction model							
model	standard deviation	Model a)	Model b)	Model c)						
Model a)	$\mathbb{E}[R-\hat{R}]$	-0.018	0.051	-0.238						
Model a)	$D(R - \hat{R})$	1.306	1.862	1.839						
Model b)	$\mathbb{E}[R-\hat{R}]$	-0.048	0.008	0.008						
Model D)	$D(R-\hat{R})$	28.812	2.366	2.388						
Model c)	$\mathbb{E}[R-\hat{R}]$	-0.129	0.022	0.020						
woder c)	$D(R - \hat{R})$	13.548	2.043	2.029						

Table 4.2: Table over the corresponding expected values and standard deviations of the simulated prediction errors shown in figure 4.1.

Expected values and standard deviations of the prediction errors

which explains why the empirical distributions for the risk margin objects are more skewed for these two models. As for model a), the parameter estimates for model c), i.e.  $\hat{f}_{j0}$  and  $\hat{\sigma}_{j0}$ , has the smallest standard deviation when the underlying data is simulated by the model itself. However, the difference among the estimates is not as large as for model a) and therefore, the empirical distributions for the risk margin objects resemble each other to a greater extent for this model.

The "true" values of the risk margin objects for each of the two models, together with the best estimate, are shown in table 4.3 for different values of the cost-of-capital rate. We call these the "true" values since they are based on the parameter estimates in table 4.1, which in turn are based on the observed triangle data. A bar chart of the total outstanding liability for each approach can be seen in figure 4.2 where we let

 $A = BE + V_0 \text{ for model a})$  B = BE + RM for model a)  $C = BE + V_0 \text{ for model c})$ D = BE + RM for model c).

Furthermore, figure 4.3 shows the predicted values of the cost-of-capital margin and the risk margin in Solvency II at time t, i.e.  $V_t$  and  $RM_t$ . The corresponding predicted solvency capital requirements at time t is seen in figure 4.4, where we denote the prediction of SCR(t) according to the expressions derived in section 2.7 by  $\widetilde{SCR(t)}$ .

Values of risk margin objects and the best estimate											
(Thousands)											
Rate	Μ	Μ	Model c)								
$\eta_0 = CoC$	BE	$V_0$	RM	BE	$V_0$	RM					
3%	16661.7	149.4	119.5	18479.5	266.5	297.8					
6%	16661.7	293.4	258.0	18479.5	523.3	626.3					
9%	16661.7	18479.5	766.0	982.9							

Table 4.3: Assigned values of the risk margin objects and the best estimate based on the parameter estimates in table 4.1.



Figure 4.2: Bar chart showing the calculated values of the total outstanding insurance liability for each method.



Figure 4.3: Predicted values of the risk margin objects at time t for model a) to the left and model c) to the right.



Figure 4.4: Predicted solvency capital requirements at time t for model a) to the left and model c) to the right, where  $w_t$  denotes the best estimate ratio, i.e. BE(t)/BE(0).

The proxy for future solvency capital requirements that is used in the costof-capital formula (2) is turning SCR(t) into a deterministic value. In the new more consistent approach for calculating a risk margin, SCR(t) is kept as a random variable as it actually is. Thus, the predicted best estimate ratio should equal the predicted value of the solvency capital requirement at time t divided by the value of the solvency capital requirement at time 0, i.e.

$$\frac{BE(t)}{BE(0)} = \frac{\widetilde{SCR}(t)}{SCR(0)},$$

where all predictions are based on the information at time 0. Table 4.4 shows the calculated weights where we let  $w_t^{SCR}$  denote the "true" weights  $\widetilde{\frac{SCR(t)}{SCR(0)}}$  and  $w_t^{BE}$  denote the proposed proportionality weights  $\frac{BE(t)}{BE(0)}$ .

	"True" weights vs.									
best estimate ratio in Solvency II										
	Mod	el a)	Mod	el c)						
t	$w_t^{SCR}$	$w_t^{BE}$	$w_t^{SCR}$	$w_t^{BE}$						
0	1.000	1.000	1.000	1.000						
1	0.911	0.702	0.645	0.719						
2	0.675	0.478	0.490	0.496						
3	0.528	0.307	0.332	0.329						
4	0.411	0.197	0.237	0.216						
5	0.357	0.120	0.146	0.133						
6	0.047	0.064	0.047	0.069						
7	0.000	0.028	0.036	0.029						
8	0.000	0.004	0.000	0.005						

Table 4.4: Table over "true" weights, i.e. the weights when the future solvency capital requirement is a random variable, compared to the weights used in the proposed approximation in Solvency II.

## 4.2 Results using data from a Swedish insurance company

We will now consider data from six lines of businesses in a Swedish non-life insurance company, which we from now on will denote by  $\text{LoB}_k$ , for  $k \in \{1, ..., 6\}$ . Here, we will mainly focus on comparing the different approaches for assigning a value to the risk margin but also to predict the future solvency capital requirements. Thus, it will be enough to look at these different calculated objects as a complement to the simulation study performed in the previous section.

Given the loss triangles for each line of business, we compute parameter estimates for each of the two prediction models. Thereafter, we are able to calculate the best estimate and assign values to the cost-of-capital margin and the risk margin in Solvency II, according to each prediction model. These values are found in table 4.5, where the cost-of-capital rate equals 6%. We have also performed the same calculations for a cost-of-capital rate of 3% and 9% respectively, these results are found in table A.2 in the Appendix. Moreover, figure 4.5 shows a bar chart over the calculated value of the total outstanding insurance liability for each line of business. Note here that the risk margin objects are all small compared to the best estimate. Furthermore, the predicted values of the risk margin objects and the corresponding solvency capital requirements at time t, is seen in figure 4.6 and 4.7 for each line of business. We have also summarized the values of the predicted best estimate ratios and the "true" weights in table A.3, A.4 and A.5 in the Appendix. Recall that we denote the "true" weights by  $w_t^{SCR} = \frac{\widetilde{SCR}(t)}{SCR(0)}$  and the best estimate ratios by  $w_t^{BE} = \frac{BE(t)}{BE(0)}$ . Note also that when studying figure 4.7 and the calculated weights in table A.3, A.4 and A.5, the proxy of future solvency capital requirements gives a better approximation when most of the liability cash flow is paid in the beginning of the runoff period, e.g for LoB<sub>1</sub> and LoB<sub>2</sub>, than when the liability cash flow declines slower, e.g for LoB<sub>3</sub> and LoB<sub>6</sub>. This is also reflected in figure 4.6 showing the value of the risk margin objects at time t.

Valu	Values of risk margin objects and the best estimate										
(Thousands)											
$CoC = \eta_0 = 6\%$											
	Ν	fodel a)	]	Model c)							
	BE	$V_0$	RM	BE	$V_0$	RM					
$LoB_1$	5644.5	201.2	201.5	5305.5	184.7	198.2					
$LoB_2$	20685.3	526.7	564.4	17086.3	367.1	402.6					
$LoB_3$	9128.4	652.1	685.4	6717.4	681.1	842.1					
$LoB_4$	28887.8	535.2	519.7	29092.1	508.9	481.2					
$LoB_5$	54192.2	1112.6	1432.2	45886.3	1194.3	1684.9					
$LoB_6$	130025.9	4315.4	3579.1	99023.0	2543.1	1712.1					

Table 4.5: Assigned values of the risk margin objects and the best estimate.



Figure 4.5: Bar chart showing the calculated values of the total outstanding insurance liability for each method.





 $\mathrm{LoB}_1$ 

























Figure 4.6: Predicted values of risk margin objects at time t for each line of business, using model a) to the left and model c) to the right.













 $LoB_3$ 

















Figure 4.7: Predicted solvency capital requirements at time t for each line of business, using model a) to the left and model c) to the right and where  $w_t$  denotes the best estimate ratio, i.e. BE(t)/BE(0).

### 4.3 Change in runoff pattern

From what we have seen until now, the assigned values of the risk margin objects has been small compared to the best estimate. Since the risk margin account for the risk, this imply that the underlying data is good and not so volatile for our insurance products. Therefore, we will here consider a scenario where the runoff pattern is changed for some of the latter accident years but the corresponding predicted ultimate claims amount is the same. For this scenario we will only apply model a) and consider  $LoB_1$ ,  $LoB_3$  and  $LoB_6$  from the previous section. A description of how this scenario is set up for each line of business is found in section A.6 in the Appendix.

As before, we have assigned values to the risk margin objects based on the parameter estimates that is obtained from the new loss triangles. The results for each line of business together with the best estimate is found in table 4.6. Furthermore, figure 4.8 shows a bar chart over the total outstanding liability for each line of business both for the original data and our considered scenario. Moreover, figure 4.9 and 4.10 shows the predicted values of the risk margin objects and the solvency capital requirements at time t.

Note that the risk margin objects have all been assigned higher values in our considered scenario than for the original data. A higher value of the risk margin is associated with a higher concentration of risk, i.e. the prediction of the future liability cash flow is more volatile. Overall, looking at figure 4.10, the approximation of future solvency capital requirements underestimate the requirement compared to the more correct valuation in the beginning of the runoff period. Moreover, the proxy is less appropriate for the considered scenarios, particularly for LoB<sub>3</sub>. This is also reflected in figure 4.9.

Values of risk margin objects and the best estimate (Thousands)										
$LoB_k$	$LoB_k$ Data $BE$ $V_0$ $RM$									
L <sub>o</sub> D	Observed	E644 E	201.2	201.5						
$LOD_1$	Scenario	3044.0	823.2	826.8						
LoP	Observed	0128 /	652.1	685.4						
$L0D_3$	Scenario	9120.4	1178.2	823.9						
LoD	Observed	120025-0	4315.4	3579.1						
LOD <sub>6</sub>	Scenario	150025.9	8248.6	7296.9						

Table 4.6: Best estimate together with the assigned values of the risk margin objects for the observed data and the considered scenario where the runoff pattern is changed.



Figure 4.8: Bar chart showing the calculated values of the total outstanding insurance liability for the observed loss triangle to the left and for the new runoff scenario to the right.

# Values of risk margin objects at time t (Thousands)



 $\mathrm{LoB}_1$ 









Figure 4.9: Predicted values of risk margin objects at time t for the observed loss triangle to the left and for the new runoff scenario to the right.





 $\mathrm{LoB}_1$ 









Figure 4.10: Predicted solvency capital requirements at time t where  $w_t$ denotes the best estimate ratio, i.e. BE(t)/BE(0). The results for the observed loss triangle is shown to the left and for the new runoff scenario to the right.

## 5 Discussion

The aim of this thesis was to assign a value to the cost-of-capital margin, i.e. the value of the residual cash flow, by using loss triangle data. For comparison, we have focused on the risk margin in the Solvency II framework which is calculated according to the cost-of-capital formula. A common way to perform these calculations in the industry, is by using a proposed approximation technique in the framework for predicting the future solvency capital requirement. Therefore, this proxy is considered throughout the thesis. Moreover, we have considered two different stochastic models in assigning values to these risk margin objects.

First of all, studying the empirical distributions of the prediction error for the stochastic model on incremental payments (model a)) and the stochastic model inspired by chain-ladder (model c)), we observed that each model had the smallest prediction error when the underlying data was simulated by the model itself, which comes naturally. However, model a) had a significantly larger standard deviation for the prediction error than model c) when varying the underlying model for simulating loss triangle data. Thus, model c) is less sensitive according to how the underlying triangle data is distributed in predicting the outstanding liability cash flow.

When comparing the assigned values of the cost-of-capital margin,  $V_0$ , and the risk margin in Solvency II, RM, by the proposed approximation, we could see that the relation between  $V_0$  and RM was similar for both model a) and c). More specifically, if  $V_0$  was smaller than RM for model a) it was the same for model c) and the other way around. Although one exception was found when we used Mack's data in section 4.1. One thing worth mentioning is that we only had one fully developed accident year for this loss triangle, which could be a reason for the difference between the model predictions. For the insurance data considered in section 4.2 several fully developed accident years was included in the loss triangle data for the different lines of businesses. Moreover, RM was larger than  $V_0$  for all lines of businesses, except for  $LoB_4$  and  $LoB_6$ . This is explained by the predicted solvency capital requirements at time t, which was strictly smaller for the approximation than for the stochastic solvency capital requirement. Henceforth, when comparing the predicted values of the risk margin objects at time t, i.e.  $V_t$  and  $RM_t$ , the risk margin in Solvency II may overestimate as well as underestimate the value of the risk compared to the more correct valuation procedure. The same goes for the prediction of the future solvency capital requirement.

The proportionality weight proposed by EIOPA assumes that the future solvency capital requirement is decreasing with the ratio of the best estimate.

By analyzing the results of the predicted future solvency capital requirements according to the different approaches, we could see that the proxy was performing well when most of the liability cash flow was expected to be paid in the beginning of the runoff period, e.g for  $LoB_1$  and  $LoB_2$ . The contrary was observed when a larger amount of the liability was expected to be paid in future years, e.g for  $LoB_3$  and  $LoB_6$ . Therefore, we also considered a scenario where the runoff pattern was changed for some of the latter accident years. The aim with this procedure was to see if the value of the risk margin objects and the predictions of the future solvency capital requirements would be different. This was done for LoB<sub>1</sub>, LoB<sub>3</sub> and LoB<sub>6</sub> using model a) and resulted in larger solvency capital requirements for future years and hence also larger values for the risk margin objects for all three lines of businesses. This result is natural since the loss triangle data varies more between the accident years for this scenario. Consequently, the prediction of the outstanding liability cash flow is more volatile which induce a larger risk. Thus, the value of the risk margin should be higher. We could also observe that the difference between the predictions of future solvency capital requirement was larger among the two methods for the considered scenario for each line of business. Thus, the approximation of the future solvency capital requirement seems to be less appropriate for insurance products that is more volatile, i.e. where the best estimate is more difficult to predict. Although the total outstanding insurance liability does not seem to be very different according to the two approaches, our results give an indication of how the approximation behaves in various situations.

The final choice of a prediction model should be based on studying the residuals after fitting a model. In our case, these were all distributed according to a standard normal distribution as they should be. As a complement, fully developed accident years can be used to perform back-testing. This makes it possible to validate the models against real data where the model that predicts the outstanding liability cash flow the best should be used.

#### 5.1 Future work

To valuate the aggregate outstanding liability for an insurance company containing several lines of businesses, dependence between loss triangles must be handled. In our work, the covariance between the value of the residual cash flows for the different lines of businesses has been assumed to equal zero. For this case, we only have to calculate the sum of the best estimates and cost-of-capital margins for each line of business to get the aggregate value of the insurance liability cash flow. This assumption is unrealistic and future work may include a closer look how to handle this. If independence between accident years is assumed, one could find bounds for the correlation between the  $\sum_{t=1}^{T} X_t^{o,(k)}, \ k \in \mathcal{K}$ .

## References

- Engsner, H., Lindholm, M. & Lindskog, F. (2017) Insurance valuation: a computable multi-period cost-of-capital approach. Insurance Mathematics and Economics 72, 250-264.
- [2] European Commission (2015) Delegated Regulation (EU) 2015/35 of 10 October 2014. Official Journal of the European Union.
- [3] EIOPA (2015) Guidelines on the valuation of the technical provisions. Official Publication of EIOPA.
- [4] IASB (2013) Exposure Draft ED/2013/7 Insurance contracts. Available online at: http://www.ifrs.org/Current-Projects/IASB-Projects/ Insurance-Contracts/Exposure-Draft-June-2013/Documents/ED-Insurance-Contracts-June-2013.pdf
- [5] Lindholm, M., Lindskog, F. & Wahl, F. (2017) Valuation of non-life insurance liabilities from loss triangles. Research Reports in Mathematical Statistics, Research Report 2017:5, Stockholm University.
- [6] Ohlsson, E. & Lauzeningks, J. (2008) The one-year non-life insurance risk. Conference paper, ASTIN Colloquium 2008, Manchester.
- [7] Mack, T. (1993) Distribution-free calculation of the standard error of chain ladder reserve estimates. ASTIN Bulletin, Vol. 23, No. 2.
- [8] Mack, T. (1993) Measuring the Variability of Chain Ladder Reserve Estimates. CAS Prize Paper Competition. Available online at: https://www.casact.org/pubs/forum/94spforum/94spf101.pdf
- [9] Merz, M. & Wüthrich, M.V. (2008) Modelling the claims development result for solvency purposes. CAS E-Forum, 542-568.
- [10] Möhr, C. (2011) Market-consistent valuation of insurance liabilities by cost of capital. ASTIN Bulletin, 41, 315-341.
- [11] Verall, R.J. (1990) Bayes and empirical Bayes estimation for the chain ladder model. ASTIN Bulletin, 20, 217-243.
- [12] Wüthrich, M.V., Embrechts, P. & Tsanakas, A. (2011) Risk margin for a non-life insurance runoff. Statistics & Risk Modeling 28, 299-317.

## A Appendix

## A.1 Residuals



Residuals

Figure A.1: Residuals calculated for each model with the parameter estimates in table 4.1. The dotted line in red marks the standard deviation equal to one.





Figure A.2: Empirical distributions of risk margin objects (in thousands) with model a) as prediction model and where  $\eta_0 = CoC = 6\%$ . The underlying data is simulated using model a) in the upper two figures, model b) for the two figures in the middle and model c) in the two bottom figures.



Figure A.3: Empirical distributions of risk margin objects (in thousands) with model c) as prediction model and where  $\eta_0 = CoC = 6\%$ . The underlying data is simulated using model a) in the upper two figures, model b) for the two figures in the middle and model c) in the two bottom figures.

## A.3 Distribution of parameter estimates

Table A.1: Expected values and standard deviations of the parameter estimates that is used for calculating  $V_0$  and RM, when  $CoC = \eta_0 = 6\%$ .

	Distribution of parameter estimates											
			Simula	ating from	m model	l a)						
j	$\mathbb{E}[\hat{eta}_j]$	$D(\hat{eta}_j)$	$\mathbb{E}[\hat{\sigma}_j]$	$D(\hat{\sigma}_j)$	$\mathbb{E}[\hat{f}_{j0}]$	$D(\hat{f}_{j0})$	$\mathbb{E}[\hat{\sigma}_{j0}]$	$D(\hat{\sigma}_{j0})$				
1	0.0	0.0	43507	10289	3.5	0.197	202389	50852				
2	-1.7	0.83	81614	22247	1.7	0.062	205927	55820				
3	0.4	0.79	187861	54952	1.5	0.051	276417	81222				
4	0.5	0.66	247114	80078	1.2	0.026	178920	57900				
5	-0.3	0.23	101444	36699	1.1	0.018	125065	45196				
6	-0.4	0.59	95444	40074	1.1	0.021	143567	60675				
7	-0.9	0.69	65527	34205	1.0	0.008	45385	23949				
8	0.3	0.14	4869	3655	1.1	0.014	52921	40178				
9	1.6	0.00	0	0	1.1	0.001	0	0				
10	0.0	0.00	0	0	-	-	-	-				
			Simula	ating from	m model	b)						
j	$\mathbb{E}[\hat{\beta}_j]$	$D(\hat{\beta}_j)$	$\mathbb{E}[\hat{\sigma}_j]$	$D(\hat{\sigma}_j)$	$\mathbb{E}[\hat{f}_{j0}]$	$D(\hat{f}_{j0})$	$\mathbb{E}[\hat{\sigma}_{j0}]$	$D(\hat{\sigma}_{j0})$				
1	0.0	0.00	43346	10401	3.5	0.221	220956	52376				
2	2.5	2.11	206154	56373	1.7	0.063	197823	50924				
3	0.8	0.37	184916	56161	1.5	0.053	268369	71760				
4	0.8	0.52	254956	84513	1.2	0.028	190410	57887				
5	0.3	0.35	178671	65614	1.1	0.028	189738	57536				
6	0.3	0.64	1170921	73511	1.1	0.023	145746	52241				
7	0.2	0.76	126257	67727	1.1	0.006	31687	12604				
8	0.2	0.73	2649	24923	1.1	0.011	40868	20732				
9	1.7	73.40	0	0	1.0	0.001	0	0				
10	0.0	0.00	0	0	-	-	-	-				
			Simula	ating fro	m mode	l c)						
j	$\mathbb{E}[\hat{\beta}_i]$	$D(\hat{\beta}_i)$	$\mathbb{E}[\hat{\sigma}_i]$	$D(\hat{\sigma}_i)$	$\mathbb{E}[\hat{f}_{i0}]$	$D(\hat{f}_{i0})$	$\mathbb{E}[\hat{\sigma}_{i0}]$	$D(\hat{\sigma}_{i0})$				
1	0.0	0.0	43352	10271	3.4	0.203	207017	52534				
2	2.4	1.93	193183	52368	1.7	0.058	189232	51432				
3	0.8	0.37	176296	51114	1.5	0.047	241927	71167				
4	0.8	0.48	231313	74811	1.2	0.026	178209	58185				
5	0.3	0.34	165104	60058	1.1	0.022	157958	57310				
6	0.3	0.59	141588	60515	1.1	0.019	124718	52879				
7	0.3	0.98	109498	56466	1.1	0.004	24078	12623				
8	0.2	0.85	28222	21298	1.1	0.008	27386	20906				
9	1.0	43.13	0	0	0.0	0.000	0	0				
10	0.0	0.00	0	0	-	-	-	-				

# A.4 Values of risk margin objects for different cost-of-capital rates

Table A.2: Assigned values of risk margin objects together with the corresponding best estimate for different values of the cost-of-capital rate.

Values of risk margin objects and the best estimate											
(Thousands)											
$CoC = \eta_0 = 3\%$											
Model a) Model c)											
	BE	$V_0$	RM	BE	$V_0$	RM					
$LoB_1$	5644.5	102.5	99.2	5305.5	94.0	97.8					
$\mathrm{LoB}_2$	20685.3	268.2	279.1	17086.3	187.0	199.3					
$\mathrm{LoB}_3$	9128.4	332.1	324.7	6717.4	346.9	403.0					
$\mathrm{LoB}_4$	28887.8	272.5	256.2	29092.1	259.2	236.8					
$LoB_5$	54192.2	566.6	712.2	45886.3	608.2	839.4					
$\mathrm{LoB}_6$	130025.9	2197.6	1661.7	99023.0	1295.1	783.5					
$CoC = \eta_0 = 6\%$											
Model a) Model c)											
	BE	$V_0$	RM	BE	$V_0$	RM					
$LoB_1$	5644.5	201.2	201.5	5305.5	184.7	198.2					
$\mathrm{LoB}_2$	20685.3	526.7	564.4	17086.3	367.1	402.6					
$\mathrm{LoB}_3$	9128.4	652.1	685.4	6717.4	681.1	842.1					
$\mathrm{LoB}_4$	28887.8	535.2	519.7	29092.1	508.9	481.2					
$LoB_5$	54192.2	1112.6	1432.2	45886.3	1194.3	1684.9					
$\mathrm{LoB}_6$	130025.9	4315.4	3579.1	99023.0	2543.1	1712.1					
		Co	$C = \eta_0 =$	9%							
	Ν	fodel a)		]	Model c)						
	BE	$V_0$	RM	BE	$V_0$	RM					
$LoB_1$	5644.5	294.6	306.7	5305.5	270.3	300.9					
$\mathrm{LoB}_2$	20685.3	771.0	855.2	17086.3	537.4	609.7					
$\mathrm{LoB}_3$	9128.4	954.6	1079.2	6717.4	997.0	1314.2					
$\mathrm{LoB}_4$	28887.8	783.4	790.0	29092.1	744.9	732.4					
$LoB_5$	54192.2	1628.6	2159.4	45886.3	1748.2	2535.7					
$LoB_6$	130025.9	6316.6	5731.4	99023.0	3722.4	2773.8					

## A.5 "True" weights vs. best estimate ratio in Solvency II

Table A.3: Table over "true" weights, i.e. the weights when the future solvency capital requirement is a random variable, compared to the weights used in the proposed approximation in Solvency II for  $LoB_1$  and  $LoB_2$ .

	"True" weights vs.											
best estimate ratio in Solvency II												
		Lo	$B_1$			Lo	$B_2$					
	Model a)		Mod	el c)	Mode	el a)	Mod	el c)				
t	$w_t^{SCR}$	$w_t^{BE}$	$w_t^{SCR}$	$w_t^{BE}$	$w_t^{SCR}$	$w_t^{BE}$	$w_t^{SCR}$	$w_t^{BE}$				
0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000				
1	0.277	0.255	0.257	0.277	0.242	0.234	0.226	0.234				
2	0.120	0.105	0.106	0.120	0.091	0.088	0.079	0.089				
3	0.086	0.047	0.048	0.086	0.039	0.035	0.039	0.035				
4	0.048	0.018	0.018	0.048	0.018	0.012	0.017	0.012				
5	0.036	0.009	0.009	0.036	0.011	0.005	0.008	0.005				
6	0.025	0.004	0.004	0.025	0.008	0.002	0.007	0.002				
7	0.021	0.002	0.002	0.021	0.005	0.001	0.005	0.001				
8	0.006	0.000	0.000	0.006	0.003	0.000	0.004	0.000				

Table A.4: Table over "true" weights, i.e. the weights when the future solvency capital requirement is a random variable, compared to the weights used in the proposed approximation in Solvency II for  $LoB_3$  and  $LoB_4$ .

	"True" weights vs.												
best estimate ratio in Solvency II													
		Lo	$B_3$			Lo	$B_4$						
	Mod	el a)	Mod	el c)	Mod	lel a)	Mod	el c)					
t	$w_t^{SCR}$	$w_t^{BE}$	$w_t^{SCR}$	$w_t^{BE}$	$w_t^{SCR}$	$w_t^{BE}$	$w_t^{SCR}$	$w_t^{BE}$					
0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000					
1	0.488	0.633	0.459	0.662	0.261	0.184	0.291	0.189					
2	0.406	0.429	0.384	0.453	0.094	0.077	0.113	0.078					
3	0.339	0.274	0.296	0.294	0.080	0.040	0.093	0.041					
4	0.283	0.167	0.171	0.182	0.039	0.021	0.036	0.022					
5	0.238	0.108	0.145	0.125	0.025	0.012	0.022	0.013					
6	0.202	0.070	0.134	0.066	0.019	0.007	0.019	0.008					
7	0.112	0.033	0.060	0.038	0.016	0.004	0.015	0.004					
8	0.060	0.021	0.049	0.019	0.014	0.002	0.012	0.002					
9	0.050	0.011	0.032	0.011	0.012	0.001	0.011	0.001					
10	0.015	0.005	0.028	0.001	0.002	0.000	0.003	0.000					

	"True" weights vs.											
best estimate ratio in Solvency II												
		Lo	$B_5$			Lo	$B_6$					
	Mod	el a)	Mod	el c)	Mod	el a)	Mod	el c)				
t	$w_t^{SCR}$	$w_t^{BE}$	$w_t^{SCR}$	$w_t^{BE}$	$w_t^{SCR}$	$w_t^{BE}$	$w_t^{SCR}$	$w_t^{BE}$				
0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000				
1	0.096	0.181	0.096	0.202	0.919	0.615	0.762	0.574				
2	0.035	0.096	0.035	0.107	0.850	0.411	0.694	0.367				
3	0.027	0.059	0.027	0.066	0.682	0.271	0.638	0.231				
4	0.017	0.037	0.017	0.041	0.165	0.174	0.495	0.142				
5	0.014	0.024	0.014	0.027	0.100	0.106	0.359	0.081				
6	0.009	0.015	0.009	0.016	0.063	0.059	0.276	0.041				
7	0.008	0.009	0.008	0.010	0.007	0.028	0.173	0.016				
8	0.007	0.005	0.007	0.005	0.001	0.010	0.097	0.004				
9	0.003	0.002	0.003	0.002	0.001	0.001	0.001	0.000				
10	0.002	0.001	0.002	0.001	0.000	0.000	0.000	0.000				

Table A.5: Table over "true" weights, i.e. the weights when the future solvency capital requirement is a random variable, compared to the weights used in the proposed approximation in Solvency II for  $LoB_5$  and  $LoB_6$ .

### A.6 Set up for changing the runoff pattern

We will here describe the set up where we consider a change in the runoff pattern for some of the latter accident years but the corresponding ultimate claims amount to be the same. The ultimate claims amount (or the ultimate loss) is the final amount that is paid for an accident year and the runoff pattern describes how this amount is distributed over the development years. The runoff pattern that characterize  $LoB_1$ ,  $LoB_3$  and  $LoB_6$  is found in table A.6 below. There the percentage of the ultimate claims amount that is paid during development year j is seen. Thus, for LoB<sub>1</sub>, about 55% of the ultimate loss is paid during year 1, 36% during year 2 and so on. Now, let us assume a scenario where the runoff pattern is changed for the last third of the observed accident years but the ultimate loss is the same, i.e. we have a breakpoint where the runoff pattern changes. This can be obtained by distributing the predicted ultimate loss, based on the observed loss triangle at time 0, for these latter accident years according to a new runoff pattern. We then obtain a new modified loss triangle having the structure shown in figure A.4. Based on this upper triangle we calculate new parameter estimates that we use to predict the risk margin objects and future solvency capital requirements. In this way we are able to see the effect of a sudden change in the runoff pattern but where the ultimate loss is considered to be the same. The new runoff pattern is set according to table A.6 where we have chosen a runoff pattern that deviates quite a lot from what is characteristic for each line of business. This is done to clearly see the consequences when there exists a larger variety in the underlying loss triangle data.



Figure A.4: Structure of data when considering a breakpoint in the runoff pattern for latter accident years, i.e. the data above the dashed line is our observed data but the data below the dashed line is now modified and distributed according to the new runoff pattern.

Runoff pattern for					New runoff pattern for			
$\mathbf{\hat{LoB}}_k$					$\hat{\mathbf{LoB}}_k$			
j	$LoB_1$	$LoB_3$	$LoB_6$	-	j	$LoB_1$	$LoB_3$	$LoB_6$
1	55%	24%	46%	-	1	20%	10%	10%
2	36%	34%	28%		2	15%	10%	15%
3	5%	10%	9%		<b>3</b>	10%	10%	10%
4	3%	10%	6%		4	10%	10%	10%
5	1%	10%	3%		5	10%	10%	5%
6	0%	4%	2%		6	10%	10%	5%
$\overline{7}$	0%	2%	2%		$\overline{7}$	10%	10%	10%
8	0%	3%	2%		8	10%	5%	5%
9	0%	1%	1%		9	5%	3%	15%
10	0%	1%	0%		10	0%	2%	10%
11	-	0%	0%		11	-	1%	5%
12	-	0%	0%		12	-	0%	0%

Table A.6: Runoff pattern that is characteristic for  $\text{LoB}_k$  to the left and the new considered runoff pattern to the right.