

# The confidence in the annuity divisor for premium pension

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#### Abstract

In Sweden the public pension premiepension is part of the national pension and determined by accumulated pension, funds returns, interest rate and annuity divisor. The annuity divisor therein is determined by life expectancy tables, interest rate and operating costs. A closer look into the current methodology used by the Swedish Pension Agency and Statistics Sweden tell if the methodology could be improved. The future mortality rates are predicted using the Lee-Carter model which are used to estimate the parameters in the Gompertz-Makeham mortality law. The force of mortality is the first step in the calculation of the annuity divisor. Assuming that the number of deaths have a binomial distribution, simulation of life expectancies result in annuity divisors close to published values. However, a closer look reveal that life expectancy simulations are in better agreement with historical values for women than men. The modelling approach performs poorer for higher ages, regardless of gender.

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# Introduction

#### 1.1 Sweden's national public pension and annuity divisor

The Swedish public pension is up of three main parts; national public pension, occupational pension and own savings, see figure 1.1 (Pensionsmyndigheten 2019). The national public pension is paid out whole life, unlike the occupational pension and own savings where pay out periods can vary. The national public pension is administered by Swedish Pensions Agency, PPM, a government body with the role to administer and pay out national pensions, as well to provide both general and individual information about pensions.



Figure 1.1: Pyramid illustrating the different parts of the Swedish pension system.

#### 1.1.1 Inkomstpension

Inkomstpension is part of the national public pension for individuals born after 1938, and contribute 16 % to your pensionable income or sickness compensation or activity compensation. The inkomstpension is paid through taxes and later paid out whole life. The inkomstpension amount is determined by accumulated pension, interest rate and annuity divisor. The annuity divisor is nongender specific and obtained from statistical life expectancy tables, time of claim and individual's age at time of claim (Pensionsmyndigheten 2018).

#### 1.1.2 Premiepension

Premiepension is part of the national public pension for individuals born after 1938, and made of 2.5 % of your pensionable income. During savings, the assets can be placed in funds of own choice or in the state fund AP7 Såfa. At retirement and time of claim, the funds can be transferred to a traditional insurance with guaranteed pay out amount or kept in fund insurance where the value

is determined by investment returns. The premiepension amount is determined by accumulated pension, funds returns, interest rate and annuity divisor. The annuity divisor is obtained from life expectancy tables published by Statistics Sweden, interest rate and operating costs (Pensionsmyn-digheten 2018).

#### 1.1.3 Annuity divisor for Premiepension

The annuity divisor,  $AD_x$ , for premium pension is based on an individual's age x. It is defined by PPM as an integral over a function with survival function and interest rate (Pensionsmyndigheten 2015, page 106).

(1.1)  

$$AD_{x} = \int_{x}^{\infty} e^{-\delta t} \frac{l_{x+t}}{l_{x}} dt$$

$$\delta = \ln(1+r) - \xi$$

$$l_{x} = \exp\left(-\int_{0}^{x} \mu(s) ds\right)$$

$$\mu_{x} = \begin{cases} a + be^{cx}, & \text{if } x \le 100\\ \mu_{100} + (x - 100) \cdot 0.01, & \text{if } x > 100 \end{cases}$$

where

$$\begin{split} \delta &= \text{rate intensity} \\ r &= \text{interest rate} \\ \xi &= \text{rate intensity for operational costs} \\ l_x &= \text{survival function for age } x \\ \mu_x &= \text{force of mortality for age } x \end{split}$$

The rate intensity is based on lending rate of 3% from 2014-03-01 before any cost deduction, resulting in an intensity  $\delta = 0.028559$  (Statistiska centralbyrån 2015, page 107). The survival function,  $l_x$ , is the probability of an individual surviving up to at least age x. The force of mortality,  $\mu_x$ , is modelled by the Gompertz-Makeham formula of rate of mortality, see section 3.4.

#### **1.2** Scope and limitation

This report follows PPM's methodology to calculate the premiepension annuity divisor, section 1.1.3, and SCB's methodology to forecast mortality rates using Lee-Carter model, section 3.2 and section 4.2.

Historical statistical data for population and number of deaths from 1995 to 2014 are obtained from SCB's open database. Three manual adjustments are made to historical deaths<sup>1</sup>, where recorded number of deaths were null. The records were changed to one number of deaths so logarithmic rules can apply for the logarithm of central mortality rate.

The data is used to forecast Sweden's population mortality rates for years 2015-2060 in the software Matlab. The forecast is simulated 10,000 times. The annuity divisor is estimated for three cohorts and for each forecast of life expectancy, hence 3 times à 10,000. The three cohorts are represented by the same generations which are used by PPM; they are 1938, 1945 and 1955 representing the cohorts 1930-ies, 1940-ies and 1950-ies. Following PPM's reporting, only age 61 to 70 are published.

<sup>&</sup>lt;sup>1</sup>Female records for number of deaths age 7 year 2006 and 2008, and age 9 in 2012.

# Swedish population data

Statistics Sweden, SCB, is an appointed government agency responsible for official statistics in areas such as population size, immigration, emigration, amongst many. The statistics are impartial and made available to the public through their website. SCB are certified according to ISO 20252:2012 for market, opinion and social research surveys, they are also certified to the international standard ISO 14001 for environmental management system. The certification confirm SCB fulfil the fundamental quality requirements in the production of statistics (Statistiska centralbyrån).

#### 2.1 Historical Swedish population data

All historical data for ages 0 to 100+ (age 100 and above) are obtained for years 1995 to 2014, the data contain information about

- Number of deaths by gender
- Number of new-borns alive by gender
- Population by gender

The data for number of deaths and number of new-borns alive are recorded weeks after the event. In the early years 1995 to 1997 the events are recorded up to 13 weeks after the occurrence, from 1998 the record time have been reduced to 4 weeks. The statistics of population by gender are recorded number of people on 31 December every year (Statistiska centralbyrån 2018).

#### 2.2 Forecasted Swedish life expectancy data

SCB publish the future population of Sweden annually. Every three years, most recent 2018, alternative forecasts are made with variations in future fertility, mortality, migration and stochastic roll-forwards to describe any uncertainties in the variations. In the years in between only population at the beginning of the year and assumptions are updated (Statistiska centralbyrån 2016, page 8). Forecasts of Sweden's population mortality rates are estimated using Lee-Carter model of mortality rates see section 3.2.

# Life expectancy modelling

Life expectancy is a statistical prediction of the average time a person is expected to live. Several factors are considered in predicting the life expectancy, most commonly used are gender and age, but also factors such as if smoker, marital status and socio-economic status (Statistiska centralbyrån 2018) can be considered. The life tables presenting the life expectancies of a certain population can be of periodical or of cohort type, the former presents the life expectancies in a given time period with no consideration of the individual's year of birth. The cohort view, used here, present the life expectancies by the year or time interval an individual is born (Bilius 2014).

#### 3.1 Formula based computation

The life expectancy for an individual of a certain population is assumed to be independent from each other. The individual's lifetime is defined as a non-negative continuous stochastic variable with the cumulative distribution function,  $F_x$ , as (Andersson 2005, page 47)

$$F_x = P(T \le x), \quad x \ge 0$$

where x is age and T is lifetime. The cumulative distribution function also describes the survival function,  $l_x$ , as

(3.1) 
$$l_x = P(T > x) = 1 - F_x = \exp(-\int_0^x \mu(s) ds)$$

where  $\mu$  is mortality rate. For remaining lifetime at age x, denoted  $T_x$ , the cumulative distribution function is defined as

$$F_{x,t} = P(T_x \le t), \quad t \ge 0$$

where lifetime and remaining lifetime have following relation

$$P(T_x > t) = P(T > x + t | T > x) = \frac{P(T > x + t)}{P(T > x)}$$

and the remaining lifetime can be described using the mortality rate as  $P(T_x > t) = \exp(-\int_x^{x+t} \mu(s) ds)$ .

In the insurance industry the companies are more interested in an individual's probability of survival from one year to the next. This is described by the survival function and the remaining lifetime  $T_x$  as

$$l_{x,t} = 1 - F_{x,t} = P(T_x > t), \quad t \ge 0$$

Correspondingly, the risk of an individual of age x dying within the year, t = 1, is referred to as the one-year death risk,  $q_x$ , defined as

$$q_x = P(T_x \le 1) = 1 - P(T_x > 1) = 1 - e^{-\int_x^{x+1} \mu(s) ds}$$

Similarly, the probability of survival for another year is

$$l_x = 1 - q_x$$

The life expectancy is commonly displayed as tables where the mortality increases with the age group. The life tables with the life expectancies use one-year death risk  $q_x$  to display estimated likelihood of death occurring in the age span of [x, x + 1).

#### 3.2 Lee-Carter model

In 1992 Ronald Lee and Lawrence Carter published their study of US death rates between 1933 and 1987 (Lee-Carter 1992). Their work known as the Lee-Carter model of mortality forecast has been widely used in the insurance industry (Girosi and King 2007). The model describes the life expectancies as a logarithm of the central mortality rate,  $\ln(m_{x,t})$ , with a general pattern of mortality,  $\alpha_x$ , the relative speed of change in mortality,  $\beta_x$ , and level of mortality at time t,  $k_t$ ,

(3.3) 
$$\ln(m_{x,t}) = \alpha_x + \beta_x k_t + \epsilon_{x,t}$$

where

 $\alpha_x = \text{constant}$  describing the general pattern of mortality at each age x  $\beta_x = \text{constant}$  describing the relative speed of change in mortality at each age x  $k_t = \text{index}$  of the level of mortality capturing the main time trends in death rates  $\epsilon_{x,t} = \text{error term}$ 

with the constraints

(3.4) 
$$\sum_{x} \beta_{x} = 1 \quad \text{and} \quad \sum_{t} k_{t} = 0$$

The error term  $\epsilon_{x,t}$ , mean 0 and variance  $\sigma_{\epsilon}^2$ , reflect the age-specific historical influences not captured by the model and captures all remaining variations. Lee and Carter discovered the results deviated from data that could be explained by the model giving same weight to old as young people. To address the logical difference of young people contributing as much as older to the total death rates, a second stage of calibration to the time varying index  $k_t$  is suggested. The re-estimation of parameter  $k_t$  take the  $a_x$  and  $b_x$  estimates from the first estimation as given, and implied number of deaths equal the actual number of deaths. The new estimates  $k_t$  are then found by an iterative search (Lee-Carter 1992, page 661).

Lee and Carter studied the correlation across ages assuming that all death rates to be a function of the same time varying index. They discovered the time varying index followed an autoregressive integrated moving average (ARIMA) model for the selected data. Each rate can be modelled by different order ARIMA process but in practice the random walk with a drift, ARIMA(0,1,0), has been used almost exclusively (Girosi and King 2007). The random walk describes the time varying index as  $k_t = k_{t-1} + \theta + \zeta_t$ , with  $\zeta_t \sim N(0, \sigma_{\zeta}^2)$ . For example, to roll forward 3 years the time varying index  $k_t$  is rolled forward by an iterative process

$$\begin{aligned} \hat{k}_{t} &= \hat{k}_{t-1} + \hat{\theta} + \zeta_{t} \\ &= (\hat{k}_{t-2} + \hat{\theta} + \zeta_{t-1}) + \hat{\theta} + \zeta_{t} \\ &= ((\hat{k}_{t-3} + \hat{\theta} + \zeta_{t-2}) + \hat{\theta} + \zeta_{t-1}) + \hat{\theta} + \zeta_{t} \\ &= \hat{k}_{t-3} + 3\hat{\theta} + (\zeta_{t-2} + \zeta_{t-1} + \zeta_{t}) \end{aligned}$$

#### 3.3 Renshaw and Haberman generalization

The Lee-Carter model have been adopted by Renshaw and Haberman with an extra parameter for cohort effects (Renshaw and Haberman 2006)

$$\ln(m_{x,t}) = \alpha_x + \beta_x k_t + \beta_x^{(1)} \gamma_{t-x} + \epsilon_{x,t}$$

where the new term  $\beta_x^{(1)}\gamma_{t-x}$  show the additional cohort effects as a function of the year of birth, t-x. Like the Lee-Carter model, restrictions are made with the new parameter considered, the restrictions are

$$\sum_{x} \beta_x = 1, \quad \sum_{x} \beta_x^{(1)} = 1, \quad \sum_{t} k_t = 0, \quad \gamma_{t-x} = 0$$

#### 3.4 Gompertz-Makeham generalization

The force of mortality, the rate which an individual is dying at the age of x, was first introduced by Benjamin Gompertz in 1825 as law of mortality and amended by William Makeham in 1860 (Hooker 1965). Makeham introduced an age independent component, a, to the law of mortality, defining the mortality rate as following Gompertz-Makeham formula (Andersson 2005, page 60)

(3.5) 
$$\mu(x) = a + be^{cx}$$

with the constraints

 $a+b>0, \qquad b>0, \qquad c\ge 0$ 

Hence, if a = 0 we have Gompertz force of mortality.

#### 3.5 Lazarus and Thiele generalization

Several generalizations have been made to Gompertz-Makeham formula (Pitacco 2016), one proposed by Lazarus 1867 was to capture infant mortality with a negative exponential term, which decreases as the age increases

$$\mu(x) = a + be^{cx} + \varphi e^{-\psi x}$$

with the constraints,

 $a+b>0, \quad b>0, \quad c\geq 0, \quad \varphi>0, \quad \psi>0$ 

Thiele generalized the formula further in 1871 (Pitacco 2016), by proposing an age-pattern of mortality over the whole life span

$$\mu(x) = a + be^{cx} + \varphi e^{-\psi x} + \lambda e^{-\delta(x-\epsilon)^2}$$

with the constraints,

$$a+b>0, \quad b>0, \quad c\geq 0, \quad \varphi>0, \quad \psi>0, \quad \lambda>0, \quad \delta>0, \quad \epsilon>0$$

# Fitting and applying the mortality rates

#### 4.1 The Gompertz-Makeham model

#### 4.1.1 Adapt the parameters in the Gompertz-Makeham model

The Swedish observed mortality rate reconciles well with Gompertz-Makeham's model, however, for higher ages the difference increases (Andersson 2005). Therefore, PPM have chosen a simplified, linear, representation of the force of mortality from age 100, outlining Gompertz-Makeham model by age as (Pensionsmyndigheten 2015, page 106)

(4.1) 
$$\mu(x) = \begin{cases} a + be^{cx}, & \text{if } x \le 100\\ \mu_{100} + (x - 100) \cdot 0.01, & \text{if } x > 100 \end{cases}$$

For data PPM use SCB life expectancy forecast for years 2015 to 2060 and ages 0 to 106, male and female. The Gompertz-Makeham model is estimated for three generations; 1938, 1945 and 1955, where each generation represent age groups; 77+, 70-76 and 60-69 years. Calculations and estimations are done individually for each generation using similar methodology. From age of 65, number of survivors for male ,  $l_{x,generation}^{\prime(m)}$ , and female,  $l_{x,generation}^{\prime(f)}$ , are estimated as

$$\begin{split} l_{x,\text{generation}}^{'(m)} &= \begin{cases} P_{2014,\text{ generation}}^{(m)}, & \text{for } x = 65\\ l_{x-1,\text{generation}}^{'(m)} (1 - q_{x-1,\text{ generation}}^{(m)}), & \text{for } x = 66, \cdots, 106 \end{cases} \\ l_{x,\text{generation}}^{'(f)} &= \begin{cases} P_{2014,\text{ generation}}^{(f)}, & \text{for } x = 65\\ l_{x-1,\text{generation}}^{'(f)} (1 - q_{x-1,\text{ generation}}^{(f)}), & \text{for } x = 66, \cdots, 106 \end{cases} \end{split}$$

where

$$\begin{split} P_{2014, \text{ generation}}^{(m)} &= \text{Male population 2014-12-31 for generation} \in \{1938, 1945, 1955\} \\ P_{2014, \text{ generation}}^{(f)} &= \text{Female population 2014-12-31 for generation} \in \{1938, 1945, 1955\} \\ q_{x, \text{ generation}}^{(m)} &= \text{Male one-year death risk at the age of } x \text{ years for generation} \in \{1938, 1945, 1955\} \\ q_{x, \text{ generation}}^{(f)} &= \text{Female one-year death risk at the age of } x \text{ years for generation} \in \{1938, 1945, 1955\} \\ \end{split}$$

Following regulations<sup>1</sup> the one-year death risk is weighted to be gender neutral (gn),  $q_{x,gn}$ , by considering the number of survivors and one-year death risk as (Pensionsmyndigheten 2016)

<sup>&</sup>lt;sup>1</sup>1998:674 Lagen om inkomstgrundad ålderspension 5 kap. 11§, and 2010:110 Socialförsäkringsbalken 62 kap. 34§

(4.2) 
$$q_{x,gn} = \frac{l_x^{\prime(m)} q_x^{(m)} + l_x^{\prime(f)} q_x^{(f)}}{l_x^{\prime(m)} + l_x^{\prime(f)}}$$

The gender neutral one-year death risk is adjusted for 6 months in the approximation for mortality intensity  $\mu_{x+0.5,gn} = -\ln(1 - q_{x+0.5,gn})$  for short durations (Andersson 2012, page 81). The Gompertz-Makeham parameters are estimated by minimizing the square difference in equation (4.3) below. Table 4.1 show reported Gompertz-Makeham's estimates for 2015 (Pensionsmyndigheten 2015, page 107).

Table 4.1: Reported Gompertz-Makeham's parameters 2015 by PPM based on forecast data for years 2015 to 2060.

Cohort	a	b	c
$1930\text{-}\mathrm{ies}$	0.00005	0.00000198	0.1239
$1940\text{-}\mathrm{ies}$	0.00460	0.00000053	0.1373
1950-ies	0.00470	0.00000019	0.1416

#### 4.1.2 Parameter estimation for Gompertz-Makeham model

The Gompertz-Makeham parameters are estimated by PPM using forecast data on life expectancies for the years 2015 to 2060 and Excel problem solver GRG Nonlinear, a non-linear generalized reduced gradient algorithm. PPM assign individuals by year of birth to a population cohort. Each cohort is estimated on so called prognosis basis with the view that the retrospective reserve and future premiums should cover future claims. This is expressed in the equation defining the retrospective reserve as (Alm 2006)

$$V_W(t) = B_W(t)A_W(t) - P_W(t)a_W(t)$$

where

 $V_W(t) =$ Retrospective reserve  $B_W(t) =$ Contracted sum insured

 $A_W(t) =$ Capital value at time t of remaining future pay out of 1 SEK according contract

 $P_W(t) =$ Contracted premium

 $a_W(t) = \text{Premium payment annuity of 1 SEK}$ 

The parameters are estimated using Excel function GRG Nonlinear solver, by taking the square difference between calculated force of mortality using SCB probability of deaths and Gompertz-Makeham mortality rate using SCB forecast data of life expectancies. The solver has the objective to minimize the difference by calibrating the variables a, b and c in Gompertz-Makeham model as following

(4.3) 
$$S = \min_{\hat{a},\hat{b},\hat{c}} \sum_{x} \left( \hat{\mu}_{x+0.5} - (\hat{a} + \hat{b}e^{\hat{c}(x+0.5)}) \right)^2$$

#### 4.2 The Lee-Carter model

#### 4.2.1 Adapt the parameters in the Lee-Carter model

SCB assume the Swedish population to be homogenous and have therefore omitted the second stage calibration of time varying index  $k_t$ , see section 3.2 (Statistiska centralbyrån 2015, page 194). This

report apply the same methodology as SCB in life expectancy forecasting, thus this step will not be carried out here.

The sophisticated approach to estimate the time varying index,  $k_t$ , as an ARIMA process is simplified for a linear approach only considering the minimum and maximum of estimated indices in vector  $\hat{\mathbf{k}}_t$ . The approach define the drift parameter,  $\hat{\theta}$ , used for forecasting as (Statistiska centralbyrån 2015, page 194),

(4.4) 
$$\hat{\theta} = \frac{\max(\hat{\mathbf{k}}) - \min(\hat{\mathbf{k}})}{n-1}$$

to forecast n number of years, the drift parameter is multiplied with number of years to forecast, hence  $n\hat{\theta}$ . For example, to estimate conditional mean of the time varying index 3 years, it would be  $\mathbf{E}[k_t + 3|k_t] = k_t + 3\hat{\theta}$ .

It has been suggested by Lundström and Qvist (Lundström and Qvist 2004) to use 25 years of historical data to capture Swedish gender mortality trends, however we adopt the SCB approach using a shorter time period from 1995 to 2014. The SCB method forecast individuals in two age groups; 0-49 years and 50-100 years. The younger age group use historical data for all ages between 0 to 106 years and the older age group consider historical data between ages 50 and 100 years. The first year of forecast, year 2015, is based on estimated mortality rate up to year 2014, by not using 2014 data smoothen the forecast (Statistiska centralbyrån 2015, page 198). For higher ages, 101 to 106, the mortality rate is not stable and therefore a smoothing method is used instead. The method multiply a factor to the mortality rate at age 100, for example the mortality rate for age 102 is  $\mu_{102} = \mu_{100} \times Factor_{102}$ . Table 4.2 are the used factors for forecast and corresponding to the differences in mortality between age groups for the period 2005 to 2014 (Statistiska centralbyrån 2015, page 199).

Table 4.2: Smoothing factors for higher ages by SCB, used in estimation of mortality rates.

Age	Factor
100	1.00
101	1.11
102	1.19
103	1.22
104	1.28
105	1.35
106	1.41

#### 4.2.2 Parameter estimation for the Lee-Carter model

Assuming the mortality rate can be described as the ratio between number of deaths and average population for one year, t, by age and gender (Statistiska centralbyrån 2015, page 191) as

$$(4.5) m_{x,t} = \frac{D_{x,t}}{N_{x,t}}$$

where

 $D_{x,t} =$  Number of deaths at age x in year-end t

 $P_{x,t}$  = Population age x in year-end t

 $P_{\text{newborn},t} = \text{Number of new-born alive in year-end } t$ 

$$N_{x,t} = \begin{cases} [(P_{x-1,t-1} + P_{x,t})/2], & \text{if } x > 0\\ [(P_{\text{new born},t}/2], & \text{if } x = 0 \end{cases}$$

Each mortality rate is saved as logarithm of the mortality rate in the matrix  $\mathbf{M}$ , with j number of rows corresponding to ages, and n number of columns corresponding to years.

(4.6) 
$$\mathbf{M} = \begin{pmatrix} \ln(m_{x,t}) & \ln(m_{x,t+1}) & \cdots & \ln(m_{x,t+n}) \\ \ln(m_{x+1,t}) & \ln(m_{x+1,t+1}) & \cdots & \ln(m_{x+1,t+n}) \\ \vdots & \vdots & \ddots & \vdots \\ \ln(m_{x+j,t}) & \ln(m_{x+j,t+1}) & \cdots & \ln(m_{x+j,t+n}) \end{pmatrix} = \mathbf{A} + \mathbf{B}\mathbf{K}^{\mathbf{T}} + \epsilon$$

where

$$\mathbf{A} = \begin{pmatrix} \alpha_{x,t} & \alpha_{x,t+1} & \cdots & \alpha_{x,t+n} \\ \alpha_{x+1,t} & \alpha_{x+1,t+1} & \cdots & \alpha_{x+1,t+n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{x+j,t} & \alpha_{x+j,t+1} & \cdots & \alpha_{x+j,t+n} \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} \beta_{x,t} & \beta_{x,t+1} & \cdots & \beta_{x,t+n} \\ \beta_{x+1,t} & \beta_{x+1,t+1} & \cdots & \beta_{x+1,t+n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{x+j,t} & \beta_{x+j,t+1} & \cdots & \beta_{x+j,t+n} \end{pmatrix}$$
$$\mathbf{K}^{\mathbf{T}} = \begin{pmatrix} k_{x,t} & k_{x,t+1} & \cdots & k_{x,t+n} \\ k_{x+1,t} & k_{x+1,t+1} & \cdots & k_{x+1,t+n} \\ \vdots & \vdots & \ddots & \vdots \\ k_{x+j,t} & k_{x+j,t+1} & \cdots & k_{x+j,t+n} \end{pmatrix} \quad \epsilon = \begin{pmatrix} \epsilon_{x,t} & \epsilon_{x,t+1} & \cdots & \epsilon_{x,t+n} \\ \epsilon_{x+1,t} & \epsilon_{x+1,t+1} & \cdots & \epsilon_{x+1,t+n} \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon_{x+j,t} & \epsilon_{x+j,t+1} & \cdots & \epsilon_{x+j,t+n} \end{pmatrix}$$

The constraints (3.4) implies the variable  $a_x$  to be average of central mortality rates over time, hence the average of each row in matrix **M** will return the general pattern of mortality at each age x as (Lee and Carter 1992, page 661)

$$\hat{\mathbf{a}}_x = \bar{\mathbf{M}}_x = \frac{1}{n} \sum_{i=1}^n \ln(m_{x,i})$$

Centralizing the matrix **M** by removing the estimated variable  $\hat{\mathbf{a}}_x$  we have

$$ilde{\mathbf{M}} = \mathbf{M} - ar{\mathbf{M}}$$

and remaining the parameters  $b_x$  and  $k_t$  to be estimated. Lee and Carter suggested applying singular value decomposition (SVD) to find the least square solution to equation (3.3). Let the centralised matrix  $\tilde{\mathbf{M}}$  be

$$\tilde{\mathbf{M}} = \mathbf{U}\mathbf{S}\mathbf{V}^T$$

where

- $\mathbf{U} = \text{The right singular vectors of } \tilde{\mathbf{M}} \text{ and a } j \times j \text{ matrix with orthogonal columns}$  so,  $\mathbf{U}^T \mathbf{U} = \mathbf{I}$
- $\mathbf{V}=$  The left singular vectors of  $\tilde{\mathbf{M}}$  and a  $n\times n$  matrix with orthogonal columns so,  $\mathbf{V}\mathbf{V}^T=\mathbf{I}$
- $\mathbf{S}=\mathbf{A}$  diagonal matrix with square roots of positive eigenvalues  $s_i$  in decreasing order along the diagonal

The estimated parameters can be described as (Lee and Carter 1992, page 661)

$$\hat{\boldsymbol{\beta}} = s_1 \mathbf{u}^{(1)}$$
$$\hat{\mathbf{k}} = \mathbf{v}^{(1)}$$

where

 $s_1 =$  First and largest element in the diagonal matrix **S** 

 $\mathbf{u}^{(1)} = \text{The first column vector in matrix } \mathbf{U}$ 

 $\mathbf{v}^{(1)} = \text{The first column vector in matrix } \mathbf{V}$ 

# Simulation and re-fitting of the Annuity divisor

#### 5.1 Simulation of the Lee-Carter model and mortality rates

Assuming the number of deaths follow a binomial distribution, Bin(N,q), with n deaths in the N population and q probability of dying, we can simulate the numerator,  $N_{x,t}$ , in mortality rate (4.5). We use historical data (h) between years 1995 and 2014 to determine the population  $N_{x,t}^{(h)}$ , for age x and year t, and let historical mortality rate  $m_{x,t}^{(h)}$  determines the probability of death  $q_{x,t}^{(h)}$  (Statistiska centralbyrån 2015, page 181).

(5.1) 
$$m_{x,t}^{i} = \frac{D_{x,t}^{i}}{N_{x,t}^{(h)}}$$

where

$$D_{x,t}^{i} \sim \operatorname{Bin}(N_{x,t}^{(h)}, q_{x,t}^{(h)})$$
$$N_{x,t}^{(h)} = [(P_{x-1,t-1} + P_{x,t})/2]$$

and  $i = 1, \dots, 10,000$  simulations. The simulation creates elements in the matrix **M**, like matrix (4.6),

$$\mathbf{M}^{i} = \begin{pmatrix} \ln(m_{0,1996}^{i}) & \ln(m_{0,1997}^{i}) & \cdots & \ln(m_{0,2014}^{i}) \\ \ln(m_{1,1996}^{i}) & \ln(m_{1,1997}^{i}) & \cdots & \ln(m_{1,2014}^{i}) \\ \vdots & \vdots & \ddots & \vdots \\ \ln(m_{106,1996}^{i}) & \ln(m_{106,1997}^{i}) & \cdots & \ln(m_{106,2014}^{i}) \end{pmatrix}$$

and for each matrix the Lee-Carter parameters  $\hat{\boldsymbol{\alpha}}^{i}$ ,  $\hat{\boldsymbol{\beta}}^{i}$  and  $\hat{\boldsymbol{k}}^{i}$ , are estimated using SVD. With the estimated vector  $\hat{\mathbf{k}}_{t}$ , the drift parameter,  $\hat{\theta}^{i}$ , in equation (4.4) is estimated for each simulation. Estimation is done for ages 0 to 100 and the years 2015 to 2060, for individuals aged 101 to 106 the mortality rates are smoothened according to section 4.2.1 using table 4.2. A matrix of forecasted logarithms of mortality rates are created as

$$(5.2) \qquad \qquad \mathbf{\hat{m}^{i}} = \begin{pmatrix} \hat{m}_{0,2015}^{i} & \hat{m}_{0,2016}^{i} & \cdots & \hat{m}_{0,2060}^{i} \\ \hat{m}_{1,2015}^{i} & \hat{m}_{1,2016}^{i} & \cdots & \hat{m}_{1,2060}^{i} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{m}_{100,2015}^{i} & \hat{m}_{100,2016}^{i} & \cdots & \hat{m}_{100,2060}^{i} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{m}_{106,2015}^{i} & \hat{m}_{106,2016}^{i} & \cdots & \hat{m}_{106,2060}^{i} \end{pmatrix}$$

This is in turn used in estimation of probability of deaths by transforming the mortality rates as

$$\hat{q}_{x,t}^{i} \begin{cases} 1 - e^{-0.5(\hat{m}_{x,t}^{i} + \hat{m}_{x-1,t}^{i})}, & \text{for } x = 1 \text{ to } x = 106\\ 1 - e^{-0.5(\hat{m}_{0,t}^{i})}, & \text{if } x = 0 \end{cases}$$

#### Re-fitting of the Annuity divisor with simulated mortal-5.2ity rates

PPM use the forecasted probability of death to estimate the one-year death risk,  $\hat{q}^i$ , and the probable future population  $\hat{l}_x^{(\cdot)i}$  by gender. This is transformed to be gender neutral (gn) as in equation (4.2)

$$\hat{q}_{x,\text{gn}}^{i} = \frac{\hat{l}_{x}^{(m),i}\hat{q}_{x}^{(m),i} + \hat{l}_{x}^{(f),i}\hat{q}_{x}^{(f),i}}{\hat{l}_{x}^{(m),i} + \hat{l}_{x}^{(f),i}}$$

Thereafter the mortality intensities are estimated by approximation

$$\hat{\mu}_{x+0.5,\text{gn}}^{i} = -\ln(1 - \hat{q}_{x+0.5,\text{gn}}^{i})$$

and used in minimizing the square error (4.3) to estimate the parameters in Gompertz-Makeham's model (4.3).

The estimated force of mortality,  $\hat{\mu}^i_{x+0.5,\mathrm{gn}}$ , is input to the survival function used in the estimation of annuity divisor, equation (5.3).

5.3)  
$$\mu_x = \exp\left(-\int_0^x \mu(s)ds\right)$$
$$\mu_x = \begin{cases} a_{1955} + b_{1955}e^{c_{1955}x}, & \text{if } 61 \le x \le 65\\ a_{1945} + b_{1945}e^{c_{1945}x}, & \text{if } 66 \le x \le 75\\ a_{1938} + b_{1938}e^{c_{1938}x}, & \text{if } 76 \le x \le 100\\ \mu_{100} + (x - 100) \cdot 0.01, & \text{if } x > 100 \end{cases}$$

(5

# Comparison of the results of the re-fitted Lee-Carter model

#### 6.1 Result of re-fitted Lee-Carter model

The mortality rates are simulated 10000 times and for each simulation i the number of deaths,  $D_{x,t}^i$ , are assumed to have a binomial distribution. The results are compared to historical mortality rates,  $m_{x,t}^{(h)}$ , for the years 1996 to 2014. The difference is constructed as

$$\hat{\epsilon}_{x,t} = m_{x,t}^{(h)} - \bar{m}_{x,t}^{(sim)}$$

where the average simulated mortality rates are

$$\bar{m}_{x,t}^{(sim)} = \frac{1}{10000} \sum_{i=1}^{10000} \hat{m}_{x,t}^{i}$$

The difference resemble the error term  $\epsilon_{x,t}$  in the Lee-Carter model, equation (3.3). Assuming the difference to be of normal distribution with expected mean 0 and variance  $\sigma_{\epsilon}^2$ , we plot the differences in QQ-plots. The QQ-plots plot the theoretical normal quantile along the horizontal x-axis and the residual data along the vertical y-axis. Where the points appear closer to a straight line, the more likely they come from the same distribution and historical and average simulation are more equivalent. The results in Appendix A.1 show the line with a bend up to the right for all males and all years, a distribution with a right skew and heavy tail, meaning the large differences are bigger than expected compared to a normal distribution. For the female results in Appendix A.2 there is no such distinct bend, there is however a minor bend on the left side, meaning the small differences are smaller than expected. For both genders and all years, we have a outliers to the right side in the QQ-plots, so there are differences larger than expected.

A closer look on the differences,  $\hat{\epsilon}_{x,t}$ , by age and time for male in Appendix A.3 and female in Appendix A.4 show in both cases larger differences for upper age group 90 to 100 years. For males the differences for ages 0 to 89 ranges between just below 0 to about 0.03, the differences increases about 10-folds for ages 90 to 100. In Appendix A.5 (male) and Appendix A.6 (female) the differences are plot in age intervals of 10 years and show increase in difference by age, regardless of gender.

For females the simulation of mortality rates is often higher than historical values, as shown in Figure A.10 and Figure A.11, where a band of negative differences occur for most ages. Same logic applies to the simulation of male mortality rates, where simulation have underestimated the mortality rates. Figure A.7 and Figure A.8 show a band of positive differences between estimated male mortality rates and historical values.

#### 6.2 Result of annuity divisor with re-fitted Lee-Carter model

Simulation of number of deaths and Lee-Carter model of mortality rates result in estimated annuity divisors in table 6.1.

Ago	PPM	Expected	Expected	Standard	2.75~%	97.5 %
Age		value	median	deviation	percentile	percentile
61	17.49	17.4858	17.4904	0.0095	17.4646	17.4961
62	17.10	17.1009	17.1057	0.0098	17.0791	17.1115
63	16.69	16.7068	16.7117	0.0100	16.6845	16.7176
64	16.28	16.3036	16.3086	0.0102	16.2808	16.3147
65	15.85	15.8915	15.8966	0.0105	15.8682	15.9028
66	15.42	15.4708	15.4760	0.0107	15.4469	15.4824
67	14.42	15.0417	15.0471	0.0109	15.0174	15.0536
68	13.97	14.6048	14.6102	0.0111	14.5800	14.6168
69	13.52	14.1603	14.1658	0.0113	14.1351	14.1726
70	13.06	13.7089	13.7145	0.0115	13.6833	13.7214

Table 6.1: Expected value and median of annuity divisor for 10000 scenarios.

Estimation is done using same methodology as PPM with estimated Makeham parameters for each cohort and simulation. The estimated annuity divisor is within 0.4 % of the reported values for ages 61 to 66 (Pensionsmyndigheten 2015, page 107). Between ages 66 and 67 the reported annuity divisor drops by one unit (from 15.42 to 14.42) because of change in Gompertz-Makeham parameters, from cohort 1950-ies to cohort 1940-ies. This also takes place for ages 75 and 76, but is outside of the reported range and not shown here. The move will not be recognized by the insured because he or she will not change his / her's year of birth.

The reported annuity divisors are within the estimated range 2.75% to 97.5% percentile for ages 61 to 63 and the differences are then not significant. However, for the higher ages the differences between estimated and reported annuity divisors falls out of the range and therefore significant.

The change in annuity divisor from one age to the next in table 6.2 are of same degree in both reported and estimated values. Rounding the expected values to same level of accuracy as reported values, show the absolute differences in change falls under 2 % for all ages except from 66 to 67. This larger change of 0.57 is explained by the change of Gompertz-Makeham parameters with PPM.

1 ~~~	PPM	Expected	Change in	Change in	Absolute
Age		value	$\mathbf{PPM}$	expected value	difference
61	17.49	17.4858	-	-	-
62	17.1	17.1009	-0.3900	-0.3849	0.01
63	16.69	16.7068	-0.4100	-0.3941	0.02
64	16.28	16.3036	-0.4100	-0.4032	0.01
65	15.85	15.8915	-0.4300	-0.4121	0.02
66	15.42	15.4708	-0.4300	-0.4207	0.01
67	14.42	15.0417	-1.0000	-0.4290	0.57
68	13.97	14.6048	-0.4500	-0.4370	0.01
69	13.52	14.1603	-0.4500	-0.4445	0.01
70	13.06	13.7089	-0.4600	-0.4514	0.01

Table 6.2: Change in annuity divisor

Age	PPM	Expected value	Difference
61	17.49	17.4858	0.0042
62	17.10	17.1009	-0.0009
63	16.69	16.7068	-0.0168
64	16.28	16.3036	-0.0236
65	15.85	15.8915	-0.0415
66	15.42	15.4708	-0.0508
67	14.42	15.0417	-0.6217
68	13.97	14.6048	-0.6348
69	13.52	14.1603	-0.6403
70	13.06	13.7089	-0.6489

Table 6.3: Difference between reported and estimated annuity divisors.

A closer look of the difference show modelled values are higher than reported in all but one case see table 6.3. Figure 6.1 illustrate the simulated median is higher for all ages, and the average is higher for all ages except age 61. Bottom and top edges of the box indicate the 25th and 75th percentiles, respectively. The lower reported annuity numbers suggest a risk averse reserve amount.



Figure 6.1: Boxplot of difference between reported and estimated annuity divisor,  $AD_x^{PPM} - AD_x^{sim}$ .

The performance of the estimate is measured with mean square error (MSE) defined as the average sum of squared error between simulated annuity divisors,  $AD_x^{sim}$ , and the annuity divisors used by PPM,  $AD_x^{PPM}$ . That is,

(6.1) 
$$MSE(AD_x^{sim}, AD_x^{PPM}) = \frac{1}{10000} \sum_{i=1}^{10000} (AD_x^{sim,i} - AD_x^{PPM})^2$$

We discover the MSE is very small, see Table 6.4, with proportion of square root of MSE in

Age	PPM	$\sqrt{MSE}$
61	17.49	0.0104
62	17.1	0.0098
63	16.69	0.0195
64	16.28	0.0257
65	15.85	0.0428
66	15.42	0.0519
67	14.42	0.6218
68	13.97	0.6349
69	13.52	0.6404
70	13.06	0.6490

Table 6.4: Square root of MSE.

parts per million of the reported annuity divisors.

#### 6.3 Conclusion

The estimated mortality rates from simulation of the number of deaths, resulted in differences between estimated mortality rates and historical values to be higher than expected for male and smaller than expected for females. The QQ-plots show the assumption of number of deaths being binomial distributed is more appropriate for females. The outliers in the QQ-plots, regardless of gender, could come from the estimated mortality rates for higher ages, where the differences are higher.

As for the simulated annuity divisor results, the differences between reported values and average simulated values appear in parts per hundreds for ages 61 to 66 and are considered small given the size of the annuity divisor. For higher ages the difference is more significant and concerns about the methodology is warrant.

Any accuracy in the simulated female mortality rates (derived from number of deaths) could have been cancelled out by the heavy tail in differences for male when deriving the annuity divisors. Results suggest using different distribution for male and female to likely narrow the gap between historical mortality rates and estimated mortality rates and in turn narrow the gap between reported annuity divisor and average simulated annuity divisor. Appendix

## A.1 Male QQ-plot of residuals in the Lee-Carter model



Figure A.1: Male QQ-plot with quantiles of residual in the Lee-Carter model (y-axis) and theoretical quantiles from normal distribution  $N(0, \sigma_{\epsilon}^2)$  (x-axis) for the years 1996 to 2003.



Figure A.2: Male QQ-plot with quantiles of residual in the Lee-Carter model (y-axis) and theoretical quantiles from normal distribution  $N(0, \sigma_{\epsilon}^2)$  (x-axis) for the years 2004 to 2011.



Figure A.3: Male QQ-plot with quantiles of residual in the Lee-Carter model (y-axis) and theoretical quantiles from normal distribution  $N(0, \sigma_{\epsilon}^2)$  (x-axis) for the years 2012 to 2014.

## A.2 Female QQ-plot of residuals in the Lee-Carter model



Figure A.4: Female QQ-plot with quantiles of residual in the Lee-Carter model (y-axis) and theoretical quantiles from normal distribution  $N(0, \sigma_{\epsilon}^2)$  (x-axis) for the years 1996 to 2003.



Figure A.5: Female QQ-plot with quantiles of residual in the Lee-Carter model (y-axis) and theoretical quantiles from normal distribution  $N(0, \sigma_{\epsilon}^2)$  (x-axis) for the years 2004 to 2011.



Figure A.6: Female QQ-plot with quantiles of residual in the Lee-Carter model (y-axis) and theoretical quantiles from normal distribution  $N(0, \sigma_{\epsilon}^2)$  (x-axis) for the years 2012 to 2014.

#### A.3 Male linear plot of residuals in the Lee-Carter model



Figure A.7: Male linear plot of residuals between estimated mortality rates and historical mortality rates. All ages from year 1995 to 2013.



Figure A.8: Male linear plot of residuals between estimated mortality rates and historical mortality rates. Ages 0 to 89 years from year 1995 to 2013.



Figure A.9: Male linear plot of residuals between estimated mortality rates and historical mortality rates. Ages 90 to 100 years from year 1995 to 2013.

### A.4 Female linear plot of residuals in the Lee-Carter model



Figure A.10: Female linear plot of residuals between estimated mortality rates and historical mortality rates. All ages from year 1995 to 2013.



Figure A.11: Female linear plot of residuals between estimated mortality rates and historical mortality rates. Ages 0 to 89 years from year 1995 to 2013.



Figure A.12: Female linear plot of residuals between estimated mortality rates and historical mortality rates. Ages 90 to 100 years from year 1995 to 2013.



# A.5 Male linear plot of residuals in the Lee-Carter model for ages 20 to 50

Figure A.13: Male linear plot of residuals between estimated mortality rates and historical mortality rates. Ages 20 to 50 from year 1995 to 2013.



Figure A.14: Male linear plot of residuals between estimated mortality rates and historical mortality rates. Ages 60 to 90 years from year 1995 to 2013.



# A.6 Female linear plot of residuals in the Lee-Carter model for ages 20 to 50

Figure A.15: Female linear plot of residuals between estimated mortality rates and historical mortality rates. Ages 20 to 50 from year 1995 to 2013.



Figure A.16: Female linear plot of residuals between estimated mortality rates and historical mortality rates. Ages 60 to 90 years from year 1995 to 2013.

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