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Fair Dynamic Valuation of Insurance Liabilities

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Abstract

This thesis study sets out to investigate the topic of fair insurance liability valuation. Considering a discrete multi-period time setting, we explore a class of fair dynamic valuations that combine market consistency, actuarial judgment as well as time consistency. Furthermore, we show how to construct a fair dynamic valuation using a backwards iteration procedure. This procedure is implemented numerically through the usage of Least Square Monte Carlo (LSMC) approximation techniques. As part of this, we focus on investigating two main aspects; the choice of underlying regression models used in the LSMC approximations as well as the choice of actuarial valuation function. Moreover, we assess how these choices affect the resulting fair dynamic valuation. Our results indicate that the choice of regression models has an evident impact on the subsequent valuations. In particular, we identify smoothing splines and LOESS regression models as promising candidates that achieve improved estimates in the LSMC approximations.

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Contents

Chapter 1

Introduction

Insurance liability valuation is a fundamental part of an insurance business. In addition, modern insurance solvency legislation, including Solvency II and the Swiss Solvency Test (SST), obligates a fair valuation of insurance liabilities. In the case of Solvency II, the regulation[1] requires that such a valuation should reflect the amount that another insurance entity would pay to take over the corresponding insurance obligations. Furthermore, the value must be consistent with financial market information as well as available underwriting risk data. In this sense, a fair valuation needs to combine techniques from both finance and actuarial mathematics.

The topic of fair valuation of insurance liabilities has been investigated in numerous studies. Pelsser & Stadje (2014) introduced a two-step procedure for market consistent evaluations, which was later used by Dhaene et al. (2017) to define a two-step fair valuation that was both market consistent and actuarial. This work was further extended by Barigou et al. (2019) who considered a multi-period setting, in which the notion of time consistency was introduced and the concept of fair dynamic valuations formalized. Additional work on the topic has been done by Barigou & Dhaene (2019), who investigated mean-variance hedging as a tool for valuation in a multi-period setting, as well as Chen et al. (2020) and Chen et al. (2021), who researched the use of convex hedging approaches.

In this thesis project, we investigate the topic of fair dynamic valuations and extend the work by Barigou et al. (2019). This includes an exploration of the mathematical theory of fair dynamic valuations, i.e. where we combine concepts of market consistency, actuarial judgment and time consistency. Also, we study the methods of constructing a fair dynamic valuation in practice.

[1]For a more detailed specification regarding the regulatory requirements of liability valuation in Solvency II, see Official Journal of the European Union: Solvency II Directive (2009/138/EC) Articles 75-84.

These techniques are implemented and used as part of a numerical analysis. For this we use concepts in risk minimization, such as those presented by Föllmer & Schweizer (1988) and Černý & Kallsen (2009). Furthermore, we utilize Least Square Monte Carlo (LSMC) methods. Originally introduced by Carriere (1996), LSMC methods have become an important tool in mathematical finance for the valuation of American-style options, see for example Longstaff & Schwartz (2001), Tsitsiklis & Van Roy (2001) and Clément et al. (2002). Additionally, we explore the usage of a cost-of-capital valuation approach, similar to that analyzed by Engsner et al. (2017).

The report is structured as follows: Chapter 2 outlines the mathematical theory and specifies the financial-actuarial framework. This part takes a theoretical approach to the topic and defines, among other thing, the concept of dynamic valuations and their related properties. Next, Chapter 3 presents the choice of setup as well as the utilized techniques, focusing more on practical methods and implementation. This section also lays the foundation for the numerical analysis results, which are given in Chapter 4. Thereafter, Chapter 5 contains a discussion, which ties back to the obtained results as well as the study's scope and objectives. Finally, Chapter 6 presents the main conclusions of the study.

Chapter 2

Mathematical Framework

In this chapter, we provide the mathematical theory underlying this thesis. This includes formalization of the combined financial-actuarial framework as well as concepts of claims, valuations and hedgers. Most importantly, we define what constitutes a fair dynamic valuation.

2.1 General framework

We consider a time horizon $T \in \mathbb{N}$ as well as the set of trading times defined as $\{0, 1, \ldots, T\}$. That is, the trading times are represented by a set of discrete points in time. Here, time zero represents today.

To model the financial-actuarial framework, we use the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Here, Ω is the sample space, $\mathcal{F}$ is a σ-algebra of Ω, and $\mathbb{P} \colon \mathcal{F} \to [0, 1]$ is the real world (sometimes also called the physical) probability measure. Moreover, $\mathbb{F} = \{\mathcal{F}_t\}_{t \in \{0,1,\ldots,T\}}$ denotes the filtration, where for each $t \in \{0, 1, \ldots, T\}$, $\mathcal{F}_t$ is the σ-algebra representing the available information at time t. The term available information here should be thought of in a rather wide sense. That is, $\mathcal{F}_t$ includes both financial information, such as stock prices and interest rates, as well as actuarial information, e.g. death/survival of life insurance policyholders, available at time t. It is also worth noting that the sequence of σ-algebras contained in $\mathbb{F}$ is non-decreasing, i.e. we have that $\{\emptyset, \Omega\} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_T = \mathcal{F}$. In other words, more information becomes available as time passes.

In this study, all introduced random variables are defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Furthermore, we will always assume that the second moment of random variables is finite. That is, for any random variable X, we always assume $\mathbb{E}\left[X^2\right] < \infty$. This guarantees that the variance of X is well-defined. Additionally, whenever we formulate relations between random variables, these should be interpreted in the $\mathbb{P}$ almost surely ($\mathbb{P}$ a.s.) sense.

For instance, if X_1 and X_2 are random variables and we write $X_1 = X_2$, what we actually mean is that $\mathbb{P}(X_1 = X_2) = 1$.

2.2 The financial market & trading strategies

The financial market, i.e. the market of tradable assets[2], is assumed to contain $n+1$ assets, where $n \in \mathbb{N}_0$. To specify, for each $i \in \{0, 1, \dots, n\}$ and $t \in \{0, 1, \dots, T\}$, $Y^{(i)}(t)$ denotes the market price of the asset with index i at time t. Moreover, we denote $\boldsymbol{Y}(t) = \left(Y^{(0)}(t), Y^{(1)}(t), \dots, Y^{(n)}(t)\right)$ as the vector of market prices at time t. Notably, we always assume that $\{\boldsymbol{Y}(t)\}_{t\in\{0,1,\dots,T\}}$ is adapted to the filtration $\mathbb{F}$. This means that $\boldsymbol{Y}(t)$ is $\mathcal{F}_t$-measurable for each $t \in \{0, 1, \dots, T\}$.

The asset with index 0 will always denote a risk free zero-coupon bond (ZCB) which pays the amount 1 at the time horizon T. It is worth noting that this asset is assumed to always exist. Moreover, we let the risk free short rate (i.e. the continuously compounded risk free interest rate) be denoted $r(t)$, $t \in [0, T]$. In this general setting, we treat this short rate as stochastic[3].

Given the formalization of the financial market, we can now introduce the concept of trading strategies, as specified in Definition 1.

Definition 1 *For a chosen $t \in \{0, 1, \dots, T-1\}$, a time t trading strategy is an $(n+1)$-dimensional and $\mathbb{F}$-predictable process $\boldsymbol{\theta}_t = \{\boldsymbol{\theta}_t(u)\}_{u\in\{t+1,\dots,T\}}$. Here, $\boldsymbol{\theta}_t(u) = \left(\theta_t^{(0)}(u), \theta_t^{(1)}(u), \dots, \theta_t^{(n)}(u)\right)$ is referred to as a position vector, where $\theta_t^{(i)}(u)$ is the number of invested units in asset with index i over the time interval $(u-1, u]$.*

The $\mathbb{F}$-predictable property here is rather important, since it implies that, for each $u \in \{t+1, \dots, T\}$, $\boldsymbol{\theta}_t(u)$ is $\mathcal{F}_{u-1}$-measurable. In other words, given the information available at time $u-1$, $\boldsymbol{\theta}_t(u)$ is fully determined.

Given a time t trading strategy $\boldsymbol{\theta}_t$, there is a possible rebalancing of positions at each time point $u \in \{t+1, \dots, T-1\}$. The portfolio values before and after this rebalancing are expressed in Equations (2.1) and (2.2) respectively. Given these equations we can define the property of self-financing trading strategies, as specified in Definition 2. We note that no rebalancing occurs at the time horizon T.

$$\boldsymbol{\theta}_t(u) \cdot \boldsymbol{Y}(u) = \sum_{i=0}^{n} \theta_t^{(i)}(u) Y^{(i)}(u), \tag{2.1}$$

[2] We do not model any financial market imperfections, such as transaction costs or any limits in the amount of bought or sold assets. In other words, the market is assumed deep, liquid and transparent. Also, we do not consider any assets as paying dividends.

[3] For some examples of stochastic short rate models, see Brigo & Mercurio (2006)

$$\boldsymbol{\theta}_t(u+1) \cdot \boldsymbol{Y}(u) = \sum_{i=0}^{n} \theta_t^{(i)}(u+1)Y^{(i)}(u). \tag{2.2}$$

Definition 2 *A time t trading strategy $\boldsymbol{\theta}_t$ is said to be self-financing if for each $u \in \{t+1, \ldots, T-1\}$ it holds that*

$$\boldsymbol{\theta}_t(u) \cdot \boldsymbol{Y}(u) = \boldsymbol{\theta}_t(u+1) \cdot \boldsymbol{Y}(u).$$

Moreover, the set of all self-financing time t trading strategies is denoted Θ_t.

The self-financing property ensures that no additional value needs to be invested or withdrawn at any rebalancing time. We can also note that the initial investment of a time t trading strategy $\boldsymbol{\theta}_t$ is given by Equation (2.3), and the final portfolio value at the time horizon T is given by Equation (2.4). Naturally, a special case of these concepts is when $t = 0$, i.e. when the trading strategy starts at time zero and the portfolio is managed over the whole time span.

$$\boldsymbol{\theta}_t(t+1) \cdot \boldsymbol{Y}(t) = \sum_{i=0}^{n} \theta_t^{(i)}(t+1)Y^{(i)}(t), \tag{2.3}$$

$$\boldsymbol{\theta}_t(T) \cdot \boldsymbol{Y}(T) = \sum_{i=0}^{n} \theta_t^{(i)}(T)Y^{(i)}(T). \tag{2.4}$$

Another important assumption is the notion of a financial market without arbitrage. This is formalized in Definition 3 below.

Definition 3 *A market is said to have no arbitrage (or said to be arbitrage-free) if, for each $t \in \{0, 1, \ldots, T-1\}$, there does not exist a self-financing time t trading strategy $\boldsymbol{\theta}_t \in \Theta_t$ such that the following criteria all hold:*

- $\boldsymbol{\theta}_t(t+1) \cdot \boldsymbol{Y}(t) = 0$,
- $\mathbb{P}\left(\boldsymbol{\theta}_t(T) \cdot \boldsymbol{Y}(T) \geq 0\right) = 1$,
- $\mathbb{P}\left(\boldsymbol{\theta}_t(T) \cdot \boldsymbol{Y}(T) > 0\right) > 0$.

For the scope of this study, we will always assume the market to have no arbitrage. This is a rather important assumption, as it is equivalent to existence of an equivalent martingale measure (EMM), denoted $\mathbb{Q}$. In many applications, $\mathbb{Q}$ is called a risk-neutral measure. For more details on this topic, see for example Björk (2009).

Consider again the short rate $r(t)$, $t \in [0, T]$, as introduced earlier. Given the existence of the $\mathbb{Q}$-measure, we have the martingale property of $\{\boldsymbol{Y}(t)\}_{t=0}^{T}$ as formulated in Equation (2.5). Furthermore, this connection implies that there is a similar martingale property for the portfolio value of any time t trading strategy $\boldsymbol{\theta}_t \in \Theta_t$, as specified in Equation (2.6).

$$\boldsymbol{Y}(t) = \mathbb{E}^{\mathbb{Q}} \left[\exp\left(- \int_t^{t+1} r(u)\,\mathrm{d}u \right) \boldsymbol{Y}(t+1) \middle| \mathcal{F}_t \right], \qquad t \in \{0, 1, \dots, T-1\}, \tag{2.5}$$

$$\boldsymbol{\theta}_t(s+1) \cdot \boldsymbol{Y}(s) = \mathbb{E}^{\mathbb{Q}} \left[\exp\left(- \int_s^{T} r(u)\,\mathrm{d}u \right) \boldsymbol{\theta}_t(T) \cdot \boldsymbol{Y}(T) \middle| \mathcal{F}_s \right], \qquad s \in \{t, t+1, \dots, T-1\}. \tag{2.6}$$

We can also specify the discounting factors $B(t, T)$, which reflect the time t value of receiving a risk free[4] payment of 1 at the time horizon T. Recalling that the financial asset with index 0 is defined to have this exact structure, we have the relationship expressed in Equation (2.7).

$$Y^{(0)}(t) = B(t, T) = \mathbb{E}^{\mathbb{Q}} \left[\exp\left(- \int_t^{T} r(u)\,\mathrm{d}u \right) \middle| \mathcal{F}_t \right], \qquad t \in \{0, 1, \dots, T\}. \tag{2.7}$$

To simplify the notation going forward, for each $t \in \{0, 1, \dots, T-1\}$ we introduce the special self financing time t trading strategy $\boldsymbol{\lambda}_t \in \Theta_t$ specified in Equation (2.8). This trading strategy is the buy-and-hold strategy of holding one unit of the asset with index 0, such that a payment of 1 is received at time T. In particular, this gives the portfolio value in Equation (2.9)

$$\boldsymbol{\lambda}_t(u) = (1, 0, \dots, 0) \in \mathbb{R}^{n+1}, \quad u \in \{t+1, \dots, T\}, \tag{2.8}$$

$$\boldsymbol{\lambda}_t(u) \cdot \boldsymbol{Y}(u) = B(u, T), \quad u \in \{t+1, \dots, T\}. \tag{2.9}$$

To illustrate the concepts introduced above, Example 1 below provides a simple example of a financial market and a trading strategy.

Example 1 *Consider the special case of the financial market when $n = 1$, i.e. where we have two tradable assets. The asset with index 0 is, as usual, assumed to be a risk free zero coupon bond paying one unit of money at maturity T, and its value at time t is expressed in Equation (2.7).*

[4]By risk free we mean absence of default risk.

The asset with index 1 is assumed to be a stock with price at time zero given by $Y^{(1)}(0) = Y_0^{(1)}$*. The dynamics of the stock price is modelled with a geometric Brownian motion with drift parameter* μ *and volatility parameter* σ*. Letting* $\{W(t)\}_{t\in[0,T]}$ *denote a standard Brownian motion (under* $\mathbb{P}$*), the stock price process can be expressed as*

$$Y^{(1)}(t) = Y_0^{(1)} \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W(t)\right), \quad t \in [0, T].$$

Next, let us consider the self financing time 0 *trading strategy* $\boldsymbol{\theta}_0 \in \Theta_0$ *that consists in buying one unit of the bond and one unit of the stock, and holding these until the time horizon* T*. That is, we consider a simple buy-and-hold strategy, where* $\boldsymbol{\theta}_0(u) = (1, 1)$ *for each* $u \in \{1, 2, \dots, T\}$*. The initial investment of this portfolio is given by*

$$\boldsymbol{\theta}_0(1) \cdot \boldsymbol{Y}(0) = B(0, T) + Y_0^{(1)},$$

and its value at the time horizon T *is given by*

$$\boldsymbol{\theta}_0(T) \cdot \boldsymbol{Y}(T) = 1 + Y_0^{(1)} \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) T + \sigma W(T)\right).$$

Notably, this final value is random as seen from time zero, and its expected value and variance are given by

$$\begin{aligned}
\mathbb{E}\left[\boldsymbol{\theta}_0(T) \cdot \boldsymbol{Y}(T)\right] &= 1 + Y_0^{(1)} e^{\mu T}, \\
\mathrm{Var}\left[\boldsymbol{\theta}_0(T) \cdot \boldsymbol{Y}(T)\right] &= \left(Y_0^{(1)}\right)^2 e^{2\mu T}\left(e^{\sigma^2 T} - 1\right).
\end{aligned}$$

2.3 t-claims

The financial-actuarial framework is assumed to contain both tradable claims and non-tradable claims. Here, tradable claims include for example claims on the assets traded in the financial market. Non-tradable claims include for example certain insurance liabilities, for which there is no market where they can be bought or sold. Notably, many types of claims are a combination of tradable and non-tradable components. To formalize what we mathematically mean by claims, we introduce the concept of t-claims in Definition 4 below.

Definition 4 *For a chosen* $t \in \{0, 1, \dots, T\}$*, a t-claim is an* $\mathcal{F}_t$*-measurable random variable, which is payable at time t. We denote the set of all t-claims as* $\mathcal{C}_t$*.*

Claims payable at the time horizon T are a special case of the above definition. These are of particular interest for us in this study, as we will consider insurance liabilities as T-claims. Henceforth, we will denote such insurance liabilities as $S \in \mathcal{C}_T$. Also, related to the notion of tradable and non-tradable claims, we introduce some special cases of T-claims, namely t-hedgeable and t-orthogonal T-claims. These are specified in Definitions 5 and 6.

Definition 5 *For a chosen $t \in \{0, 1, \ldots, T-1\}$, a T-claim $S_h \in \mathcal{C}_T$ is said to be t-hedgeable if there exists a self-financing time t trading strategy $\boldsymbol{\theta}_t \in \Theta_t$, which we call a replicating t-hedge of S_h, such that $S_h = \boldsymbol{\theta}_t(T) \cdot \boldsymbol{Y}(T)$. We denote the set of all t-hedgeable T-claims as $\mathcal{H}_T^t \subseteq \mathcal{C}_T$.*

Definition 6 *For a chosen $t \in \{0, 1, \ldots, T-1\}$, a T-claim $S_\perp \in \mathcal{C}_T$ is said to be t-orthogonal if it is $\mathbb{P}$-independent of $\{\boldsymbol{Y}(u)\}_{u \in \{t+1, \ldots, T\}}$. We denote the set of all t-orthogonal T-claims as $\mathcal{O}_T^t \subseteq \mathcal{C}_T$.*

From these definitions we find that if a T-claim is t-hedgeable, then it is also $(t+1)$-hedgeable. Similarly, if a T-claim is t-orthogonal, it must also be $(t+1)$-orthogonal. In mathematical terms, we have $\mathcal{H}_T^0 \subseteq \mathcal{H}_T^1 \subseteq \ldots \subseteq \mathcal{H}_T^{T-1}$ and $\mathcal{O}_T^0 \subseteq \mathcal{O}_T^1 \subseteq \ldots \subseteq \mathcal{O}_T^{T-1}$.

It is also worth noting that there exist no T-claims that are both t-hedgeable and t-orthogonal. However, there do exist T-claims that are neither t-hedgeable nor t-orthogonal. Some concrete examples of claims are given in Example 2 below.

Example 2 *Let us again return to the special case considered in Example 1, i.e. where the financial market consists of two assets; the usual zero-coupon bond paying 1 unit of money at time T, and a stock modelled with a geometric Brownian motion.*

Let us also consider a population of life insurance policyholders. At time zero there are N_0 policyholders alive. The number of survivors at each future time point t is denoted N_t. Given the information available at time zero, this is a random variable since some policyholders might die before time t.

(a) *The T-claim $S = Y^{(1)}(T)$, which we can interpret as a liability directly linked to the value of the stock, is clearly t-hedgeable, i.e. $S \in \mathcal{H}_T^t$, for each $t \in \{0, 1, \ldots, T-1\}$. This can be seen as the simple trading strategy of holding one unit of stock will perfectly replicate the claim.*

(b) *Next, we consider now the T-claim $S = N_T$. This can be thought of as the liability paying one unit of money to every surviving life insurance policyholder at time T. Under the assumption that policyholder death events are independent of the financial market, we find that this T-claim is t-orthogonal, i.e. $S \in \mathcal{O}_T^t$, for each $t \in \{0, 1, \ldots, T-1\}$.*

(c) *Let us now consider the slightly more sophisticated T-claim defined* *as* $S = N_T \max\big(K, Y^{(1)}(T)\big)$*, where* K *is a non-negative constant. Such a claim can be interpreted as an equity-linked insurance liability paying the value of the stock, with guaranteed level* K*, to every surviving policyholder at time* T*. It is clear that such a* T*-claim depends on both the evolution of the stock value as well as the mortality of the insured. In other words, it is affected by both equity risk and mortality risk. In extension, we find that is is neither* t*-hedgeable nor* t*-orthogonal, i.e.* $S \notin \mathcal{H}_T^t \cup \mathcal{O}_T^t$*, for each* $t \in \{0, 1, \ldots, T-1\}$*.*

2.4 t-valuations & t-hedgers

Having specified the financial market, the concept of trading strategies and t-claims, as well as some important properties of these, we are now ready to introduce t-valuations and t-hedgers. Here, for a chosen $t \in \{0, 1, \ldots, T-1\}$, a t-valuation assigns an $\mathcal{F}_t$-measurable random variable to each T-claim $S \in \mathcal{C}_T$. In other words, given the information available at time t, we assign a time t value to the T-claim. Similarly, a t-hedger assigns a self-financing time t trading strategy that is designed to hedge the future liability $S \in \mathcal{C}_T$. As will be shown, there is a natural link between t-valuations and t-hedgers.

We begin with the concept of t-valuations, as defined in Definition 7. Related to this, Definition 8 introduces some properties of certain t-valuations.

Definition 7 *For a chosen* $t \in \{0, 1, \ldots, T-1\}$*, a* t*-valuation is a function* $\varphi_t \colon \mathcal{C}_T \to \mathcal{C}_t$*, which attaches to every* T*-claim* $S \in \mathcal{C}_T$ *a* t*-claim* $\varphi_t[S] \in \mathcal{C}_t$*, such that* φ_t *is:*

- *normalized, i.e.* $\varphi_t[0] = 0$*, and*
- *translation invariant, i.e. for any* $S \in \mathcal{C}_T$ *and* $a \in \mathcal{C}_t$ *it holds that* $\varphi_t[S + a] = \varphi_t[S] + B(t,T)a$*.*

Definition 8 *Given a* t*-valuation* φ_t*, we say that:*

- φ_t *is market consistent (MC) if any* t*-hedgeable part of any* T*-claim is marked-to-market, i.e. for any* $S \in \mathcal{C}_T$ *and* $S_h \in \mathcal{H}_T^t$ *it holds that:*

$$\varphi_t[S + S_h] = \varphi_t[S] + \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(u)\,\mathrm{d}u\right) S_h \,\middle|\, \mathcal{F}_t\right].$$

- φ_t *is actuarial if any* t*-orthogonal* T*-claim is marked-to-model, i.e. for any* $S_\perp \in \mathcal{O}_T^t$ *it holds that:*

$$\varphi_t[S_\perp] = B(t,T)\pi_t[S_\perp],$$

where $\pi_t \colon \mathcal{O}_T^t \to \mathcal{C}_t$ *is a* t*-valuation that is* $\mathbb{P}$*-law invariant as well as* $\mathbb{P}$*-independent of* $\{\boldsymbol{Y}(u)\}_{u \in \{t, \ldots, T\}}$*.*

- φ_t *is fair if it is both market consistent and actuarial.*

Here, it is worth commenting on the terms *marked-to-market* and *marked-to-model*. The former is a widely used term in finance and accounting, which refers to assigning values based on current market prices. This means that a value is assigned based on the financial market players' view of current market conditions. In contrast, the latter term refers to assigning a value based on some underlying model. Such models are often used when current market prices are not available or when a market is deemed too illiquid. It can be claimed that such a valuation is somewhat more subjective since there may be numerous choices of underlying model, and the choice of model will affect the resulting value. In our framework, the function π_t is reflecting this underlying model. We will refer to π_t as the actuarial valuation function. In this general setting, we do not specify this function explicitly, however some concrete options are presented later in Section 3.4.

For a chosen T-claim $S \in \mathcal{C}_T$, we interpret $\varphi_t[S]$ as the value of S at time t. Notably, given the information available at time t, this value is deterministic. To simplify notation going forward, we introduce $\widetilde{\varphi}_t[S]$ as defined in Equation (2.10). This is the value given at time T, if the amount $\varphi_t[S]$ is invested in the asset with index 0 at time t. We will think of $\widetilde{\varphi}_t[S]$ as a T-claim, though it is worth noting that it is not only $\mathcal{F}_T$-measurable, but in fact $\mathcal{F}_t$-measurable as well.

$$\widetilde{\varphi}_t[S] = \frac{\varphi_t[S]}{B(t,T)}, \quad S \in \mathcal{C}_T. \tag{2.10}$$

Next, we introduce the notion of t-hedgers in Definition 9, as well as related properties in Definition 10.

Definition 9 *For a chosen $t \in \{0, 1, \ldots, T-1\}$, a t-hedger is a function $\vartheta_t \colon \mathcal{C}_T \to \Theta_t$, which attaches to every T-claim $S \in \mathcal{C}_T$ a self-financing time t trading strategy $\vartheta_t[S] = \boldsymbol{\theta}_{t,S} \in \Theta_t$, where $\boldsymbol{\theta}_{t,S}$ is called the t-hedge of S, such that ϑ_t is:*

(i) normalized, i.e. $\vartheta_t[0] = \boldsymbol{\theta}_{t,0} = \mathbf{0}_t$ where $\mathbf{0}_t = \{\mathbf{0}_{\boldsymbol{t}}(u)\}_{u \in \{t+1,\ldots,T\}}$ and for each $u \in \{t+1, \ldots, T\}$, $\mathbf{0}_{\boldsymbol{t}}(u)$ is the null vector in $\mathbb{R}^{n+1}$, and

(ii) translation invariant, i.e. for any $S \in \mathcal{C}_T$ and $a \in \mathcal{C}_t$ it holds that $\vartheta_t[S + a] = \boldsymbol{\theta}_{t,S+a} = \boldsymbol{\theta}_{t,S} + a\boldsymbol{\lambda}_t$.

Definition 10 *Given a t-hedger ϑ_t, we say that:*

- *ϑ_t is market consistent (MC) if any t-hedgeable part of any T-claim is marked-to-market, i.e. for any $S \in \mathcal{C}_T$ and $S_h \in \mathcal{H}_T^t$ it holds that:*

$$\vartheta_t[S + S_h] = \boldsymbol{\theta}_{t,S+S_h} = \boldsymbol{\theta}_{t,S} + \boldsymbol{\theta}_{t,S_h},$$

where $\boldsymbol{\theta}_{t,S_h}$ is a replicating t-hedge of S_h, i.e. $S_h = \boldsymbol{\theta}_{t,S_h}(T) \cdot \boldsymbol{Y}(T)$.

- *ϑ_t is actuarial, with the underlying actuarial t-valuation φ_t, if any t-orthogonal T-claim is marked-to-model, i.e. for any $S_\perp \in \mathcal{O}_T^t$ it holds that:*
$$\vartheta_t[S_\perp] = \boldsymbol{\theta}_{t,S_\perp} = \widetilde{\varphi}_t[S_\perp]\boldsymbol{\lambda}_t.$$

- *ϑ_t is fair if it is both market consistent and actuarial.*

It is worth emphasizing that we here distinguish between t-hedger and t-hedge. While the former is a function mapping a T-claim to a self-financing time t trading strategy, the latter is the actual time t trading strategy that the t-hedger takes given a certain T-claim.

Comparing Definitions 11 and 13 as well as Definitions 12 and 14, there are noticeable similarities between t-valuations and t-hedgers. In fact, we have a natural relationship between the two concepts. This relationship is formulated in Theorem 1. As a first step however, Lemma 1 lists some properties of a certain class of t-hedgers, which are essential to the proof of the theorem.

Lemma 1 *Let φ_t be a t-valuation, ϑ_t be a t-hedger, and let $\Psi_t\colon \mathcal{C}_T \to \Theta_t$ be defined by*

$$\Psi_t[S] = \boldsymbol{\psi}_{t,S} = \boldsymbol{\theta}_{t,S} + \widetilde{\rho}_t[S - \boldsymbol{\theta}_{t,S}(T)\cdot \boldsymbol{Y}(T)]\boldsymbol{\lambda}_t, \quad S \in \mathcal{C}_T.$$

Then, Ψ_t is a t-hedger, with t-hedges $\Psi_t[S] = \boldsymbol{\psi}_{t,S}$ for $S \in \mathcal{C}_T$, such that:

1. *If ϑ_t is market consistent, then Ψ_t is a market consistent and $\Psi_t[S_h] = \boldsymbol{\psi}_{t,S_h} = \boldsymbol{\theta}_{t,S_h} = \vartheta_t[S_h]$ for any $S_h \in \mathcal{H}_T^t$.*
2. *If ϑ_t is actuarial and φ_t is actuarial, then Ψ_t is actuarial, with underlying actuarial t-valuation φ_t.*
3. *If ϑ_t is fair and φ_t is actuarial, then Ψ_t is fair, with underlying actuarial t-valuation φ_t.*

Proof: See Appendix A.1 for a proof of Lemma 1.

Theorem 1 *Let φ_t be a t-valuation. Then,*

1. *φ_t is market consistent if and only if there exists a market consistent t-hedger ϑ_t^m, with t-hedges $\vartheta_t^m[S] = \boldsymbol{\theta}_{t,S}^m$ for $S \in \mathcal{C}_T$, such that $\varphi_t[S] = \boldsymbol{\theta}_{t,S}^m(t+1)\cdot \boldsymbol{Y}(t)$ for any $S \in \mathcal{C}_T$.*
2. *φ_t is actuarial if and only if there exists an actuarial t-hedger ϑ_t^a, with t-hedges $\vartheta_t^a[S] = \boldsymbol{\theta}_{t,S}^a$ for $S \in \mathcal{C}_T$, such that $\varphi_t[S] = \boldsymbol{\theta}_{t,S}^a(t+1)\cdot \boldsymbol{Y}(t)$ for any $S \in \mathcal{C}_T$.*

3. *φ_t is fair if and only if there exists a fair t-hedger ϑ_t^f, with t-hedges $\vartheta_t^f[S] = \boldsymbol{\theta}_{t,S}^f$ for $S \in \mathcal{C}_T$, such that $\varphi_t[S] = \boldsymbol{\theta}_{t,S}^f(t+1) \cdot \boldsymbol{Y}(t)$ for any $S \in \mathcal{C}_T$.*

Proof: See Appendix A.2 for a proof of Theorem 1.

As a final part of this section, Example 3 below presents some examples of t-valuations and t-hedgers.

Example 3 *Let us consider a chosen time $t \in \{0, 1, \ldots, T-1\}$, at which we have a t-hedger ϑ_t, with t-hedges $\vartheta_t[S] = \boldsymbol{\theta}_{t,S}$ for $S \in \mathcal{C}_T$, and an associated t-valuation φ_t defined as*

$$\varphi_t[S] = \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t), \quad S \in \mathcal{C}_T.$$

Now, we consider a few example setups and discuss their respective properties.

(a) *Let the t-hedger be defined by*

$$\boldsymbol{\theta}_{t,S} = \mathbb{E}\left[S \,|\, \mathcal{F}_t\right] \boldsymbol{\lambda}_t, \quad S \in \mathcal{C}_T,$$

and the associated t-valuations thus being given by

$$\varphi_t[S] = B(t,T)\mathbb{E}\left[S \,|\, \mathcal{F}_t\right], \quad S \in \mathcal{C}_T.$$

If we let the actuarial valuation function π_t be defined by the conditional expectation above, it is clear that this choice of t-hedger then is actuarial. Likewise, the t-valuation φ_t is actuarial as well.

(b) *Let us now instead consider a market consistent t-hedger, given as*

$$\boldsymbol{\theta}_{t,S} = \arg\min_{\boldsymbol{\theta}_t \in \Theta_t} \mathbb{E}\left[(S - \boldsymbol{\theta}_t(T) \cdot \boldsymbol{Y}(T))^2 \,\middle|\, \mathcal{F}_t\right], \quad S \in \mathcal{C}_T,$$

which we refer to as the quadratic minimization hedge. Notably, the associated t-valuation is also market consistent.

(c) *As a third case, we now construct a fair t-hedger (and associated fair t-valuation) by defining our t-hedger as a mix of the two cases above, namely as*

$$\boldsymbol{\theta}_{t,S} = \begin{cases} \mathbb{E}\left[S \,|\, \mathcal{F}_t\right] \boldsymbol{\lambda}_t, & \text{if } S \in \mathcal{O}_T^t, \\ \arg\min_{\boldsymbol{\theta}_t \in \Theta_t} \mathbb{E}\left[(S - \boldsymbol{\theta}_t(T) \cdot \boldsymbol{Y}(T))^2 \,\middle|\, \mathcal{F}_t\right], & \text{otherwise.} \end{cases}$$

It is clear that this choice of t-hedger satisfies the actuarial property, given our reasoning in (a). In a similar manner, it satisfies the market consistent property.

2.5 Dynamic valuations & dynamic hedgers

In the previous section, we investigated t-valuations and t-hedgers for a chosen time point $t \in \{0, 1, \ldots, T-1\}$. However, we did not require any connection between t-valuations or t-hedgers for different times. This connection will be introduced in this section, where we present the concepts of dynamic valuations and dynamic hedgers.

Here, the dynamic valuation is defined in Definition 11, with associated properties given in Definition 12. In a similar manner, the dynamic hedger is defined in Definition 13 and its related properties in Definition 14.

Definition 11 *A dynamic valuation is an ordered sequence of t-valuations $\{\varphi_t\}_{t=0}^{T-1}$.*

Definition 12 *Given a dynamic valuation $\{\varphi_t\}_{t=0}^{T-1}$, we say that:*

- *$\{\varphi_t\}_{t=0}^{T-1}$ is market consistent if every t-valuation φ_t is market consistent.*
- *$\{\varphi_t\}_{t=0}^{T-1}$ is actuarial if every t-valuation φ_t is actuarial.*
- *$\{\varphi_t\}_{t=0}^{T-1}$ is time consistent if for any $S \in \mathcal{C}_T$ and $t \in \{0, 1, \ldots, T-2\}$ it holds that $\varphi_t[S] = \varphi_t[\widetilde{\varphi}_{t+1}[S]]$.*
- *$\{\varphi_t\}_{t=0}^{T-1}$ is fair if it is market consistent, actuarial and time consistent.*

Definition 13 *A dynamic hedger is an ordered sequence of t-hedgers $\{\vartheta_t\}_{t=0}^{T-1}$.*

Definition 14 *Given a dynamic hedger $\{\vartheta_t\}_{t=0}^{T-1}$, we say that:*

- *$\{\vartheta_t\}_{t=0}^{T-1}$ is market consistent if every t-hedger ϑ_t is market consistent.*
- *$\{\vartheta_t\}_{t=0}^{T-1}$ is actuarial if every t-hedger ϑ_t is actuarial.*
- *$\{\vartheta_t\}_{t=0}^{T-1}$ is time consistent if for any $S \in \mathcal{C}_T$ and $t \in \{0, 1, \ldots, T-2\}$ it holds that $\vartheta_t[S] = \boldsymbol{\theta}_{t,S} = \boldsymbol{\theta}_{t,\widetilde{\varphi}_{t+1}[S]}$, where $\varphi_{t+1}[S] = B(t+1, T)\widetilde{\varphi}_{t+1}[S]$ is the initial investment of $\vartheta_{t+1}[S] = \boldsymbol{\theta}_{t+1,S}$, i.e.*

$$\varphi_{t+1}[S] = \boldsymbol{\theta}_{t+1,S}(t+2) \cdot \boldsymbol{Y}(t+1).$$

- *$\{\vartheta_t\}_{t=0}^{T-1}$ is fair if it is market consistent, actuarial and time consistent.*

In this dynamic framework, the term *fair* has a slightly more extended meaning than in the context of t-valuations and t-hedgers, since we here also require time consistency. The time consistent property here creates a natural connection between different time points. In particular, it ensures that the same value is assigned independent of whether a valuation is done through one or multiple steps. For a similar concept of time consistency, see for example Acciaio & Penner (2011).

Just like the definitions of t-valuations and t-hedgers could be compared, we can compare the definitions of dynamic valuations and dynamic hedgers. Furthermore, we again have a natural connection between the two concepts. This connection is specified by Theorem 2 below. This theorem is important for how we in practice can construct a fair dynamic valuation through determining a fair dynamic hedger.

Theorem 2 *A dynamic valuation $\{\varphi_t\}_{t=0}^{T-1}$ is fair if and only if there exists a fair dynamic hedger $\{\Psi_t\}_{t=0}^{T-1}$, where each t-hedger Ψ_t has t-hedges $\Psi_t[S] = \boldsymbol{\psi}_{t,S}$ for $S \in \mathcal{C}_T$, such that for every $t \in \{0, 1, \ldots, T-1\}$ and $S \in \mathcal{C}_T$ it holds that*

$$\varphi_t[S] = \boldsymbol{\psi}_{t,S}(t+1) \cdot \boldsymbol{Y}(t).$$

Proof: See Appendix A.3 for a proof of Theorem 2.

Chapter 3

Setup & Methods

In this chapter, we present the choice of setup and methods utilized in the study. This part takes its starting point in the mathematical framework outlined in Chapter 2. However, in contrast to the mathematical framework, which is rather theoretical in nature, we here adopt a more practical approach. That is, we select a representative choice of setting and present the methods and approximations needed to construct a fair dynamical valuation. In extension, this chapter lays the foundation for the numerical analysis results presented in Chapter 4.

3.1 Choice of financial-actuarial setting & T-claim

Henceforth, we limit ourselves to a specific financial-actuarial setting as well as a particular choice of T-claim. In other words, we take the general mathematical framework presented in Chapter 2 and apply it to a representative example. Here, we select the setting as related to equity-linked life insurance, where the underlying risk drivers are equity risk and mortality risk. As a first step, we specify how these underlying risk drivers fit in our mathematical framework. Thereafter, we formulate our specific choice of T-claim.

Firstly, the equity risk driver is captured through our formalization of the financial market of tradable assets. Similar to Examples 1 and 2, we here limit the financial market to contain two assets. That is, the asset with index 0 is a zero-coupon bond paying one unit of money at maturity time T, and the asset with index 1 is a stock whose value process is modelled with a geometric Brownian motion. Additionally, we consider the short rate r as constant and deterministic. To clarify, the asset value processes of this financial market are specified in Equation (3.1). Here, $Y_0^{(1)}$ is the initial price of the stock, μ is a drift parameter, σ is a volatility parameter, and

$\{W(t)\}_{t\in[0,T]}$ is a standard Brownian motion under $\mathbb{P}$.

$$\begin{cases} Y^{(0)}(t) = e^{-r(T-t)}, \\ Y^{(1)}(t) = Y_0^{(1)} \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W(t)\right), \end{cases} \quad t \in [0,T]. \tag{3.1}$$

Secondly, we specify the model for the mortality risk driver. We consider a life insurance portfolio and let N_0 denote the initial number of policyholders, all of which are alive and x years old at time zero. Moreover, we let their associated mortality intensity be denoted $\mu_x(t)$, $t \in [0,T]$. Here, the mortality intensity is assumed to be deterministic[5] and given by the Makeham formula as formulated in Equation (3.2). Here, a, b and c are constant model parameters. We denote the one-year survival probability at time t, i.e. the probability of surviving to time $t+1$ given survival to time t, as $p_x(t)$. With our chosen form for the mortality intensity, this probability is given by Equation (3.3).

$$\mu_x(t) = a + be^{c(x+t)}, \quad t \in [0,T]. \tag{3.2}$$

$$p_x(t) = \exp\left(-\int_t^{t+1} \mu_x(u)\,\mathrm{d}u\right) = \exp\left(-a - \frac{be^{c(x+t)}(e^c - 1)}{c}\right), \quad t \in [0,T]. \tag{3.3}$$

Letting N_t denote the number of surviving policyholders at time t, the number of survivors after each time step can be expressed as a sequence of conditional Binomial distributions, as specified in Equation (3.4). These random variables are assumed to be independent from the stock value process as defined in Equation (3.1)

$$N_{t+1} \,|\, N_t \sim \mathrm{Bin}(N_t, p_x(t)), \quad t \in \{0, 1 \ldots, T-1\}. \tag{3.4}$$

Finally, having defined the financial-actuarial setting, we are now ready to present our choice of T-claim. This T-claim is denoted S and is specified in Equation (3.5). Here, N_T is the number of surviving life insurance policyholders at time T, $Y^{(1)}(T)$ is the value of the stock at time T, and K is a positive constant. This type of claim can be interpreted as an equity-linked insurance liability that pays the value of a stock, with guaranteed level K, to every surviving policyholder at time T. As outlined in Example 2, this type of T-claim is neither t-hedgeable nor t-orthogonal for any $t \in \{0, 1, \ldots, T-1\}$.

$$S = N_T \max\left(K, Y^{(1)}(T)\right). \tag{3.5}$$

[5]Notably, there are more complex models, which instead model the mortality intensity as stochastic so as to capture the uncertainty in future mortality, see for example Cairns et al. (2008). However, for the scope of this study we opt to use a deterministic model.

3.2 The backwards iteration procedure

We now consider the challenge of constructing a fair dynamic valuation and applying it to the choice of T-claim S as specified in Equation (3.5). That is, given S we want to determine a dynamic valuation that is both market consistent, actuarial and time consistent. In mathematical terms, what we seek to determine is the sequence $\{\varphi_t[S]\}_{t=0}^{T-1}$. To make for easier notation[6] going forward, we will henceforth denote $\varphi_T[S] = S$. We note that $\varphi_T[S]$ is, strictly speaking, not part of the valuation sequence.

We construct the sequence $\{\varphi_t[S]\}_{t=0}^{T-1}$ using the backwards iteration procedure formulated by Barigou et al. (2019). This procedure consists in iterating over the time points, starting at time $T-1$, thereafter considering time $T-2$ and so on, to eventually arrive at time 0. At each time point $t \in \{0, 1, \ldots, T-1\}$, we first determine the t-hedge[7] $\boldsymbol{\theta}_{t,S}$, taken as the optimal quadratic hedge of $\varphi_{t+1}[S]$, i.e. as specified in Equation (3.6). Having calculated this t-hedge, we then valuate the residual, denoted $\Delta_{t+1,S}$ and specified in Equation (3.7), with an actuarial value function π_t. The t-valuation of S, i.e. $\varphi_t[S]$, is then defined as the sum of the hedge contribution and the actuarial contribution, as given in Equation (3.8).

$$\boldsymbol{\theta}_{t,S}(t+1) = \arg\min_{\boldsymbol{\theta}_t \in \Theta_t} \mathbb{E}\left[\left(\varphi_{t+1}[S] - \boldsymbol{\theta}_t(t+1) \cdot \boldsymbol{Y}(t+1)\right)^2 \,\middle|\, \mathcal{F}_t \right] =$$
$$\arg\min_{\boldsymbol{\theta}_t \in \Theta_t} \mathbb{E}\left[\left(\varphi_{t+1}[S] - \theta_t^{(0)}(t+1) e^{-r(T-t-1)} - \theta_t^{(1)}(t+1) Y^{(1)}(t+1) \right)^2 \,\middle|\, \mathcal{F}_t \right]. \tag{3.6}$$

$$\Delta_{t+1,S} = \varphi_{t+1}[S] - \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t+1), \tag{3.7}$$

$$\varphi_t[S] = \underbrace{\boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t)}_{\text{Hedge contribution}} + \underbrace{e^{-r} \pi_t \left[\Delta_{t+1,S}\right]}_{\text{Actuarial contribution}} . \tag{3.8}$$

It is worth noting that by this construction, φ_t is ensured to be both market consistent and actuarial. Furthermore, because of the backwards iteration scheme, the time consistency property is also satisfied. Consequently, $\{\varphi_t[S]\}_{t=0}^{T-1}$ is a fair dynamic valuation of S. These properties can also be seen in the light of Theorem 2.

We now take a step back and examine the expression in Equation (3.6) more closely. Notably, this is a quadratic minimization problem, to which we can

[6]This notation will be used throughout Chapters 3 and 4.

[7]It is worth noting that we are in fact only interested in estimating $\theta_{t,S}^{(1)}(t+1)$ and $\theta_{t,S}^{(0)}(t+1)$. The remaining components of $\boldsymbol{\theta}_{t,S}$ are not specified.

find an analytical solution. This fact is formulated in Proposition 1, which presents the analytical solution of the minimization problem in a slightly more general setting.

Proposition 1 *Consider a chosen $t \in \{0, 1, \dots, T-1\}$ and a deterministic constant short rate r. Let $\boldsymbol{A}_{t+1}$ and L_{t+1} be random variables taking values in $\mathbb{R}^n$ and $\mathbb{R}$ respectively. Let us denote*

$$\begin{aligned}\operatorname{Cov}\left[\boldsymbol{A}_{t+1} \,|\, \mathcal{F}_t\right] &= \mathbb{E}\left[\boldsymbol{A}_{t+1}\boldsymbol{A}_{t+1}^{\intercal} \,\middle|\, \mathcal{F}_t\right] - \mathbb{E}\left[\boldsymbol{A}_{t+1} \,|\, \mathcal{F}_t\right]\mathbb{E}\left[\boldsymbol{A}_{t+1} \,|\, \mathcal{F}_t\right]^{\intercal},\\ \operatorname{Cov}\left[\boldsymbol{A}_{t+1}, L_{t+1} \,|\, \mathcal{F}_t\right] &= \mathbb{E}\left[\boldsymbol{A}_{t+1}L_{t+1} \,|\, \mathcal{F}_t\right] - \mathbb{E}\left[\boldsymbol{A}_{t+1} \,|\, \mathcal{F}_t\right]\mathbb{E}\left[L_{t+1} \,|\, \mathcal{F}_t\right],\end{aligned}$$

and assume that the inverse of $\operatorname{Cov}\left[\boldsymbol{A}_{t+1} \,|\, \mathcal{F}_t\right]$ *exists. Then, the optimization problem*

$$\min_{a_t \in \mathbb{R},\, \boldsymbol{b}_t \in \mathbb{R}^n} \mathbb{E}\left[\left(L_{t+1} - a_t e^{-r(T-t-1)} - \boldsymbol{b}_t^{\intercal}\boldsymbol{A}_{t+1}\right)^2 \,\middle|\, \mathcal{F}_t\right],$$

has the optimal solution

$$\begin{aligned}\widehat{a}_t &= e^{r(T-t-1)}\left(\mathbb{E}\left[L_{t+1} \,|\, \mathcal{F}_t\right] - \widehat{\boldsymbol{b}}_t^{\intercal}\mathbb{E}\left[\boldsymbol{A}_{t+1} \,|\, \mathcal{F}_t\right]\right),\\ \widehat{\boldsymbol{b}}_t &= \left(\operatorname{Cov}\left[\boldsymbol{A}_{t+1} \,|\, \mathcal{F}_t\right]\right)^{-1}\operatorname{Cov}\left[\boldsymbol{A}_{t+1}, L_{t+1} \,|\, \mathcal{F}_t\right].\end{aligned}$$

Proof: See Appendix A.4 for a proof of Proposition 1.

We can now use Proposition 1 for our specific problem. From this we obtain the optimal hedge components $\theta_{t,S}^{(1)}(t+1)$ and $\theta_{t,S}^{(0)}(t+1)$ as expressed in Equations (3.9) and (3.10). In addition, we recall that the asset with index 1 is representing a stock, whose value process is modelled with a geometric Brownian motion according to Equation (3.1). Thus, its conditional expectation and conditional variance, which are both used in the expressions for the optimal hedge, can be formulated as in Equations (3.11) and (3.12).

$$\begin{aligned}\theta_{t,S}^{(1)}(t+1) &= \frac{\operatorname{Cov}\left[\varphi_{t+1}[S], Y^{(1)}(t+1) \,\middle|\, \mathcal{F}_t\right]}{\operatorname{Var}\left[Y^{(1)}(t+1) \,\middle|\, \mathcal{F}_t\right]} = \\ &\frac{\mathbb{E}\left[\varphi_{t+1}[S]Y^{(1)}(t+1) \,\middle|\, \mathcal{F}_t\right] - \mathbb{E}\left[\varphi_{t+1}[S] \,|\, \mathcal{F}_t\right]\mathbb{E}\left[Y^{(1)}(t+1) \,\middle|\, \mathcal{F}_t\right]}{\operatorname{Var}\left[Y^{(1)}(t+1) \,\middle|\, \mathcal{F}_t\right]},\end{aligned} \tag{3.9}$$

$$\theta_{t,S}^{(0)}(t+1) = e^{r(T-t-1)}\left(\mathbb{E}\left[\varphi_{t+1}[S] \,|\, \mathcal{F}_t\right] - \theta_{t,S}^{(1)}(t+1)\mathbb{E}\left[Y^{(1)}(t+1) \,\middle|\, \mathcal{F}_t\right]\right), \tag{3.10}$$

$$\mathbb{E}\left[Y^{(1)}(t+1) \,\middle|\, \mathcal{F}_t\right] = Y^{(1)}(t)e^{\mu}, \tag{3.11}$$

$$\operatorname{Var}\left[Y^{(1)}(t+1)\,\middle|\,\mathcal{F}_t\right] = \left(Y^{(1)}(t)\right)^2 e^{2\mu}\left(e^{\sigma^2}-1\right). \tag{3.12}$$

In the expressions for the optimal hedge we also find the conditional expectations $\mathbb{E}\left[\varphi_{t+1}[S]\,|\,\mathcal{F}_t\right]$ and $\mathbb{E}\left[\varphi_{t+1}[S]Y^{(1)}(t+1)\,\middle|\,\mathcal{F}_t\right]$. These pose a bigger challenge and we are generally not able to find analytical expressions for these terms. Consequently, approximation techniques will in practice be needed to calculate these conditional expectations. This is outlined more in detail in Section 3.3.

As a final point, we also take a closer look at the residual $\Delta_{t+1,S}$. Using the expressions for the optimal hedge, we find that its conditional expectation is in fact zero, as given in Equation (3.13). As a result, its conditional variance can be expressed as in Equation (3.14).

$$\mathbb{E}\left[\Delta_{t+1,S}\,|\,\mathcal{F}_t\right] = \mathbb{E}\left[\varphi_{t+1}[S]\,|\,\mathcal{F}_t\right] - \boldsymbol{\theta}_{t,S}(t+1)\cdot\mathbb{E}\left[\boldsymbol{Y}(t+1)\,|\,\mathcal{F}_t\right] = 0, \tag{3.13}$$

$$\operatorname{Var}\left[\Delta_{t+1,S}\,|\,\mathcal{F}_t\right] = \mathbb{E}\left[\Delta^2_{t+1,S}\,\middle|\,\mathcal{F}_t\right]. \tag{3.14}$$

3.3 Approximating conditional expectations

We now turn to discuss the calculation of the conditional expectations, i.e. the terms $\mathbb{E}\left[\varphi_{t+1}[S]\,|\,\mathcal{F}_t\right]$ and $\mathbb{E}\left[\varphi_{t+1}[S]Y^{(1)}(t+1)\,\middle|\,\mathcal{F}_t\right]$. As previously mentioned, we generally have no explicit formula for these terms. Consequently, we need to apply some approximation method.

One approach is to use a multinomial tree model to estimate these through simulation, see for example Černý (2004). This method is however computationally complex and quickly becomes unfeasible in practice. Another method, which is the one we opt for in this study, is to use a Least Square Monte Carlo (LSMC) technique. The LSMC method was initially introduced in Carriere (1996) and has become an important tool for the valuation of American-style options. For some examples on how this method has been applied, see Longstaff & Schwartz (2001) and Clément et al. (2002).

The idea in LSMC is to first simulate Monte Carlo (MC) scenarios for our risk drivers, i.e. the stock value and the policyholder mortality, across all time points. Thereafter, for each time point t, we regress $\varphi_{t+1}[S]$ and $\varphi_{t+1}[S]Y^{(1)}(t+1)$ on the risk driver information available at time t. From such a regression, we then get estimates for the conditional expectations. Notably, this approach ties in rather neatly with the backwards iteration procedure already introduced.

Here, we need to make a choice on what type of regression model we would like to use. That is, we need to decide whether to use e.g. a linear model, a quadratic model, or something more sophisticated. Notably, this a trade-off between bias and complexity. We also need to take into account the payoff structure of the T-claim. For instance, with our choice of T-claim S, as defined in Equation (3.5), a linear model is not well adapted to the non-linear structure imposed by the guarantee level. In Barigou et al. (2019), quadratic regression models are used to approximate the conditional expectations. That is, models as in Equations (3.15) and (3.16) are proposed. Here, $\alpha_{t,0}, \alpha_{t,1}, \alpha_{t,2}, \beta_{t,0}, \beta_{t,1}$ and $\beta_{t,2}$ are model parameters that we seek to estimate. Moreover, $\epsilon_{t,\alpha}$ and $\epsilon_{t,\beta}$ are independent error terms with zero mean and constant variance.

$$\varphi_{t+1}[S] = \alpha_{t,0} + \alpha_{t,1} N_t Y^{(1)}(t) + \alpha_{t,2} \left(N_t Y^{(1)}(t)\right)^2 + \epsilon_{t,\alpha}, \tag{3.15}$$

$$\varphi_{t+1}[S] Y^{(1)}(t+1) = \beta_{t,0} + \beta_{t,1} N_t (Y^{(1)}(t))^2 + \beta_{t,2} \left(N_t (Y^{(1)}(t))^2\right)^2 + \epsilon_{t,\beta}. \tag{3.16}$$

Given these regression models, the sum of squared residuals is minimized in order to find parameter estimates $\widehat{\alpha}_{t,0}, \widehat{\alpha}_{t,1}, \widehat{\alpha}_{t,2}, \widehat{\beta}_{t,0}, \widehat{\beta}_{t,1}$ and $\widehat{\beta}_{t,2}$. Using these, the estimates for the conditional expectation are given by Equations (3.17) and (3.18).

$$\mathbb{E}\left[\varphi_{t+1}[S] \,|\, \mathcal{F}_t\right] \approx \widehat{\alpha}_{t,0} + \widehat{\alpha}_{t,1} N_t Y^{(1)}(t) + \widehat{\alpha}_{t,2} \left(N_t Y^{(1)}(t)\right)^2, \tag{3.17}$$

$$\mathbb{E}\left[\varphi_{t+1}[S] Y^{(1)}(t+1) \,\middle|\, \mathcal{F}_t\right] \approx \widehat{\beta}_{t,0} + \widehat{\beta}_{t,1} N_t (Y^{(1)}(t))^2 + \widehat{\beta}_{t,2} \left(N_t (Y^{(1)}(t))^2\right)^2. \tag{3.18}$$

In the hope of constructing better regression models, we also consider alternative setups using smoothing spline models as well as LOESS[8] regression models. Here, the former alternative fits a smooth predictor function using penalized least squares minimization, while the latter alternative fits local regression models, i.e. for each fitting point, the fit is constructed using data observations in a chosen neighborhood of the point. For additional background and details on these models, see for example Wood (2017) and Chambers et al. (1992).

As a final remark, we note that the time point $t = 0$ here is special in the sense that N_0 and $Y^{(1)}(0)$ are deterministic. This means that, at this time point, we are not able to uniquely determine the parameters in the proposed regression models. Instead, we simply fit a constant model, i.e. calculate the averages, to estimate the conditional expectations $\mathbb{E}\left[\varphi_1[S] \,|\, \mathcal{F}_0\right] = \mathbb{E}\left[\varphi_1[S]\right]$ and $\mathbb{E}\left[\varphi_1[S] Y^{(1)}(1) \,|\, \mathcal{F}_0\right] = \mathbb{E}\left[\varphi_1[S] Y^{(1)}(1)\right]$.

[8] LOESS is short for locally estimated scatterplot smoothing.

3.4 Choice of actuarial valuation function

So far, we have not considered any explicit forms for the actuarial valuation function π_t. In this section however, we will present some specific choices that can be used in practice. A rather simple approach, which is used by Barigou et al. (2019), is the standard deviation form π_t^{SD} as specified in Equation (3.19). Here, μ_t and σ_t denote the conditional mean and conditional standard deviation of $\Delta_{t+1,S}$, which by Equations (3.13) and (3.14) are found to be given as in Equation (3.20). Moreover, α is a constant parameter.

$$\pi_t^{SD}(\Delta_{t+1,S}) = \mu_t + \alpha\sigma_t, \tag{3.19}$$

$$\mu_t = 0, \quad \sigma_t = \sqrt{\mathbb{E}\left[\Delta_{t+1,S}^2 \,\middle|\, \mathcal{F}_t\right]}. \tag{3.20}$$

Here, $\mathbb{E}\left[\Delta_{t+1,S}^2 \,\middle|\, \mathcal{F}_t\right]$ can be estimated using the LSMC technique in a similar way as done for the other conditional expectations discussed in Section 3.3. For example, if we choose a quadratic regression, we get a model as formulated in Equation (3.21). Here, similar as with previous setups, $\gamma_{t,0}, \gamma_{t,1}$ and $\gamma_{t,2}$ are parameters estimated in the regression, and $\epsilon_{t,\gamma}$ is an independent error term with zero mean and constant variance.

$$\Delta_{t+1,S}^2 = \gamma_{t,0} + \gamma_{t,1} N_t Y^{(1)}(t) + \gamma_{t,2}\left(N_t Y^{(1)}(t)\right)^2 + \epsilon_{t,\gamma}. \tag{3.21}$$

We emphasize that a problem here is that estimates for the conditional expectation are not a priori guaranteed to be non-negative. This in turn implies that the square root in Equation (3.20) might be undefined. In our implementation, we avert such errors by applying a floor of zero in the estimation. As will be presented in Chapter 4, this problem with negative values is however largely resolved by using the alternative regression setups, i.e. smoothing splines or LOESS regression models, instead of the quadratic models.

Some criticism against π_t^{SD} can be formulated. For one, the choice of numerical value for α is rather arbitrary. In addition, while π_t^{SD} is normalized and translation invariant, it does not satisfy the property of monotonicity. As a result, we consider also the alternative actuarial valuation function π_t^{CoC} as specified in Equation (3.22). Here, $\mathrm{VaR}_{t,p}$ denotes the conditional value-at-risk measure given by Equation (3.23). Moreover, η is a constant cost-of-capital rate[9] and p is a chosen confidence level. For a background

[9] A slightly more generalized approach is to consider the cost-of-capital rates as a sequence of numbers $\{\eta_t\}_{t=0}^{T-1}$, which is the case in for example Engsner et al. (2017). However, for simplicity we consider it as a single number here.

and motivation of the expression used for π_t^{CoC}, see for example Palmborg et al. (2021).

$$\pi_t^{CoC}(\Delta_{t+1,S}) = \mathrm{VaR}_{t,p}(-\Delta_{t+1,S}) - \frac{1}{1+\eta}\mathbb{E}\left[(\mathrm{VaR}_{t,p}(-\Delta_{t+1,S}) - \Delta_{t+1,S})^+ \,\middle|\, \mathcal{F}_t\right], \tag{3.22}$$

$$\mathrm{VaR}_{t,p}(-\Delta_{t+1,S}) = \operatorname{ess\,inf}\left\{y \in \mathbb{R} \,:\, \mathbb{P}\left(\Delta_{t+1,S} \leq y \,|\, \mathcal{F}_t\right) \geq 1-p\right\}. \tag{3.23}$$

As shown by Engsner et al. (2017), π_t^{CoC} has the property of being normalized, translation invariant as well as monotonic. Furthermore, the parameter η is more interpretable and can, since it is a cost-of-capital rate, be estimated from market data. Similarly, the confidence level p can be chosen in accordance with regulation requirements, e.g. $p = 0.5\%$ as in Solvency II.

The expression for π_t^{CoC} is notably more complex than that for π_t^{SD}. In particular, we cannot use the LSMC technique as easily to approximate the terms in Equation (3.22). Instead, one option is to simulate inner Monte Carlo scenarios. That is, given the information available at time t, we perform an inner sampling of N_{t+1} and $Y^{(1)}(t+1)$, which yield simulated values for the residual $\Delta_{t+1,S}$ conditioned on $\mathcal{F}_t$. However, a problem with this approach is the notable increase in computational complexity. Also, recalling that $\Delta_{t+1,S} = \varphi_{t+1}[S] - \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t+1)$, an additional issue is that we have no explicit expression for $\varphi_{t+1}[S]$ in terms of N_{t+1} and $Y^{(1)}(t+1)$. To overcome this, an extra regression model can be constructed, e.g. using a smoothing spline model, to estimate $\varphi_{t+1}[S]$ as a function of $N_{t+1}Y^{(1)}(t+1)$.

Another alternative is to make an assumption about the distribution of the residual $\Delta_{t+1,S}$. For instance, if we assume that $\Delta_{t+1,S}$ is conditionally normally distributed, the expression for $\pi_t^{CoC}(\Delta_{t+1,S})$ decomposes to a form reminiscent of π_t^{SD} as expressed in Equation (3.19). This fact is formulated in Proposition 2.

Proposition 2 *Consider a chosen $t \in \{0, 1, \ldots, T-1\}$ and assume that $\Delta_{t+1,S} \,|\, \mathcal{F}_t \sim N(\mu_t, \sigma_t^2)$. Then, $\pi_t^{CoC}(\Delta_{t+1,S})$ as specified in Equation (3.22) can be expressed as*

$$\pi_t^{CoC}(\Delta_{t+1,S}) = \mu_t + \kappa(\eta, p)\sigma_t.$$

Here, the function κ is given for any η and p as

$$\kappa(\eta, p) = \Phi^{-1}(1-p) - \frac{1}{1+\eta}\left((1-p)\Phi^{-1}(1-p) + \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\Phi^{-1}(1-p)\right)^2\right)\right),$$

where Φ is the distribution function of the standard normal distribution.

Proof: See Appendix A.5 for a proof of Proposition 2.

Thus, given such a normal distribution assumption, we can again utilize Equation (3.20) and estimate σ_t with the LSMC technique, similar to what was done previously, in order to calculate $\pi_t^{CoC}(\Delta_{t+1,S})$. This decreases the computational complexity compared to the approach of simulation inner scenarios. As a final point, we present numerical values of $\kappa(\eta, p)$ for representative choices of η and p in Table 3.1. Notably, we find that $\kappa(\eta, p)$ is not always positive.

Table 3.1: Numerical values for $\kappa(\eta, p)$.

			p		
η	10%	5%	1%	0.5%	0.1%
0	-0.05	-0.02	0.00	0.00	0.00
0.03	-0.01	0.03	0.06	0.07	0.09
0.06	0.03	0.07	0.13	0.14	0.17
0.1	0.07	0.13	0.21	0.23	0.28
0.2	0.17	0.26	0.38	0.43	0.51

3.5 Wrap up, specification of cases & numerical values

To wrap up, we recall our choice of T-claim S as specified in Equation (3.5) as well as our goal to construct a fair dynamical valuation of S, i.e. to construct the sequence $\{\varphi_t[S]\}_{t=0}^{T-1}$. To put this into practice, we conduct a numerical analysis that utilizes the setup and methods presented throughout this chapter. In this numerical analysis, we use the backwards iteration scheme outlined in Section 3.2. We analyze different setups for the regressions used in the LSMC approximations, as described in Section 3.3, as well as different choices of actuarial valuation, as introduced in Section 3.4. This is done through a selection of cases. Here, each case uses a particular combination of regression setups and actuarial valuation choice. In this report, we focus on the following 4 cases:

Case 1: Quadratic regression to approximate conditional expectations $\mathbb{E}\left[\varphi_{t+1}[S] \,|\, \mathcal{F}_t\right]$ and $\mathbb{E}\left[\varphi_{t+1}[S]Y^{(1)}(t+1) \,\middle|\, \mathcal{F}_t\right]$ + Standard deviation based actuarial valuation π_t^{SD} using a quadratic regression to approximate $\mathbb{E}\left[\Delta_{t+1,S}^2 \,\middle|\, \mathcal{F}_t\right]$.

Case 2: Quadratic regression to approximate conditional expectations $\mathbb{E}\left[\varphi_{t+1}[S] \,|\, \mathcal{F}_t\right]$ and $\mathbb{E}\left[\varphi_{t+1}[S]Y^{(1)}(t+1) \,\middle|\, \mathcal{F}_t\right]$ + Standard deviation based actuarial valuation π_t^{SD} using a LOESS regression[10]

[10]With smoothing parameter 0.1 and polynomial degree 2.

to approximate $\mathbb{E}\left[\Delta^2_{t+1,S} \,\middle|\, \mathcal{F}_t\right]$.

Case 3: Smoothing spline regression[11] to approximate conditional expectations $\mathbb{E}\left[\varphi_{t+1}[S] \,|\, \mathcal{F}_t\right]$ and $\mathbb{E}\left[\varphi_{t+1}[S]Y^{(1)}(t+1) \,\middle|\, \mathcal{F}_t\right]$ + Standard deviation based actuarial valuation π_t^{SD} using a LOESS regression[10] to approximate $\mathbb{E}\left[\Delta^2_{t+1,S} \,\middle|\, \mathcal{F}_t\right]$.

Case 4: Smoothing spline regression[11] to approximate conditional expectations $\mathbb{E}\left[\varphi_{t+1}[S] \,|\, \mathcal{F}_t\right]$ and $\mathbb{E}\left[\varphi_{t+1}[S]Y^{(1)}(t+1) \,\middle|\, \mathcal{F}_t\right]$ + Cost-of-capital based actuarial valuation π_t^{CoC} using the inner simulation approach[12] to estimate $\pi_t^{CoC}(\Delta_{t+1,S})$.

This set of cases forms the foundation for the results presented in Chapter 4. Furthermore, these cases constitute a natural order for attempts of modeling enhancements. In other words, each case tries to improve on the previous one. Notably, Case 1 uses a very similar setup to that suggested by Barigou et al. (2019), thus making it a suitable starting point for our analysis.

Here, we note that the Case 3 setup is in fact analogoues to using an actuarial valuation π_t^{CoC} under the assumption that the residual risk is conditionally normally distributed, as given by Proposition 2. In particular, the setups are identical if $\alpha = \kappa(\eta, p)$. As a result, we use this relation in the specification of the numerical parameter values, so as to make for more comparable results.

The parameter values used in the numerical analysis are listed in Table 3.2. Given this set of parameter values, it is worth noting that the simulation of underlying risk drivers, i.e. the stock value and the survival of the insured, is the same across all cases. This sampling is based on the dynamics specified in Equations (3.1) and (3.4). Simulated trajectories for these are visualized in Figure 3.1.

The numerical analysis is conducted through a code implementation in `R`. All computations are done using `R`'s default libraries `base` and `stats`. Additionally, we use the libraries `ggplot2`, `ggExtra` and `scales` for plotting. Here, all considered cases have fairly similar implementations. As an example, Appendix B presents the code used to construct the fair dynamical valuation in Case 3. We run the calculations on a computer with 8 Intel(R) Core(TM) i7-8550U 1.80GHz processors.

[11]With 10 degrees of freedom

[12]Including a smoothing spline regression with 50 degrees of freedom to estimate $\varphi_{t+1}[S]$ as a function of $N_{t+1}Y^{(1)}(t+1)$.

Table 3.2: Numerical values

Parameter	Description	Value
N	Number of MC scenarios	50000
T	Time horizon [years]	10
r	Risk free short rate	0.01
$Y_0^{(1)}$	Stock initial price	1
μ	Stock drift parameter	0.02
σ	Stock volatility parameter	0.1
K	Guarantee level	1
N_0	Initial number of insured	1000
x	Initial age of insured [years]	60
a	Mortality intensity parameter[13]	1e-3
b	Mortality intensity parameter[13]	1.2e-5
c	Mortality intensity parameter[13]	0.101314
α	Standard deviation factor	$\kappa(0.06, 0.005)$
η	Cost-of-capital rate[14]	0.06
p	Confidence level[15]	0.005

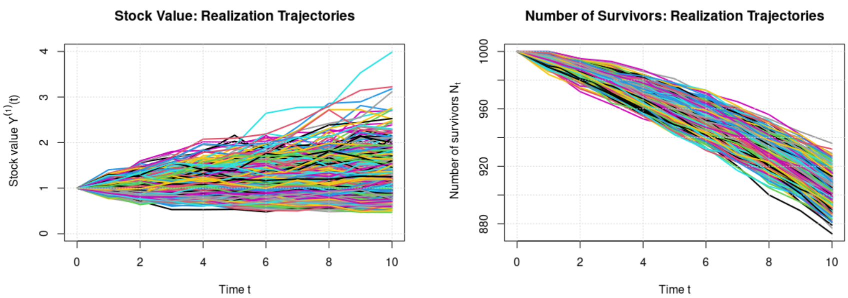

Figure 3.1: Simulated trajectories for: *(left)* stock value, and *(right)* number of surviving policyholders.

[13]These values are obtained from the Swedish mortality table M90 for males.
[14]This is the cost-of-capital rate used in Solvency II.
[15]This is the confidence level used in Solvency II.

Chapter 4

Results

This chapter presents the results of the numerical analysis, based on the cases and numerical parameter values specified in Section 3.5. Here, Sections 4.1-4.4 are all structured in a similar way. That is, we visualize the regression fits, structure of the hedge components and residuals as well as the resulting valuations. Plots are shown for time points $t = 9$, $t = 5$ and $t = 1$, which are deemed representative. Moreover, Section 4.5 compares the results between the different cases.

4.1 Case 1

Figures 4.1 and 4.2 present the regression fits for the estimation of the conditional expectations $\mathbb{E}\left[\varphi_{t+1}[S] \,\middle|\, \mathcal{F}_t\right]$ and $\mathbb{E}\left[\varphi_{t+1}[S]Y^{(1)}(t+1) \,\middle|\, \mathcal{F}_t\right]$. These are then used to determine the hedge components $\theta_t^{(1)}(t+1)$ and $\theta_t^{(0)}(t+1)$, which are illustrated in Figures 4.3 and 4.4. Next, the residuals and regression fits for the squared residuals are presented in Figures 4.5 and 4.6 respectively. Combining these results, we get the valuations as shown in Figure 4.9, with hedge contributions and actuarial contributions presented in Figures 4.7 and 4.8.

Here, a number of observations can be made. First, the quality of the regression fit for $\varphi_{t+1}[S]$ at time $t = 9$ in Figure 4.1 is somewhat questionable, and can potentially cause problems in the estimation of the conditional expectations. Also, we find slightly odd shapes of the hedge components. That is, we observe very large positions (both long and short) in certain intervals. Moving on, we note that the quadratic regressions used for the squared residual risk $\Delta_{t+1,S}^2$ shown in Figure 4.6 do not appear to give a good fit. In particular, we find that that some estimates for $\mathbb{E}\left[\Delta_{t+1,S}^2 \,\middle|\, \mathcal{F}_t\right]$ are negative, which causes the actuarial valuation to be undefined (and thus set to zero) in such cases, which can be seen in Figure 4.8.

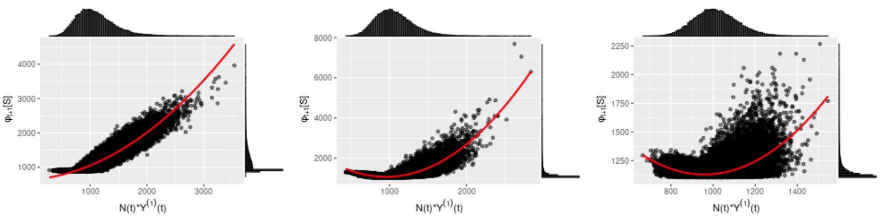

Figure 4.1: Case 1 - Regression fit of $\varphi_{t+1}[S]$ as a function of $N_t Y^{(1)}(t)$ at: *(left)* $t = 9$, *(middle)* $t = 5$, and *(right)* $t = 1$.

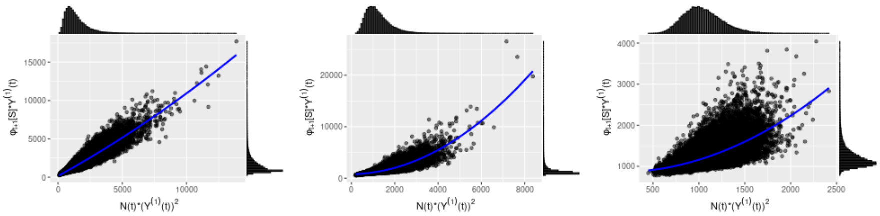

Figure 4.2: Case 1 - Regression fit of $\varphi_{t+1}[S]Y^{(1)}(t+1)$ as a function of $N_t \left(Y^{(1)}(t)\right)^2$ at: *(left)* $t = 9$, *(middle)* $t = 5$, and *(right)* $t = 1$.

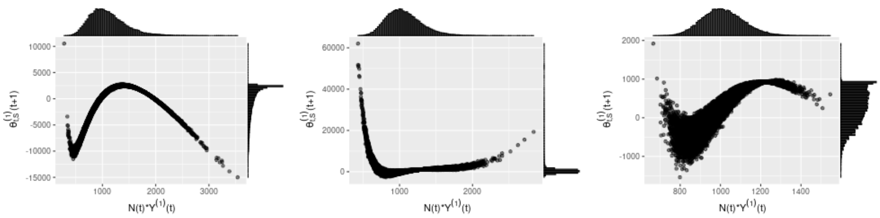

Figure 4.3: Case 1 - Hedge component $\theta^{(1)}_{t,S}(t+1)$ as a function of $N_t Y^{(1)}(t)$ at: *(left)* $t = 9$, *(middle)* $t = 5$, and *(right)* $t = 1$.

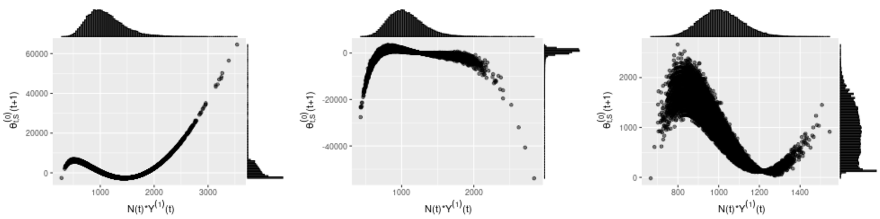

Figure 4.4: Case 1 - Hedge component $\theta^{(0)}_{t,S}(t+1)$ as a function of $N_t Y^{(1)}(t)$ at: *(left)* $t = 9$, *(middle)* $t = 5$, and *(right)* $t = 1$.

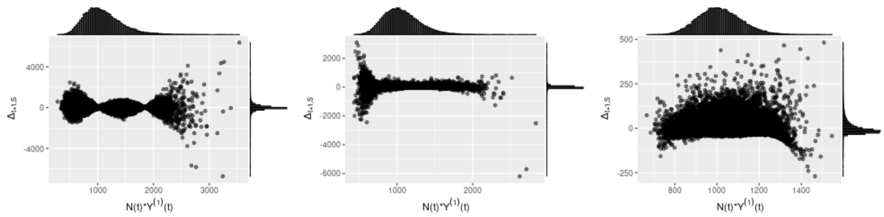

Figure 4.5: Case 1 - Residual $\Delta_{t+1,S}$ as a function of $N_tY^{(1)}(t)$ at: *(left)* $t = 9$, *(middle)* $t = 5$, and *(right)* $t = 1$.

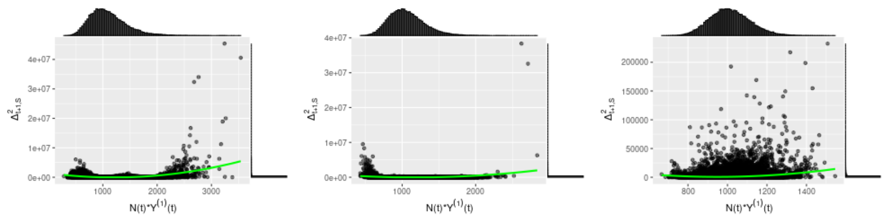

Figure 4.6: Case 1 - Regression fit of $\Delta^2_{t+1,S}$ as a function of $N_tY^{(1)}(t)$ at: *(left)* $t = 9$, *(middle)* $t = 5$, and *(right)* $t = 1$.

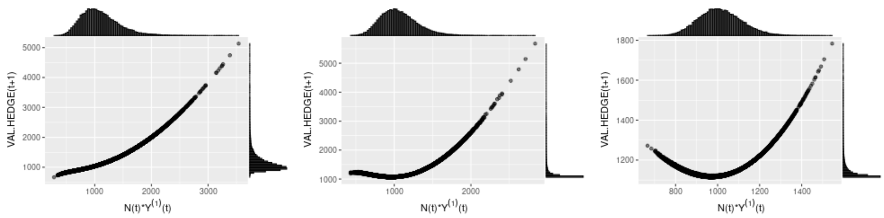

Figure 4.7: Case 1 - Hedge contribution of valuation $\varphi_t[S]$ as a function of $N_tY^{(1)}(t)$ at: *(left)* $t = 9$, *(middle)* $t = 5$, and *(right)* $t = 1$.

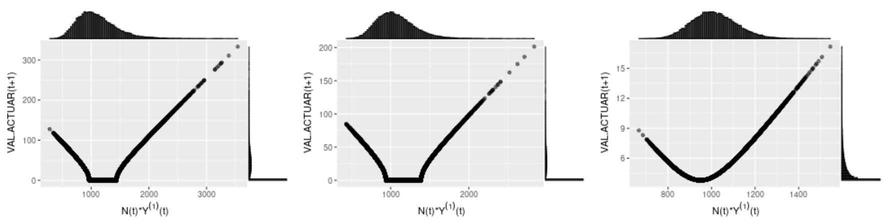

Figure 4.8: Case 1 - Actuarial contribution of valuation $\varphi_t[S]$ as a function of $N_tY^{(1)}(t)$ at: *(left)* $t = 9$, *(middle)* $t = 5$, and *(right)* $t = 1$.

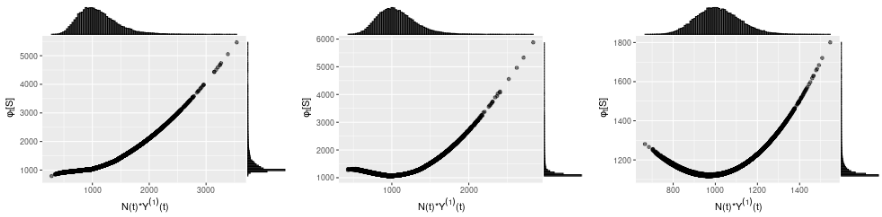

Figure 4.9: Case 1 - Valuation $\varphi_t[S]$ as a function of $N_t Y^{(1)}(t)$ at: *(left)* $t = 9$, *(middle)* $t = 5$, and *(right)* $t = 1$.

4.2 Case 2

Following the same outline as the previous Section, Figures 4.10 and 4.11 present the regression fits used to estimate the conditional expectations, which in turn are used to calculate the hedge components illustrated in Figures 4.12 and 4.13. The residual plots and squared residual fits are shown in Figures 4.14 and 4.15. Finally, the valuations, as well as corresponding hedge contributions and actuarial contributions, are presented in Figures 4.16-4.18.

We can again make a number of observations from these plots. In Figure 4.1 we identify an issue with the regression fit for $\varphi_{t+1}[S]$ at time $t = 9$, similar to what we observed for Case 1. We also notice similar shapes for the hedge components. Next, for the regression fit of $\Delta^2_{t+1,S}$ we observe a slightly better fit using the LOESS regression model, compared to the previous quadratic model. In particular, the LOESS regression setup does not produce any negative estimates for $\mathbb{E}\left[\Delta^2_{t+1,S} \,\middle|\, \mathcal{F}_t\right]$. From comparing Figures 4.14 and 4.17, we observe that the actuarial contribution is higher in the intervals where the residual variance appears higher.

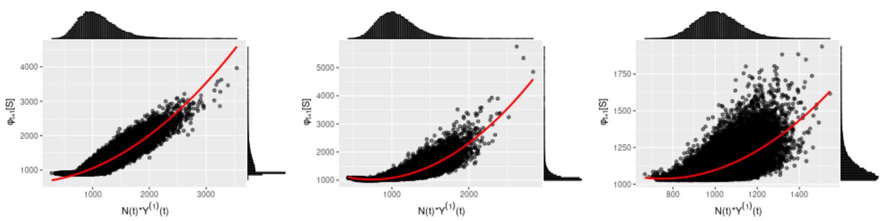

Figure 4.10: Case 2 - Regression fit of $\varphi_{t+1}[S]$ as a function of $N_t Y^{(1)}(t)$ at: *(left)* $t = 9$, *(middle)* $t = 5$, and *(right)* $t = 1$.

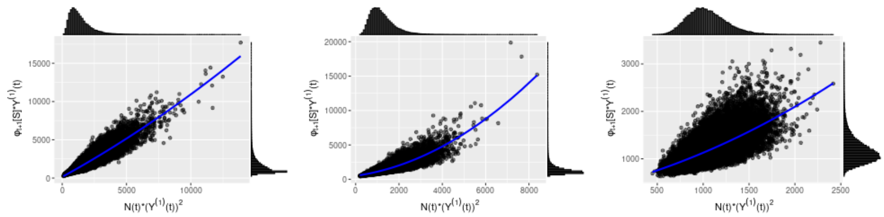

Figure 4.11: Case 2 - Regression fit of $\varphi_{t+1}[S]Y^{(1)}(t+1)$ as a function of $N_t\left(Y^{(1)}(t)\right)^2$ at: *(left)* $t=9$, *(middle)* $t=5$, and *(right)* $t=1$.

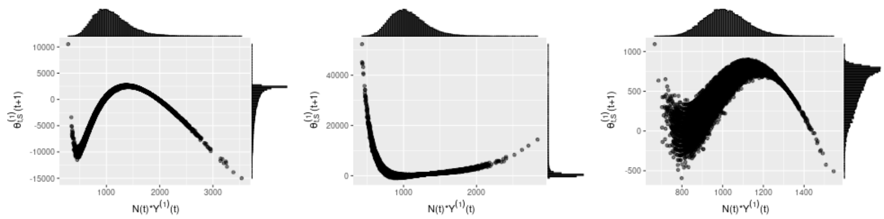

Figure 4.12: Case 2 - Hedge component $\theta^{(1)}_{t,S}(t+1)$ as a function of $N_tY^{(1)}(t)$ at: *(left)* $t=9$, *(middle)* $t=5$, and *(right)* $t=1$.

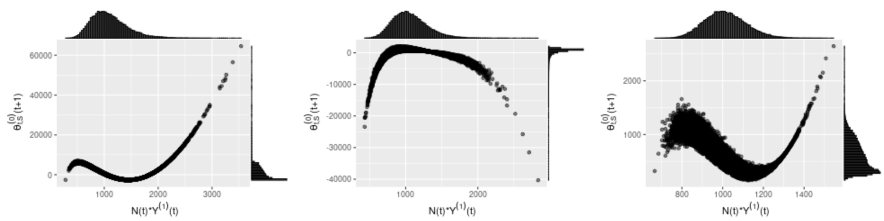

Figure 4.13: Case 2 - Hedge component $\theta^{(0)}_{t,S}(t+1)$ as a function of $N_tY^{(1)}(t)$ at: *(left)* $t=9$, *(middle)* $t=5$, and *(right)* $t=1$.

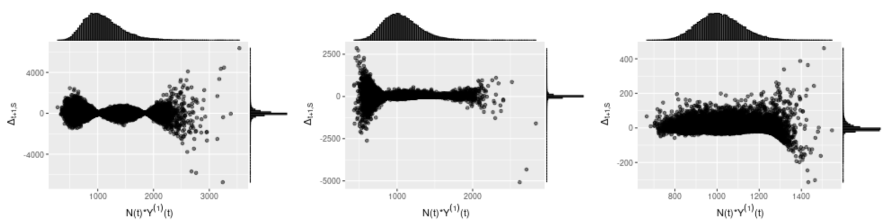

Figure 4.14: Case 2 - Residual $\Delta_{t+1,S}$ as a function of $N_tY^{(1)}(t)$ at: *(left)* $t=9$, *(middle)* $t=5$, and *(right)* $t=1$.

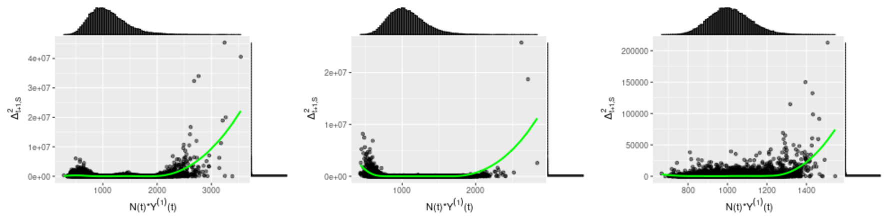

Figure 4.15: Case 2 - Regression fit of $\Delta^2_{t+1,S}$ as a function of $N_tY^{(1)}(t)$ at: *(left)* $t = 9$, *(middle)* $t = 5$, and *(right)* $t = 1$.

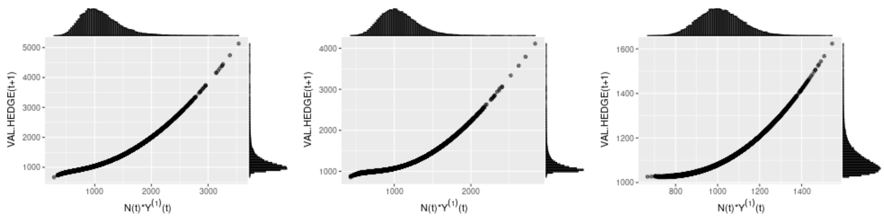

Figure 4.16: Case 2 - Hedge contribution of valuation $\varphi_t[S]$ as a function of $N_tY^{(1)}(t)$ at: *(left)* $t = 9$, *(middle)* $t = 5$, and *(right)* $t = 1$.

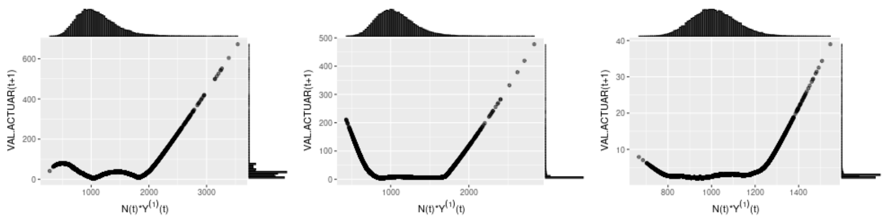

Figure 4.17: Case 2 - Actuarial contribution of valuation $\varphi_t[S]$ as a function of $N_tY^{(1)}(t)$ at: *(left)* $t = 9$, *(middle)* $t = 5$, and *(right)* $t = 1$.

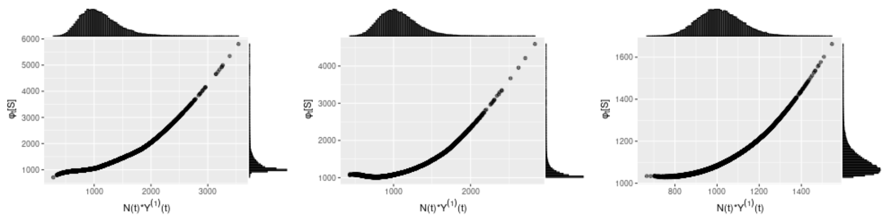

Figure 4.18: Case 2 - Valuation $\varphi_t[S]$ as a function of $N_tY^{(1)}(t)$ at: *(left)* $t = 9$, *(middle)* $t = 5$, and *(right)* $t = 1$.

4.3 Case 3

For Case 3, the regression fits are shown in Figures 4.19 and 4.20. Moreover, the hedge components are presented in Figures 4.21 and 4.22, from which we get the residual related plots in Figures 4.23 and 4.24. The valuations and associated contributions are shown in Figures 4.25-4.27.

For this case, we find that using the smoothing splines setup for the regressions of $\varphi_{t+1}[S]$, as shown in Figure 4.19, appears to provide better fits compared to the previous quadratic models. We also find that the shapes of the hedge components, as seen in Figures 4.21 and 4.22, are rather different compared to Cases 1 and 2. In extension, we also find different shapes for the residuals. Here, we do however notice a number of outliers that later cause issues in the regression of the squared residuals. Similar to Case 2, we observe that the actuarial contributions are higher in intervals where the residual is seen to have higher variance.

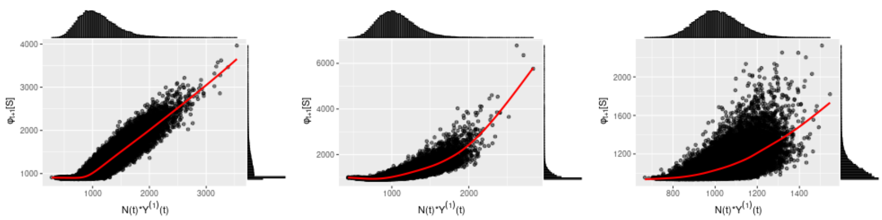

Figure 4.19: Case 3 - Regression fit of $\varphi_{t+1}[S]$ as a function of $N_t Y^{(1)}(t)$ at: *(left)* $t = 9$, *(middle)* $t = 5$, and *(right)* $t = 1$.

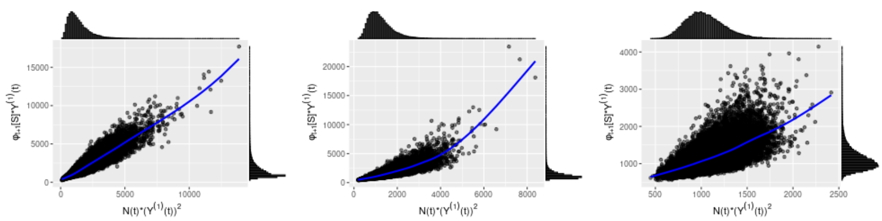

Figure 4.20: Case 3 - Regression fit of $\varphi_{t+1}[S]Y^{(1)}(t+1)$ as a function of $N_t \left(Y^{(1)}(t)\right)^2$ at: *(left)* $t = 9$, *(middle)* $t = 5$, and *(right)* $t = 1$.

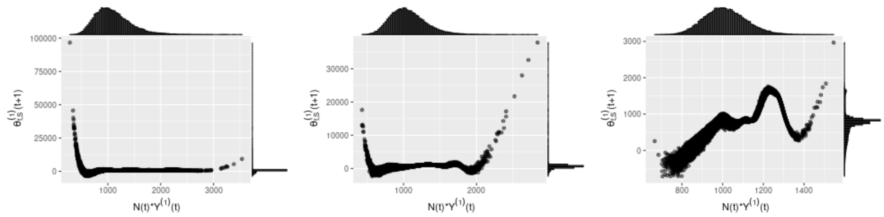

Figure 4.21: Case 3 - Hedge component $\theta^{(1)}_{t,S}(t+1)$ as a function of $N_tY^{(1)}(t)$ at: *(left)* $t = 9$, *(middle)* $t = 5$, and *(right)* $t = 1$.

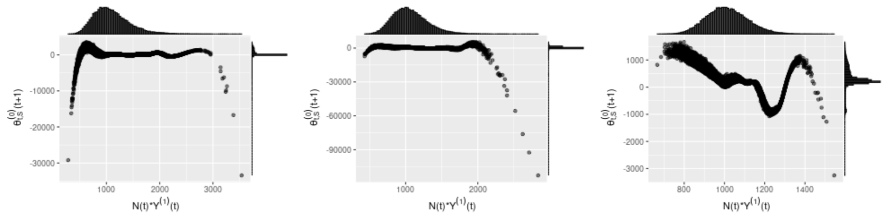

Figure 4.22: Case 3 - Hedge component $\theta^{(0)}_{t,S}(t+1)$ as a function of $N_tY^{(1)}(t)$ at: *(left)* $t = 9$, *(middle)* $t = 5$, and *(right)* $t = 1$.

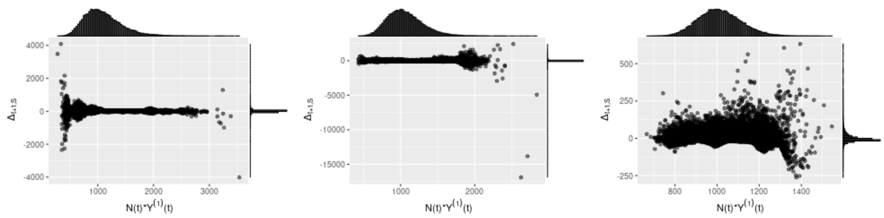

Figure 4.23: Case 3 - Residual $\Delta_{t+1,S}$ as a function of $N_tY^{(1)}(t)$ at: *(left)* $t = 9$, *(middle)* $t = 5$, and *(right)* $t = 1$.

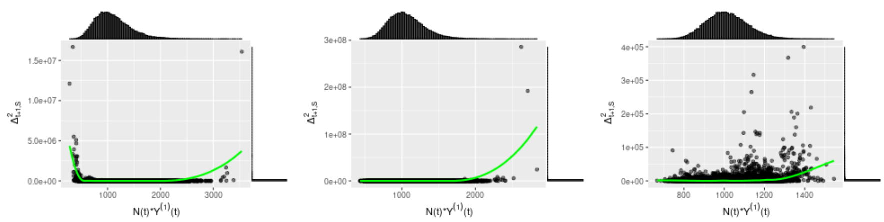

Figure 4.24: Case 3 - Regression fit of $\Delta^2_{t+1,S}$ as a function of $N_tY^{(1)}(t)$ at: *(left)* $t = 9$, *(middle)* $t = 5$, and *(right)* $t = 1$.

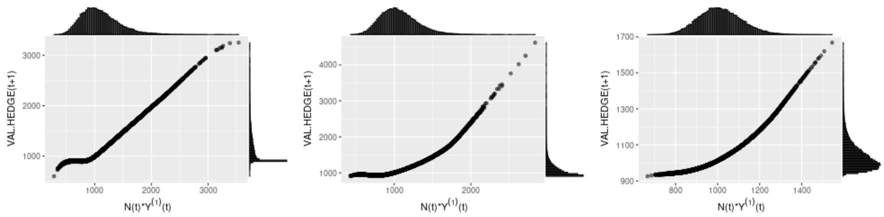

Figure 4.25: Case 3 - Hedge contribution of valuation $\varphi_t[S]$ as a function of $N_t Y^{(1)}(t)$ at: *(left)* $t = 9$, *(middle)* $t = 5$, and *(right)* $t = 1$.

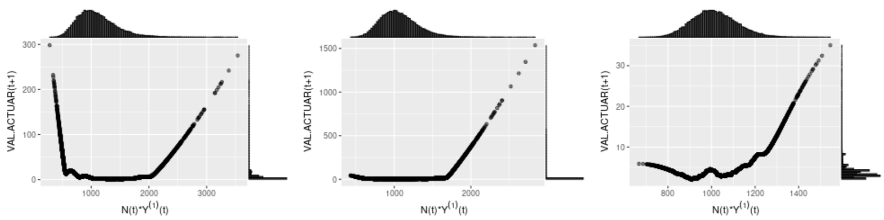

Figure 4.26: Case 3 - Actuarial contribution of valuation $\varphi_t[S]$ as a function of $N_t Y^{(1)}(t)$ at: *(left)* $t = 9$, *(middle)* $t = 5$, and *(right)* $t = 1$.

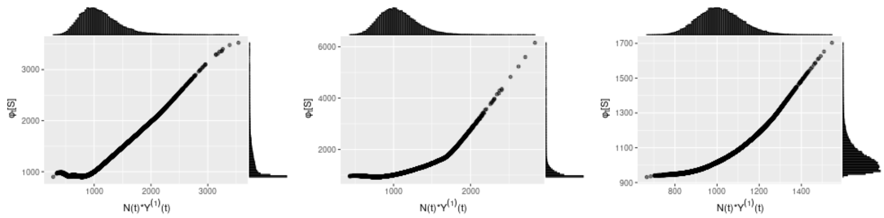

Figure 4.27: Case 3 - Valuation $\varphi_t[S]$ as a function of $N_t Y^{(1)}(t)$ at: *(left)* $t = 9$, *(middle)* $t = 5$, and *(right)* $t = 1$.

4.4 Case 4

Following a similar structure as for previous cases, we present the regression fits in Figures 4.28 and 4.29. The hedge components are shown in Figures 4.30 and 4.31, which yield the residuals in Figure 4.32. Notably, due the usage of the inner simulation technique, no regression models for the squared residuals are constructed. The hedge contributions and actuarial contributions are shown in Figures 4.33 and 4.34, with the total valuations shown in Figure 4.35.

Overall, we find the results here to be similar to those obtained in Case 3. That is, the regressions appear to give a similar fit and the hedge components are found to have a similar structure. We do notice a slight difference in the shape of the actuarial contributions to the valuation. However, given its small size relative to the hedge contributions, the total valuations appear quite similar.

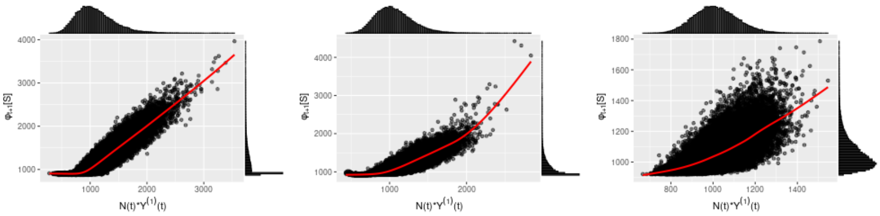

Figure 4.28: Case 4 - Regression fit of $\varphi_{t+1}[S]$ as a function of $N_t Y^{(1)}(t)$ at: *(left)* $t = 9$, *(middle)* $t = 5$, and *(right)* $t = 1$.

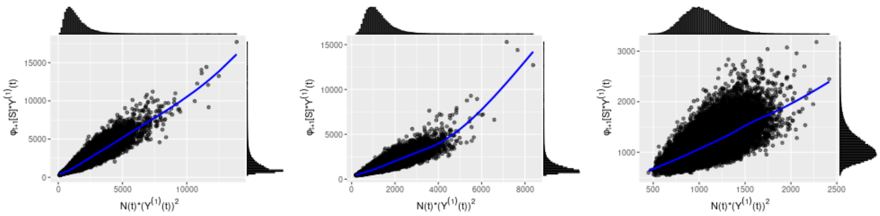

Figure 4.29: Case 4 - Regression fit of $\varphi_{t+1}[S]Y^{(1)}(t+1)$ as a function of $N_t \left(Y^{(1)}(t)\right)^2$ at: *(left)* $t = 9$, *(middle)* $t = 5$, and *(right)* $t = 1$.

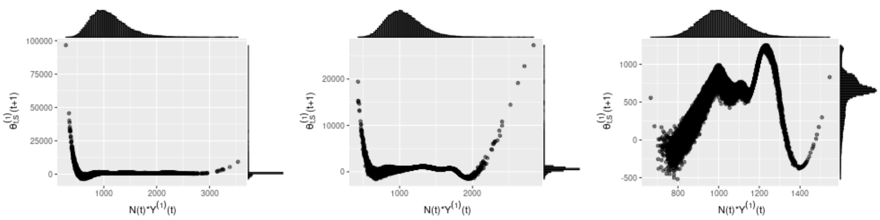

Figure 4.30: Case 4 - Hedge component $\theta^{(1)}_{t,S}(t+1)$ as a function of $N_t Y^{(1)}(t)$ at: *(left)* $t = 9$, *(middle)* $t = 5$, and *(right)* $t = 1$.

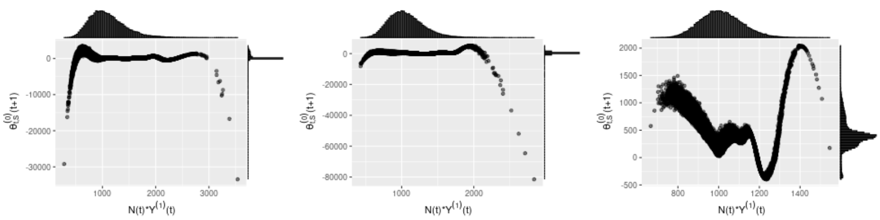

Figure 4.31: Case 4 - Hedge component $\theta^{(0)}_{t,S}(t+1)$ as a function of $N_t Y^{(1)}(t)$ at: *(left)* $t = 9$, *(middle)* $t = 5$, and *(right)* $t = 1$.

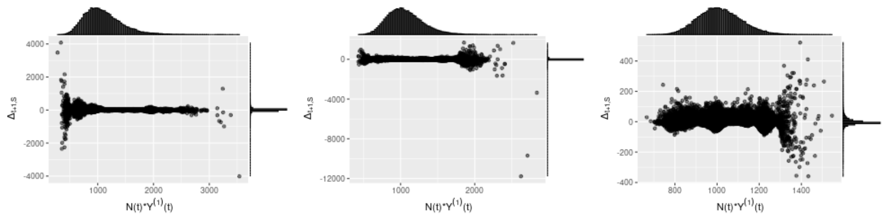

Figure 4.32: Case 4 - Residual $\Delta_{t+1,S}$ as a function of $N_t Y^{(1)}(t)$ at: *(left)* $t = 9$, *(middle)* $t = 5$, and *(right)* $t = 1$.

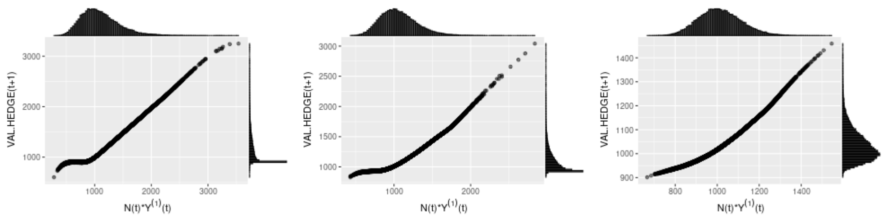

Figure 4.33: Case 4 - Hedge contribution of valuation $\varphi_t[S]$ as a function of $N_t Y^{(1)}(t)$ at: *(left)* $t = 9$, *(middle)* $t = 5$, and *(right)* $t = 1$.

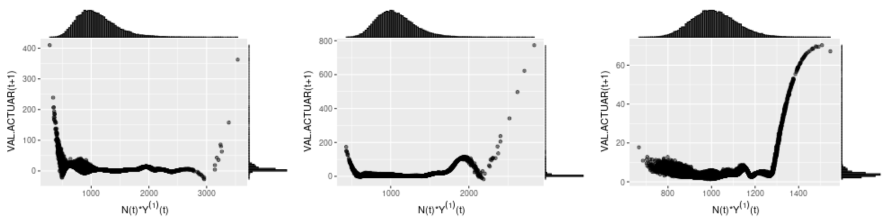

Figure 4.34: Case 4 - Actuarial contribution of valuation $\varphi_t[S]$ as a function of $N_t Y^{(1)}(t)$ at: *(left)* $t = 9$, *(middle)* $t = 5$, and *(right)* $t = 1$.

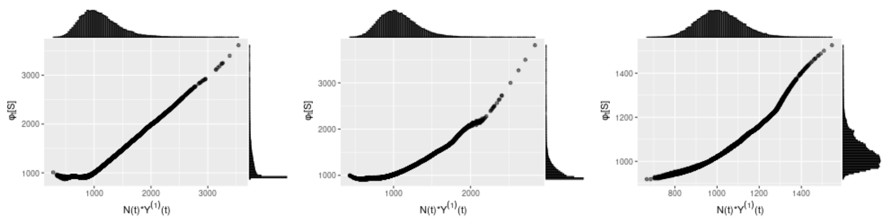

Figure 4.35: Case 4 - Valuation $\varphi_t[S]$ as a function of $N_t Y^{(1)}(t)$ at: *(left)* $t = 9$, *(middle)* $t = 5$, and *(right)* $t = 1$.

4.5 Comparison of cases

Given the results from each case, we now turn to examine some more aggregate numbers in order to compare the cases. Here, a fundamental component is the structure of the dynamic valuations over time, i.e. to analyze $\varphi_t[S]$ as a function of time t. Notably, given the information available at time zero, $\varphi_0[S]$ is deterministic while the future valuations $\varphi_1[S], \ldots, \varphi_T[S]$ are stochastic. As a result, we first examine the means of valuations, i.e. $\mathbb{E}[\varphi_t[S]]$, as presented in Figure 4.36. Here, Case 1 is seen to have the overall highest values. Case 2 has slightly lower mean valuations, though still higher than those obtained for Cases 3 and 4, which are seen to be almost identical. It is worth noting that all mean valuations intersect at the time horizon $T = 10$ years. This is due to the fact that $\varphi_T[S] = S$ is the same for all cases.

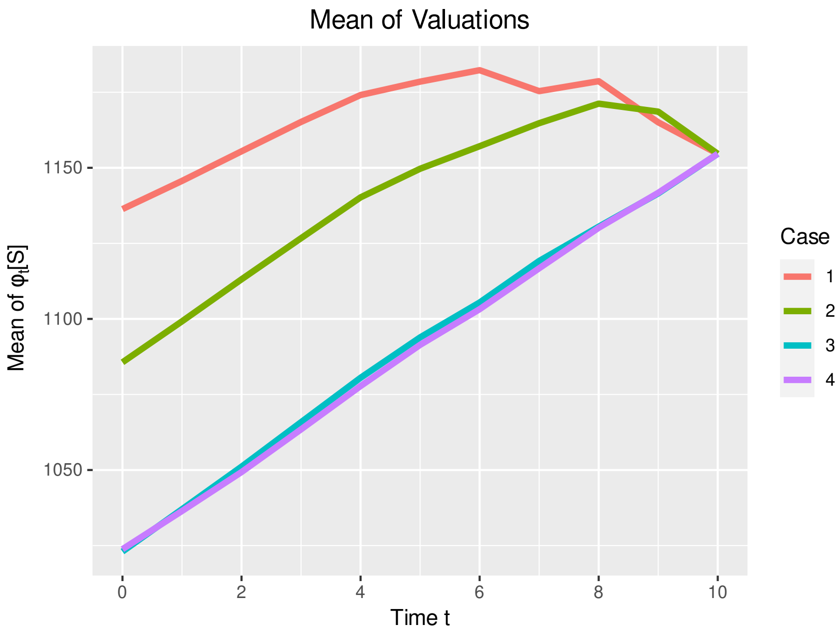

Figure 4.36: Comparison of cases - Means of valuations $\varphi_t[S]$ across time.

To analyze further, the means of the underlying hedge contributions and actuarial contributions are shown in Figure 4.37. Here, it is worth noting that we have no contribution decompositions at time T. We find the hedge contributions to be higher than the actuarial contributions in all cases. Additionally, we observe mostly higher actuarial contributions for Cases 1 and 2 compared to Cases 3 and 4. From the backwards iteration procedure used to construct the dynamic fair valuation, these higher actuarial contributions propagate backwards, resulting in higher hedge contributions and higher overall valuations for earlier times.

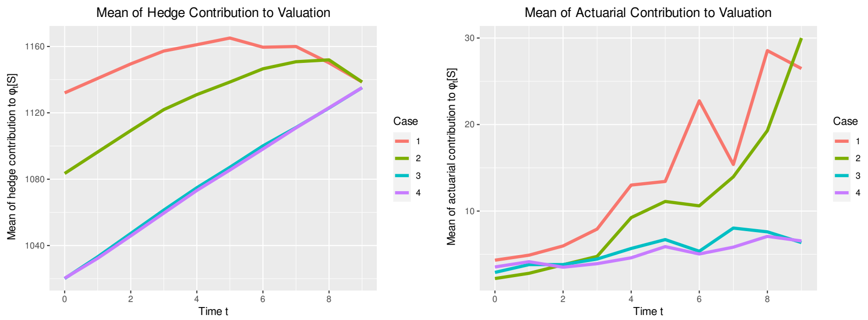

Figure 4.37: Comparison of cases - Means of contributions to valuations $\varphi_t[S]$ across time: *(left)* hedge contributions, and *(right)* actuarial contributions.

In order to investigate the valuations beyond the means, we take a closer look at the results for Case 3. Its fair dynamic valuation is illustrated in Figure 4.38, where the solid line represents the mean valuations (i.e. the same as shown in Figure 4.36) and the shaded area shows an 80% confidence interval, i.e. with lower and upper bounds given by the 10% and 90% percentiles respectively. In addition, the dashed lines illustrate three simulated trajectories for the fair dynamic valuation. As expected, we here observe $\varphi_0[S]$ to be deterministic, while the future valuations are stochastic as seen from time zero. Moreover, the confidence interval is seen to widen for later time points. This reflects the increase in uncertainty the further we look into the future from time zero.

As a last point, Table 4.1 presents the computation times[16] needed to calculate the fair dynamic valuation in each case. Here, we find that Case 4 require significantly more time than the first three cases. This difference is due to the inner Monte Carlo simulation technique employed in Case 4.

Table 4.1: Computation time in each case.

Case	Computation Time
1	4 seconds
2	33 seconds
3	35 seconds
4	1.4 hours

[16] Faster computations can potentially be achieved by employing parallel programming techniques. For the scope of this study, we have not investigated this.

Figure 4.38: Case 3 fair dynamic valuation: The solid line shows the mean, the shaded area the 80% two-sided confidence interval, and the dashed lines three simulated trajectories.

Chapter 5

Discussion

In this chapter, we present a discussion of the study. More specifically, we discuss the obtained results and what inferences can be made from these. We also reflect back on the scope and objectives of the project as a whole. In addition, we give a number of suggestions for future work.

From the results presented in Section 4.5 it is seen that the constructed fair dynamic valuations vary between the cases. Examining the mean valuation plots in Figures 4.36 and 4.37, we find that the actuarial valuations in Cases 1 and 2 are higher than those in Cases 3 and 4. In extension, this results in higher overall valuations for Cases 1 and 2. We find that this is largely due to the choice of regression models used in the LSMC approximation. That is, we recall that the first two cases utilize quadratic models, while the last two cases instead use smoothing splines models. Here, the smoothing splines option provides more flexibility in the regressions and thus better fits, e.g. as seen by comparing Figures 4.10 and 4.19. These improved regression models make for better hedges and lower residual variance, which in turn yield lower actuarial contributions and lower valuations overall. In this sense, the higher valuations obtained in Cases 1 and 2 are a consequence of sub-optimal hedges obtained from the use of the quadratic regression models. Notably, the quadratic regression setup is also problematic for the squared residual regressions, for which it is found to produce some negative estimates. In this sense, we claim to have identified a weakness in the quadratic regression approaches suggested by Barigou et al. (2019). We here identify the regression setups with smoothing splines and LOESS regression models as promising alternatives, which provide additional flexibility while still remaining computationally time-efficient.

Next, we can observe that the regressions for the squared residuals, as visualized in Figures 4.6, 4.15 and 4.24, are found to be rather problematic in all cases. Despite the better fits using the LOESS models, there are still some outliers that cause problems in the regressions. This in turn causes a few high estimates in the actuarial contribution to the valuation, as seen from Figures 4.17 and 4.26. We note this weakness in the model and recognize it as a component where further research is warranted.

It is also worth discussing the results of Cases 3 and 4 more closely. We recall that the setup in Case 3 is equivalent to using the cost-of-capital choice of actuarial valuation function, under the assumption that the residuals are conditionally normally distributed. Notably, Case 4 uses the cost-of-capital approach as well, however does not make any assumption for the distribution of the residuals. Because of this, the comparison of these two cases partly assesses if the normal distribution assumption is reasonable. We also get an indication on whether the increased computational complexity of Case 4 is justifiable, i.e. whether we can achieve more appropriate results with this method. From the valuation means presented in Figure 4.36 we find that there is little difference between the two cases. However, if we examine the actuarial contributions more in detail, we can notice some differences, as seen in the right plot of Figure 4.37 as well as by comparing Figures 4.26 and 4.34. Nevertheless, the actuarial contributions are notably smaller than the hedge contributions, which causes the valuations to still be very similar. As a result, given the overall similarity in results and the notable difference in computational complexity, we find Case 3 to be the preferable choice.

Next, we return to reflect on the overall scope and objectives. For this study, we set out to investigate the topic of fair dynamic valuations and to extend the previous work on the topic. We have done this through a number of steps. First, we outlined the mathematical theory of fair dynamic valuations. Notably, this constitutes a valuable component on its own and can, because of its general formulation, be employed in many different applications. Thereafter, we presented the choice of setup and methods. Here, we took a more practical approach and presented techniques that can be used in practice. In particular, we provided suggestions for alternative setups, such as the use of smoothing splines and LOESS regression models as well as the cost-of-capital valuation approach. These aimed to extend the previous research. Furthermore, we conducted a numerical analysis, which forms the basis for our presented results. This allowed us to apply the mathematical concepts as well evaluate and compare our proposed methods.

As a last point, based on the outcome of this study we provide some suggestions for future work on the topic. These include the following:

- **Investigating the impact of regression hyperparameters**
  In our analysis, we have investigated the use of smoothing splines and LOESS regression models as part of the LSMC approximation. No-

tably though, we have only used one set of hyperparameters[17] and in extension, not analyzed the impact of changing these parameters. This is something that ought to be researched more in detail.

- **Dynamics of underlying risk drivers**
  The underlying risk drivers, i.e. the stock value and survival of the insured, are modelled with fairly simple dynamics. For example, the stock value process is modelled with a geometric Brownian motion, which makes for easy calculations. However, such a model can be argued to not fully capture the dynamics observed in real equity markets. For instance, we could instead use a jump-diffusion model, resulting in a more advanced setup. Using such alternatives for the underlying dynamics, we could then analyze how the resulting fair dynamic valuations are affected.

- **Considering alternative claim types**
  All our numerical results are related to a particular choice of insurance liability claim, i.e. a particular choice of T-claim. However, similar methods and implementation can be used for other types of claims. This means that an extended analysis can be conducted, where results for different claim types can be compared. In particular, one could analyze whether the usage of smoothing splines and LOESS regression models also provide a good tool in such alternative cases.

[17] In the case of smoothing splines, 10 degrees of freedom, and in the case of LOESS regression, smoothing parameter 0.1 and degree 2.

Chapter 6

Conclusion

To summarize, in this study we have investigated the topic of fair dynamic valuation of insurance liabilities. We have outlined the underlying mathematical framework and described how this class of valuations combines concepts of market consistency, actuarial judgment and time consistency. Furthermore, we have presented a backwards iteration procedure that can be used in practice for the construction of a fair dynamic valuation. Considering a representative choice of insurance claim and financial-actuarial setting, we have implemented this procedure by utilizing risk minimization techniques and Least Square Monte Carlo (LSMC) approximation methods.

Based on our implementation, we have conducted a numerical analysis. As part of this, we have investigated various setups for the underlying regression models used in the LSMC approximations as well as different choices of actuarial valuation function. In particular, we have assessed the effect that the choice of setup has on the resulting fair dynamic valuations. Our results indicate that the choice of regression models has a notable impact on the resulting valuations. More specifically, we have identified a potential in using models such as smoothing splines and LOESS regressions. These options are found to provide the additional flexibility desired in the regression models, while still remaining computationally time-efficient.

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Appendix A

Mathematical Proofs

A.1 Proof of Lemma 1

Firstly, we show that Ψ_t is a t-hedger. It is clear from the definition that it is a mapping from $\mathcal{C}_T$ to Θ_t. Also, we find that

$$\Psi_t[0] = \boldsymbol{\psi}_{t,0} = \boldsymbol{\theta}_{t,0} + \widetilde{\varphi}_t[0 - \boldsymbol{\theta}_{t,0}(T) \cdot \boldsymbol{Y}(T)]\boldsymbol{\lambda}_t = \mathbf{0}_t + 0\boldsymbol{\lambda}_t = \mathbf{0}_t,$$

and for each $S \in \mathcal{C}_T$ and $a \in \mathcal{C}_t$ we obtain

$$\begin{aligned}\Psi_t[S+a] = \boldsymbol{\psi}_{t,S+a} = \boldsymbol{\theta}_{t,S+a} + \widetilde{\varphi}_t[S + a - \boldsymbol{\theta}_{t,S+a}(T) \cdot \boldsymbol{Y}(T)]\boldsymbol{\lambda}_t =\\ \boldsymbol{\theta}_{t,S} + a\boldsymbol{\lambda}_t + \widetilde{\varphi}_t[S - \boldsymbol{\theta}_{t,S}(T) \cdot \boldsymbol{Y}(T) + a - a\boldsymbol{\lambda}_t(T) \cdot \boldsymbol{Y}(T)]\boldsymbol{\lambda}_t =\\ \boldsymbol{\theta}_{t,S} + \widetilde{\varphi}_t[S - \boldsymbol{\theta}_{t,S}(T) \cdot \boldsymbol{Y}(T)]\boldsymbol{\lambda}_t + a\boldsymbol{\lambda}_t = \boldsymbol{\psi}_{t,S} + a\boldsymbol{\lambda}_t = \Psi_t[S] + a\boldsymbol{\lambda}_t,\end{aligned}$$

so we can conclude that Ψ_t indeed is a t-hedger.

Secondly, we show that if $\boldsymbol{\vartheta}_t$ if market consistent, then Ψ_t is market consistent and $\Psi_t[S_h] = \vartheta_t[S_h]$ for any $S_h \in \mathcal{H}_T^t$. To do so, consider arbitrary $S \in \mathcal{C}_T$ and $S_h \in \mathcal{H}_T^t$, then we have

$$\begin{aligned}\Psi_t[S+S_h] = \boldsymbol{\psi}_{t,S+S_h} = \boldsymbol{\theta}_{t,S+S_h} + \widetilde{\varphi}_t[S + S_h - \boldsymbol{\theta}_{t,S+S_h}(T) \cdot \boldsymbol{Y}(T)]\boldsymbol{\lambda}_t =\\ \boldsymbol{\theta}_{t,S} + \boldsymbol{\theta}_{t,S_h} + \widetilde{\varphi}_t[S - \boldsymbol{\theta}_{t,S}(T) \cdot \boldsymbol{Y}(T) + S_h - \boldsymbol{\theta}_{t,S_h}(T) \cdot \boldsymbol{Y}(T)]\boldsymbol{\lambda}_t =\\ \boldsymbol{\theta}_{t,S} + \widetilde{\varphi}_t[S - \boldsymbol{\theta}_{t,S}(T) \cdot \boldsymbol{Y}(T)]\boldsymbol{\lambda}_t + \boldsymbol{\theta}_{t,S_h} =\\ \boldsymbol{\psi}_{t,S} + \boldsymbol{\psi}_{t,S_h} = \Psi_t[S] + \Psi_t[S_h],\end{aligned}$$

where we have used that

$$\begin{aligned}\Psi_t[S_h] = \boldsymbol{\psi}_{t,S_h} = \boldsymbol{\theta}_{t,S_h} + \widetilde{\varphi}_t[S_h - \boldsymbol{\theta}_{t,S_h}(T) \cdot \boldsymbol{Y}(T)]\boldsymbol{\lambda}_t =\\ \boldsymbol{\theta}_{t,S_h} + \widetilde{\varphi}_t[0]\boldsymbol{\lambda}_t = \boldsymbol{\theta}_{t,S_h} = \vartheta_t[S_h],\end{aligned}$$

so we conclude that Ψ_t is indeed market consistent.

Thirdly, we consider the case when ϑ_t is actuarial and φ_t is actuarial. Then, for any $S_\perp \in \mathcal{O}_T^t$ we have

$$\begin{aligned}\Psi[S_\perp] = \boldsymbol{\psi}_{t,S_\perp} = \boldsymbol{\theta}_{t,S_\perp} + \widetilde{\varphi}_t[S_\perp - \boldsymbol{\theta}_{t,S_\perp}(T) \cdot \boldsymbol{Y}(T)]\boldsymbol{\lambda}_t = \\ \widetilde{\varphi}_t[S_\perp]\boldsymbol{\lambda}_t + \widetilde{\varphi}_t[S_\perp - \widetilde{\varphi}_t[S_\perp]\boldsymbol{\lambda}_t(T) \cdot \boldsymbol{Y}(T)]\boldsymbol{\lambda}_t = \\ \widetilde{\varphi}_t[S_\perp]\boldsymbol{\lambda}_t + \widetilde{\varphi}_t[S_\perp - \widetilde{\varphi}_t[S_\perp]]\boldsymbol{\lambda}_t = \widetilde{\varphi}_t[S_\perp]\boldsymbol{\lambda}_t,\end{aligned}$$

so we find that Ψ_t is actuarial with underlying actuarial t-valuation φ_t.

Lastly, in the case when ϑ_t is fair, i.e. both market consistent and actuarial, and φ_t is actuarial, the above conclusions imply that Ψ_t is both market consistent and actuarial. Thus, Ψ_t is fair with underlying actuarial t-valuation φ_t. $\square$

A.2 Proof of Theorem 1

Firstly, we prove the property related to *market consistency*. Assume φ_t to be a market consistent t-valuation, and ϑ_t, with t-hedges $\vartheta_t[S] = \boldsymbol{\theta}_{t,S}$ for $S \in \mathcal{C}_T$, to be a market consistent t-hedger. Then, for any $S \in \mathcal{C}_T$ we have

$$\begin{aligned}\varphi_t[S] = \varphi_t[S - \boldsymbol{\theta}_{t,S}(T) \cdot \boldsymbol{Y}(T) + \boldsymbol{\theta}_{t,S}(T) \cdot \boldsymbol{Y}(T)] = \\ \varphi_t[S - \boldsymbol{\theta}_{t,S}(T) \cdot \boldsymbol{Y}(T)] + \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(u)\,\mathrm{d}u\right)\boldsymbol{\theta}_{t,S}(T) \cdot \boldsymbol{Y}(T)\,\middle|\,\mathcal{F}_t\right] = \\ \varphi_t[S - \boldsymbol{\theta}_{t,S}(T) \cdot \boldsymbol{Y}(T)] + \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t) = \\ (\widetilde{\varphi}_t[S - \boldsymbol{\theta}_{t,S}(T) \cdot \boldsymbol{Y}(T)]\boldsymbol{\lambda}_t(t+1) + \boldsymbol{\theta}_{t,S}(t+1)) \cdot \boldsymbol{Y}(t) = \\ \boldsymbol{\theta}_{t,S}^m \cdot \boldsymbol{Y}(t),\end{aligned}$$

i.e. where we have defined t-hedger ϑ_t^m by

$$\vartheta_t^m[S] = \boldsymbol{\theta}_{t,S}^m = \boldsymbol{\theta}_{t,S} + \widetilde{\varphi}_t[S - \boldsymbol{\theta}_{t,S}(T) \cdot \boldsymbol{Y}(T)]\boldsymbol{\lambda}_t, \quad S \in \mathcal{C}_T,$$

which by Lemma 1 is market consistent.

Conversely, assume ϑ_t^m to be a market consistent t-hedger and define the t-valuation φ_t as $\varphi_t[S] = \boldsymbol{\theta}_{t,S}^m(t+1) \cdot \boldsymbol{Y}(t)$ for $S \in \mathcal{C}_T$. Then, for any $S \in \mathcal{C}_T$ and $S_h \in \mathcal{H}_T^t$ it holds that

$$\begin{aligned}\varphi_t[S + S_h] = \boldsymbol{\theta}_{t,S+S_h}^m(t+1) \cdot \boldsymbol{Y}(t) = \\ \boldsymbol{\theta}_{t,S}^m(t+1) \cdot \boldsymbol{Y}(t) + \boldsymbol{\theta}_{t,S_h}^m(t+1) \cdot \boldsymbol{Y}(t) = \\ \varphi_t[S] + \varphi_t[S_h],\end{aligned}$$

which shows that φ_t is a market consistent t-valuation.

Secondly, we consider the *actuarial* property. If φ_t is an actuarial t-valuation, then for any $S \in \mathcal{C}_T$ it holds that

$$\varphi_t[S] = \widetilde{\varphi}_t[S]B(t,T) = \widetilde{\varphi}_t[S]\boldsymbol{\lambda}_t(t+1) \cdot \boldsymbol{Y}(t) = \boldsymbol{\theta}^a_{t,S}(t+1) \cdot \boldsymbol{Y}(t),$$

where we have defined t-hedger ϑ^a_t, with t-hedges $\boldsymbol{\theta}^a_{t,S} = \widetilde{\varphi}_t[S]\boldsymbol{\lambda}_t$ for $S \in \mathcal{C}_T$. This is clearly an actuarial t-hedger.

Next, let ϑ^a_t be an actuarial t-hedger, with t-hedges $\vartheta^a_t[S] = \boldsymbol{\theta}^a_{t,S}$ for $S \in \mathcal{C}_T$, and underlying actuarial t-valuation $B(t,T)\pi_t$. Let the t-valuation φ_t be defined as $\varphi_t[S] = \boldsymbol{\theta}^a_{t,S}(t+1) \cdot \boldsymbol{Y}(t)$, $S \in \mathcal{C}_T$. Then, for any $S_\perp \in \mathcal{O}^t_T$ it holds that

$$\varphi_t[S_\perp] = \boldsymbol{\theta}^a_{t,S_\perp}(t+1) \cdot \boldsymbol{Y}(t) = \pi_t[S_\perp]\boldsymbol{\lambda}_t(t+1) \cdot \boldsymbol{Y}(t) = B(t,T)\pi_t[S_\perp],$$

so we find that φ_t is an actuarial t-valuation.

Lastly, we prove the *fair* property. Consider a fair t-valuation φ_t and some fair t-hedger ϑ_t, with t-hedges $\vartheta_t[S] = \boldsymbol{\theta}_{t,S}$ for $S \in \mathcal{C}_T$ and underlying actuarial t-valuation $B(t,T)\pi_t$. Since φ_t is fair it is also market consistent. Thus we know that there exists a market consistent t-hedger (as defined earlier in the proof) ϑ^m_t, with t-hedges $\vartheta^m_t[S] = \boldsymbol{\theta}^m_{t,S}$, such that for any $S \in \mathcal{C}_T$ it holds that $\varphi_t[S] = \boldsymbol{\theta}^m_{t,S}(t+1) \cdot \boldsymbol{Y}(t)$. For any $S_\perp \in \mathcal{O}^t_T$ it holds that

$$\begin{aligned}\vartheta^m_t[S_\perp] = \boldsymbol{\theta}^m_{t,S_\perp} = \boldsymbol{\theta}_{t,S_\perp} + \widetilde{\varphi}_t[S_\perp - \boldsymbol{\theta}_{t,S_\perp}(T) \cdot \boldsymbol{Y}(T)]\boldsymbol{\lambda}_t =& \\ \pi_t[S_\perp]\boldsymbol{\lambda}_t + \widetilde{\varphi}_t[S_\perp - \pi_t[S_\perp]]\boldsymbol{\lambda}_t =& \\ \widetilde{\varphi}_t[S_\perp]\boldsymbol{\lambda}_t,&\end{aligned}$$

which shows that ϑ^m_t (so we can define $\vartheta^f_t = \vartheta^m_t$) is actuarial. Since ϑ^f_t is both market consistent and actuarial it is fair.

Conversely, assume ϑ^f_t to be a fair t-hedger, with t-hedges $\vartheta^f_t[S] = \boldsymbol{\theta}^f_{t,S}$ for $S \in \mathcal{C}_T$. Let the t-valuation φ_t be defined as $\varphi_t[S] = \boldsymbol{\theta}^f_{t,S}(t+1) \cdot \boldsymbol{Y}(t)$. By the steps above we know that since ϑ^f_t is market consistent, φ_t is market consistent. Similarly, since ϑ^f_t is actuarial, φ_t must be actuarial. This means that φ_t is fair. This concludes the proof. □

A.3 Proof of Theorem 2

First, assume that $\{\varphi_t\}_{t=0}^{T-1}$ is a fair dynamic valuation. By definition, for each $t \in \{0, 1, \ldots, T-1\}$ it holds that φ_t is fair a t-valuation. For a chosen $t \in \{0, 1, \ldots, T-1\}$, we know from Theorem 1 that since φ_t is fair there exists a fair t-hedger ϑ_t, with t-hedges $\vartheta_t[S] = \boldsymbol{\theta}_{t,S}$ for $S \in \mathcal{C}_T$, such that

$\varphi_t[S] = \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t)$ for arbitrary $S \in \mathcal{C}_T$. In other words, ϑ_t is market consistent and actuarial. This means that the dynamic hedger $\{\vartheta_t\}_{t=0}^{T-1}$ is market consistent and actuarial. However, $\{\vartheta_t\}_{t=0}^{T-1}$ does not necessarily satisfy the time consistent property and so is not fair.

We consider instead the dynamic hedger $\{\Psi_t\}_{t=0}^{T-1}$ such that for each $t \in \{0, 1, \ldots, T-1\}$, the t-hedger Ψ_t is defined in the following way:

$$\Psi_t[S] = \boldsymbol{\psi}_{t,S} = \begin{cases} \boldsymbol{\theta}_{T-1,S}, & \text{if } t = T-1, \\ \boldsymbol{\theta}_{t,\widetilde{\varphi}_{t+1}[S]}, & \text{otherwise.} \end{cases} \quad S \in \mathcal{C}_T.$$

As we will show, this dynamic hedger is in fact both market consistent, actuarial and time consistent, i.e. fair. We do the proof by showing that each t-hedger Ψ_t is fair. Thereafter, we show the dynamic hedger satisfies the time consistency requirement.

Consider first the special case when $t = T-1$, for which we have the $(T-1)$-hedger Ψ_{T-1}, with $(T-1)$-hedges $\Psi_{T-1}[S] = \boldsymbol{\psi}_{T-1,S} = \boldsymbol{\theta}_{T-1,S}$. From the reasoning earlier we know this to be a fair $(T-1)$-hedger.

Next, for each $t \in \{0, 1, \ldots, T-2\}$, we need to show that Ψ_t is fair. For the market consistent check, consider arbitrary $S \in \mathcal{C}_T$ and $S_h \in \mathcal{H}_T^t$. Then, using the fact that φ_{t+1} is market consistent, $\boldsymbol{\vartheta}_t$ is market consistent and any t-hedgeable claim is also $(t+1)$-hedgeable, it holds that

$$\begin{aligned} \Psi_t[S + S_h] = \boldsymbol{\psi}_{t,S+S_h} = \boldsymbol{\theta}_{t,\widetilde{\varphi}_{t+1}[S+S_h]} = \\ \boldsymbol{\theta}_{t,\widetilde{\varphi}_{t+1}[S]+\widetilde{\varphi}_{t+1}[S_h]} = \boldsymbol{\theta}_{t,\widetilde{\varphi}_{t+1}[S]} + \boldsymbol{\theta}_{t,\widetilde{\varphi}_{t+1}[S_h]} = \\ \Psi_t[S] + \Psi_t[S_h], \end{aligned}$$

which shows Ψ_t to be market consistent. For the actuarial check, consider $S_\perp \in \mathcal{O}_T^t$, for which we have

$$\begin{aligned} \Psi_t[S_\perp] = \boldsymbol{\psi}_{t,S_\perp} = \boldsymbol{\theta}_{t,\widetilde{\varphi}_{t+1}[S_\perp]} = \boldsymbol{\theta}_{t,\pi_{t+1}[S_\perp]} = \\ \pi_t\left[\pi_{t+1}[S_\perp]\right] \boldsymbol{\lambda}_t = \pi_t[S_\perp]\boldsymbol{\lambda}_t, \end{aligned}$$

where we used the time consistent property of $\{\varphi_t\}_{t=0}^{T-1}$ as well as the fact that $\pi_{t+1}[S_\perp]$ is $\mathbb{P}$-independent of $\{\boldsymbol{Y}(u)\}_{u\in\{t+1,\ldots,T\}}$, and therefore is t-orthogonal. We find that Ψ_t is actuarial and thus also fair.

Let us now investigate the time consistency of $\{\Psi_t\}_{t=0}^{T-1}$. Using the property of translation invariance, we get for any $t \in \{0, 1, \ldots, T-2\}$ and $S \in \mathcal{C}_T$ that

$$\Psi_t[\widetilde{\varphi}_{t+1}[S]] = \boldsymbol{\psi}_{t,\widetilde{\varphi}_{t+1}[S]} = \boldsymbol{\theta}_{t,\widetilde{\varphi}_{t+1}[\widetilde{\varphi}_{t+1}[S]]} = \boldsymbol{\theta}_{t,\widetilde{\varphi}_{t+1}[S]} = \boldsymbol{\psi}_{t,S} = \Psi_t[S],$$

so $\{\Psi_t\}_{t=0}^{T-1}$ is time consistent, and thus also fair. Additionally, we find that for any $S \in \mathcal{C}_T$ it holds that

$$\varphi_t[S] = \varphi_t[\widetilde{\varphi}_{t+1}[S]] = \boldsymbol{\theta}_{t,\widetilde{\varphi}_{t+1}[S]}(t+1) \cdot \boldsymbol{Y}(t) = \boldsymbol{\psi}_{t,S}(t+1) \cdot \boldsymbol{Y}(t).$$

Let us now consider the other direction of the proof. That is, let $\{\Psi_t\}_{t=0}^{T-1}$ be a fair dynamic hedger. This means that each t-hedger Ψ_t, with t-hedges $\Psi_t[S] = \boldsymbol{\psi}_{t,S}$ for $S \in \mathcal{C}_T$, is fair. For each $t \in \{0, 1, \dots, T-1\}$, by Theorem 1 there exists a fair t-valuation φ_t such that $\varphi_t[S] = \boldsymbol{\psi}_{t,S}(t+1) \cdot \boldsymbol{Y}(t)$ for any $S \in \mathcal{C}_T$. Also we find that

$$\varphi_t[S] = \boldsymbol{\psi}_{t,S}(t+1) \cdot \boldsymbol{Y}(t) = \boldsymbol{\psi}_{t,\widetilde{\varphi}_{t+1}[S]}(t+1) \cdot \boldsymbol{Y}(t) = \varphi_t[\widetilde{\varphi}_{t+1}[S]].$$

This means that the dynamic valuation $\{\varphi_t\}_{t=0}^{T-1}$ is time consistent, in addition to being market consistent and actuarial. Thus it is also fair. □

A.4 Proof of Proposition 1

To prove the optimal solution we first define the function f_t such that for any $a_t \in \mathbb{R}$ and $\boldsymbol{b}_t \in \mathbb{R}^n$ we have

$$\begin{aligned} f_t(a_t, \boldsymbol{b}_t) = \\ \mathbb{E}\left[\left(L_{t+1} - a_t e^{-r(T-t-1)} - \boldsymbol{b}_t^\intercal \boldsymbol{A}_{t+1}\right)^2 \,\middle|\, \mathcal{F}_t\right] = \\ \mathbb{E}\left[L_{t+1}^2 \,\middle|\, \mathcal{F}_t\right] + a_t^2 e^{-2r(T-t-1)} + \boldsymbol{b}_t^\intercal \mathbb{E}\left[\boldsymbol{A}_{t+1}\boldsymbol{A}_{t+1}^\intercal \,\middle|\, \mathcal{F}_t\right] \boldsymbol{b}_t - \\ 2a_t e^{-r(T-t-1)} \mathbb{E}\left[L_{t+1} \,|\, \mathcal{F}_t\right] - \\ 2\boldsymbol{b}_t^\intercal \mathbb{E}\left[\boldsymbol{A}_{t+1} L_{t+1} \,|\, \mathcal{F}_t\right] + \\ 2a_t e^{-r(T-t-1)} \boldsymbol{b}_t^\intercal \mathbb{E}\left[\boldsymbol{A}_{t+1} \,|\, \mathcal{F}_t\right] \end{aligned}$$

From this we then obtain

$$\frac{\partial f_t}{\partial a_t}(a_t, \boldsymbol{b}_t) = 2e^{-r(T-t-1)}\left(a_t e^{-r(T-t-1)} - \mathbb{E}\left[L_{t+1} \,|\, \mathcal{F}_t\right] + \boldsymbol{b}_t^\intercal \mathbb{E}\left[\boldsymbol{A}_{t+1} \,|\, \mathcal{F}_t\right]\right)$$

and

$$\begin{aligned} \nabla_{\boldsymbol{b}_t} f_t(a_t, \boldsymbol{b}_t) = \\ 2\left(\mathbb{E}\left[\boldsymbol{A}_{t+1}\boldsymbol{A}_{t+1}^\intercal \,\middle|\, \mathcal{F}_t\right] \boldsymbol{b}_t - \mathbb{E}\left[\boldsymbol{A}_{t+1} L_{t+1} \,|\, \mathcal{F}_t\right] + a_t e^{-r(T-t-1)} \mathbb{E}\left[\boldsymbol{A}_{t+1} \,|\, \mathcal{F}_t\right]\right). \end{aligned}$$

Setting these expression to zero yields the optimal solution $\left(\widehat{a}_t, \widehat{\boldsymbol{b}}_t\right)$. The first expression can then be simplified to

$$\widehat{a}_t = e^{r(T-t-1)}\left(\mathbb{E}\left[L_{t+1} \,|\, \mathcal{F}_t\right] - \widehat{\boldsymbol{b}}_t^\intercal \mathbb{E}\left[\boldsymbol{A}_{t+1} \,|\, \mathcal{F}_t\right]\right),$$

which inserted in the second expression gives

$$\left(\mathbb{E}\left[\boldsymbol{A}_{t+1}\boldsymbol{A}_{t+1}^{\intercal} \,\middle|\, \mathcal{F}_t\right] - \mathbb{E}\left[\boldsymbol{A}_{t+1} \,|\, \mathcal{F}_t\right]\mathbb{E}\left[\boldsymbol{A}_{t+1}^{\intercal} \,\middle|\, \mathcal{F}_t\right]\right)\widehat{\boldsymbol{b}}_t = \\ \mathbb{E}\left[\boldsymbol{A}_{t+1}L_{t+1} \,|\, \mathcal{F}_t\right] - \mathbb{E}\left[\boldsymbol{A}_{t+1} \,|\, \mathcal{F}_t\right]\mathbb{E}\left[L_{t+1} \,|\, \mathcal{F}_t\right].$$

Using the covariance expressions, this equation can be reformulated as

$$\operatorname{Cov}\left[\boldsymbol{A}_{t+1} \,|\, \mathcal{F}_t\right]\widehat{\boldsymbol{b}}_t = \operatorname{Cov}\left[\boldsymbol{A}_{t+1}, L_{t+1} \,|\, \mathcal{F}_t\right].$$

Given the assumption that the inverse of $\operatorname{Cov}\left[\boldsymbol{A}_{t+1} \,|\, \mathcal{F}_t\right]$ exists, we obtain the optimal solution

$$\widehat{\boldsymbol{b}}_t = \left(\operatorname{Cov}\left[\boldsymbol{A}_{t+1} \,|\, \mathcal{F}_t\right]\right)^{-1}\operatorname{Cov}\left[\boldsymbol{A}_{t+1}, L_{t+1} \,|\, \mathcal{F}_t\right],$$

which completes the proof. $\square$

A.5 Proof of Proposition 2

As a first step, we note that since $\Delta_{t+1,S} \,|\, \mathcal{F}_t \sim N(\mu_t, \sigma_t^2)$, we have

$$\operatorname{VaR}_{t,p}(-\Delta_{t+1,S}) = \mu_t + \sigma_t\Phi^{-1}(1-p),$$

where Φ is the distribution function of the standard normal distribution.

From this, we then get the following

$$\begin{aligned}
&\mathbb{E}\left[(\operatorname{VaR}_{t,p}(-\Delta_{t+1,S}) - \Delta_{t+1,S})^+ \,\middle|\, \mathcal{F}_t\right] = \\
&\mathbb{E}\left[\left(\mu_t + \sigma_t\Phi^{-1}(1-p) - \Delta_{t+1,S}\right)^+ \,\middle|\, \mathcal{F}_t\right] = \\
&\int_{-\infty}^{\infty}\left(\mu_t + \sigma_t\Phi^{-1}(1-p) - z\right)^+ \frac{1}{\sqrt{2\pi}\sigma_t}e^{-\frac{(z-\mu_t)^2}{2\sigma_t^2}}\,\mathrm{d}z = \\
&\int_{-\infty}^{\mu_t+\sigma_t\Phi^{-1}(1-p)}\left(\mu_t + \sigma_t\Phi^{-1}(1-p) - z\right)\frac{1}{\sqrt{2\pi}\sigma_t}e^{-\frac{(z-\mu_t)^2}{2\sigma_t^2}}\,\mathrm{d}z = \\
&\sigma_t\int_{-\infty}^{\Phi^{-1}(1-p)}\left(\Phi^{-1}(1-p) - z'\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{z'^2}{2}}\,\mathrm{d}z' = \\
&\sigma_t\left(\Phi^{-1}(1-p)\Phi\left(\Phi^{-1}(1-p)\right) + \left[\frac{1}{\sqrt{2\pi}}e^{\left(-\frac{z'^2}{2}\right)}\right]_{z'=-\infty}^{z'=\Phi^{-1}(1-p)}\right) = \\
&\sigma_t\left((1-p)\,\Phi^{-1}(1-p) + \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\Phi^{-1}(1-p)\right)^2\right)\right).
\end{aligned}$$

Using the introduced function κ, as well as the two expressions above, we get the final result:

$$\pi_t(\Delta_{t+1,S}) = \mu_t + \kappa(\eta, p)\sigma_t$$

$\square$

Appendix B

Code Implementation Example

This appendix presents the R code implementation where a fair dynamic valuation is constructed using the backwards iteration procedure. The code uses the model setup of Case 3 as well as numerical parameter values defined in Section 3.5, i.e. it produces the results presented in Section 4.3. Notably, this code can be easily modified, e.g. to consider alternative regression setups or modified model parameters.

```r
#### PRELIMINARIES

# DEFINE PARAMETERS
N     <- 50000
TT    <- 10
r     <- 0.01
mu    <- 0.02
sigma <- 0.1
K     <- 1
Y0    <- 1
N0    <- 1000
a     <- 0.001
b     <- 0.000012
c     <- 0.101314
x     <- 60
eta   <- 0.06
p     <- 0.005
alpha <- qnorm(1-p) - (1/(1+eta))*((1-p)*qnorm(1-p) + exp(-0.5*qnorm(1-p)^2)/sqrt(2*pi))

# SAMPLE STOCK VALUE TRAJECTORIES
set.seed(123)
W.year      <- matrix(rnorm(N*TT), nrow = N, ncol = TT)
Y.year      <- exp(mu-0.5*sigma^2 + sigma*W.year)
Y           <- Y0*cbind(rep(1, N), t(apply(Y.year, MARGIN = 1, FUN = cumprod)))
colnames(Y) <- as.character(0:TT)

# SAMPLE NUMBER OF SURVIVORS TRAJECTORIES
ages        <- x:(x+TT-1)
p_x         <- exp(-(a+(b/c)*(exp(c)-1)*exp(c*ages)))
names(p_x)  <- as.character(0:(TT-1))
```

```r
N.surv        <- matrix(NA, nrow = N, ncol = TT+1, dimnames = list(NULL, as.character(0:TT)))
N.surv[,"0"] <- N0

for(t in 1:TT){
  N.surv[,as.character(t)] <- rbinom(n = N,
                                     size = N.surv[,as.character(t-1)],
                                     prob = p_x[as.character(t-1)])
}


### INITIALIZE OBJECTS TO STORE INFORMATION

# VALUATIONS
VAL     <- matrix(NA, nrow = N, ncol = TT+1, dimnames = list(NULL, as.character(0:TT)))
VAL.MC  <- matrix(NA, nrow = N, ncol = TT, dimnames = list(NULL, as.character(0:(TT-1))))
VAL.ACT <- matrix(NA, nrow = N, ncol = TT, dimnames = list(NULL, as.character(0:(TT-1))))

# REGRESSION FITS
ALPHA.fit  <- list()
BETA.fit   <- list()
RESID2.fit <- list()

# ESTIMATED CONDITIONAL EXPECTATIONS
ALPHA.fitted  <- matrix(NA, nrow = N, ncol = TT, dimnames = list(NULL, as.character(0:(TT-1))))
BETA.fitted   <- matrix(NA, nrow = N, ncol = TT, dimnames = list(NULL, as.character(0:(TT-1))))
RESID2.fitted <- matrix(NA, nrow = N, ncol = TT, dimnames = list(NULL, as.character(0:(TT-1))))

# HEDGE COMPONENTS
COV    <- matrix(NA, nrow = N, ncol = TT, dimnames = list(NULL, as.character(0:(TT-1))))
THETA0 <- matrix(NA, nrow = N, ncol = TT, dimnames = list(NULL, as.character(0:(TT-1))))
```

```
THETA1 <- matrix(NA, nrow = N, ncol = TT, dimnames = list(NULL, as.character(0:(TT-1))))

# RESIDUALS
RESID <- matrix(NA, nrow = N, ncol = TT, dimnames = list(NULL, as.character(0:(TT-1))))


### CALCULATIONS

# APPLY CHOICE OF T-CLAIM
VAL[,as.character(TT)] <- N.surv[,as.character(TT)] * pmax(K, Y[,as.character(TT)])

# BACKWARDS ITERATION THORUGH TIME
for(t in seq(from = TT-1, to = 0, by = -1)){

  # ALPHA MODEL - USING SMOOTHING SPLINE REGRESSION
  alpha.y <- VAL[,as.character(t+1)]
  alpha.x <- N.surv[,as.character(t)] * Y[,as.character(t)]
  if(t == 0){
    ALPHA.fit[[as.character(t)]]    <- lm(alpha.y ~ 1)
    ALPHA.fitted[,as.character(t)] <- ALPHA.fit[[as.character(t)]]$fitted.values
  }else{
    ALPHA.fit[[as.character(t)]]    <- smooth.spline(x = alpha.x, y = alpha.y, df = 10)
    ALPHA.fitted[,as.character(t)] <- predict(ALPHA.fit[[as.character(t)]], x = alpha.x)$y
  }

  # BETA MODEL - USING SMOOTHING SPLINE REGRESSION
  beta.y <- VAL[,as.character(t+1)] * Y[,as.character(t+1)]
  beta.x <- N.surv[,as.character(t)] * Y[,as.character(t)]^2
  if(t == 0){
    BETA.fit[[as.character(t)]]    <- lm(beta.y ~ 1)
```

```
  BETA.fitted[,as.character(t)] <- BETA.fit[[as.character(t)]]$fitted.values
}else{
  BETA.fit[[as.character(t)]]    <- smooth.spline(x = beta.x, y = beta.y, df = 10)
  BETA.fitted[,as.character(t)] <- predict(BETA.fit[[as.character(t)]], x = beta.x)$y
}

# HEDGE COMPONENTS
COV[,as.character(t)]      <- BETA.fitted[,as.character(t)] -
  ALPHA.fitted[,as.character(t)] * Y[,as.character(t)]*exp(mu)
THETA1[,as.character(t)] <- COV[,as.character(t)] /
  (Y[,as.character(t)]^2 * exp(2*mu) * (exp(sigma^2)-1))
THETA0[,as.character(t)] <- exp(r*(TT-t-1)) *
  (ALPHA.fitted[,as.character(t)] - THETA1[,as.character(t)] * Y[,as.character(t)]*exp(mu))

# RESIDUAL RISK
RESID[,as.character(t)] <- VAL[,as.character(t+1)] -
  (THETA0[,as.character(t)]*exp(-r*(TT-t-1)) + THETA1[,as.character(t)]*Y[,as.character(t+1)])

# SQUARED RESIDUAL RISK MODEL - USING LOESS REGRESSION
gamma.y <- RESID[,as.character(t)]^2
gamma.x <- N.surv[,as.character(t)] * Y[,as.character(t)]
if(t == 0){
  RESID2.fit[[as.character(t)]]    <- lm(gamma.y ~ 1)
  RESID2.fitted[,as.character(t)] <- RESID2.fit[[as.character(t)]]$fitted.values
}else{
  RESID2.fit[[as.character(t)]]    <- loess(gamma.y ~ gamma.x, span = 0.1)
  RESID2.fitted[,as.character(t)] <- pmax(predict(RESID2.fit[[as.character(t)]], gamma.x), 0)
}

# HEDGE CONTRIBUTION TO VALUATION
```

```
  VAL.MC[,as.character(t)] <- THETA0[,as.character(t)] * exp(-r*(TT-t)) +
    THETA1[,as.character(t)] * Y[,as.character(t)]

  # ACTUARIAL CONTRIBUTION TO VALUATION
  VAL.ACT[,as.character(t)] <- exp(-r)*alpha*sqrt(RESID2.fitted[,as.character(t)])

  # VALUATION
  VAL[,as.character(t)] <- VAL.MC[,as.character(t)] + VAL.ACT[,as.character(t)]
}
```