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Ruin Theory in the Presence of Heavy-Tailed Claims

Victor Schön

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Matematisk statistik
Matematiska institutionen
Stockholms universitet
106 91 Stockholm

Mathematical Statistics
Stockholm University
Master Thesis **2023:18**
http://www.math.su.se

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Abstract

The present thesis studies the behaviour of the ruin probability of a portfolio of insurable risks in the framework of the Cramér-Lundberg model. As is well-known, this behaviour is different depending on whether the severity distribution of the individual portfolio risks can be considered light-tailed or heavy-tailed. In particular, the overall behaviour of the ruin probability of the portfolio is to a large extent influenced by the presence of heavy-tailed claims. This is confirmed through a detailed numerical study, which estimates the ruin probability of a portfolio of insurable risks of a stylised insurance company. Various measures available to the insurance company to mitigate the impact of this influence by e.g. purchasing excess-of-loss reinsurance or increasing premiums are then explored. The overall conclusion is that careful capital and risk management of the portfolio is called for in the presence of heavy-tailed risks in order to avoid ruin.

*Postal address: Mathematical Statistics, Stockholm University, SE-106 91, Sweden. E-mail: vschon@kth.se. Supervisor: Kristoffer Lindensjö.

Acknowledgements

The present thesis was initially written as part of Svenska Aktuarieföreningen's curriculum[1]. It was at the suggestion of my supervisor, Associate Professor Kristoffer Lindensjö, that I should endeavor to make the necessary adjustments to the initial work in order for it to conform to the standards of a master's thesis at the Department of Mathematics at Stockholm University. I am therefore very thankful to him for providing me with this guidance and also suggestions on how to improve the initial work. Naturally, any remaining errors, omissions or inconsistencies are my own.

Victor Schön
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[1]Please see https://aktuarieforeningen.se/bli-medlem/ for more information about the requirements. Amongst these are the *diplomarbete*, which is an independent work that has bearing on the actuarial field.

Contents

1 Introduction

The foundations of risk theory was laid by the actuary Filip Lundberg in his PhD thesis from 1903, Lundberg [18], in which he introduced a simple, yet very useful, model capable of describing the basic dynamics of a homogeneous insurance portfolio. As mentioned in Mikosch [20] p. 3, risk theory is a synonym for non-life insurance mathematics and, in its most simple form, concerned with the amount of premium to be charged by the insurance company in order to avoid ruin. Lundberg realised that the Poisson process is an indispensable tool for modelling the arrival of claims in the portfolio, but it was then Harald Cramér, the distinguished mathematician and actuary, who developed and made Lundberg's ideas mathematically precise, see e.g. Cramér [5] and Cramér [6]. The resulting model of these efforts are known as the *Cramér-Lundberg model*, also known as the collective risk model, or the compound Poisson model, for the modelling of the total claim amount process, $S(t)$, experienced by the insurance company:

$$S(t) = \sum_{n=1}^{N(t)} X_n,$$

where $N(t)$ is a homogeneous Poisson process and $\{X_n, n \geq 1\}$ is a sequence of i.i.d. of random variables independent of $N(t)$. The *surplus* of the insurance company over time, $U(t)$, is then modelled according to

$$U(t) = u + p(t) - S(t), \tag{1.1}$$

where, u is the amount of initial capital, $p(t)$ is the premium charged and $S(t)$ is as above. In this setting, the aim of ruin theory is to study the behaviour of the probability of the surplus becoming negative, that is,

$$\psi(u) = P(\{U(t) < 0,\ t > 0\})$$

for an initial value u.

Since the initial works reference above a vast literature on ruin theory has developed generalising and extending the Cramér-Lundberg model in various directions. Important contributions are e.g. Feller [10], Grandell [12] and Rolski et al [22]. An extension of (1.1) analysing the impact on $\psi(u)$ in the presence of general investment strategies can e.g. be found in Hult and Lindskog [14].

The purpose of this thesis is to investigate the behaviour of the ruin probability of a portfolio of risks insured by an insurance company. It will be seen that the behaviour of the ruin probability is significantly different depending on whether the probability distribution of the insured risks can be classified as, colloquially, light-tailed or heavy-tailed. In the former instance, it is possible to obtain an upper bound on the ruin probability, which decays exponentially fast to null, whereas in the latter instance this is not possible, see for example Embrechts et al [8] or Rolski et al [22]. Consequently, the behaviour of the ruin probability is to a large extent influenced by the presence of risks whose probability distribution can be considered heavy-tailed. In addition, ruin in the presence of heavy-tailed risks is due to the single large claim affecting the portfolio and essentially wiping out the surplus in one single stroke, see e.g. Embrechts and Veraverbeke [7] for more information on this interpretation.

There exists no single definition by which the probability distribution can be classified as light-tailed or heavy-tailed, see e.g. Embrechts et al [8] or Mikosch [20]. Instead, various notions exist which try to capture the properties it should possess if it is heavy-tailed, see e.g. Goldie and Klüppelberg [11] for further reference. These properties could, for example, be

- For a heavy-tailed distribution, its right tail decreases more slowly than any exponential tail.
- The mean excess function of a heavy-tailed distribution is unbounded.
- Heavy tails can not be "diversified" away in the sense of the Central Limit Theorem.
- For a heavy-tailed distribution, the tail of the distribution of $S_n = X_1 + \cdots + X_n$ is determined by the tail of the distribution of $M_n = max(X_1, \ldots, X_n)$. This implies that S_n is large due to one of the X_i being large.
- For a heavy-tailed distribution, the conditional probability of a very large loss given the occurrence of an already large loss tends to 1 as the threshold increases. In the light-tailed case, this probability is smaller than 1.

The above considerations are treated in Chapter 2. In the same chapter, two brief sections on regular variation and subexponential distributions are included, both illustrating properties desirable in a heavy-tailed distribution. The subexponential class of distribution forms a particularly useful class of distributions, from the moderately heavy-tailed lognormal distribution, to more heavy-tailed ones, such as the Pareto or Burr distributions.

Chapter 3 constitutes a very brief introduction to ruin theory and presents two classical theorems: the Lundberg bound and Cramér's ruin bound, which are both concerned with the light-tailed case. Here, light-tailed should be understood as the existence of the moment generating function of the severity distribution. Theorem 3.5 states the asymptotic behaviour of the ruin probability in the heavy-tailed case. As already alluded to, this behaviour is significantly different from the light-tailed case.

Chapter 4 is an application of the ideas presented in the two previous chapters in the context of modelling the ruin probability of a portfolio of three lines of business; each with its own severity distribution. The three distributions are, respectively: Exponential, $\text{Exp}(\gamma)$, Gamma, $\Gamma(\alpha, \beta)$, and Pareto, $\text{Pa}(\alpha, \sigma)$. The reason for including the Exponential distribution is that, in some sense, it constitutes a reference distribution when determining whether a severity distribution is light-tailed or heavy-tailed. The Gamma and Pareto distributions are standard choices for modelling claims severity in non-life insurance mathematics, with the former being considered light-tailed and the latter heavy-tailed. If not evident earlier, this chapter hopefully demonstrates clearly the dangers of heavy-tailed severity distributions and the implications on risk management as well as capital management.

The following parameterisation of the severity distributions have been used throughout the text:

1. Exponential distribution. The probability density function is given by

$$f_X(x; \gamma) = \gamma e^{-\gamma x}, \ \gamma > 0.$$

2. Gamma distribution. The probability density function is given by

$$f_X(x; \alpha, \beta) = \frac{1}{\Gamma(\beta)} x^{\alpha-1} e^{-\beta x}, \ \alpha, \beta > 0.$$

3. Pareto distribution. The probability density function is given by

$$f_X(x; \alpha, \sigma) = \frac{\alpha \sigma^\alpha}{(x + \sigma)^{\alpha+1}}, \ \alpha, \sigma > 0.$$

Unless otherwise stated, $\gamma = 1$, $\alpha = 5$, $\beta = 2$, $\alpha = 4$ and $\sigma = 4$. The arrival rate of the Poisson process has been assumed to be equal 1 throughout the text, i.e. $\lambda = 1$. The usual and often difficult problem of parameter estimation is not treated here. The parameter values in this thesis have simply been chosen to illustrate the key concepts.

2 Light-tailed or Heavy-tailed

This chapter begins with a introduction to the topic of classifying a severity distribution as light-tailed or heavy-tailed. Although there exists no general procedure for this, it is possible to think of a number of properties a severity distribution should possess depending on whether it is light-tailed or heavy-tailed. In the first section, it is suggested that one such property is to study the right tail of the severity distribution and investigate if it decays faster to null than the right tail of the exponential distribution. If it decays faster, then it is considered light-tailed and vice-versa. Another such property might that the mean excess function, $e_F(u)$, see Definition 2.1, tends to infinity if the severity distribution is heavy-railed, i.e. $e_F(u) \to \infty$; if not, it could be considered light-tailed. Yet another way of determining whether the a severity distribution is light-tailed or heavy-tailed could be to see if its mean excess function is bounded or unbounded. In the first instance, it could be considered light-tailed; in the second, heavy-tailed. The concept of regular variation and, more specifically, a distribution function whose right tail is regularly varying is then studied. These classes of functions exhibit properties that a heavy-tailed distribution intuitively have, especially when studying $S_n = X_1 + \cdots + X_n$ for which the approximation given by the Central Limit Theorem is inadequate in the right tail of the distribution, i.e. for $P(S_n > x)$ when x is large. This is a consequence of the closure property of both families of distributions, which states that if $\{X_n, n \geq 1\}$ is an i.i.d. sample of regularly varying or subexponential distributions, then $S_n = X_1 + \cdots + X_n$ is also regularly varying or subexponential. Although distribution functions, whose right tail exhibits regular variation, exhibit some properties desirable in a heavy-tailed distribution and could be one alternative for classification, it would be too restrictive to define a severity distribution as heavy-tailed as it excludes typical distribution functions thought of as heavy-tailed. The class of subexponential distribution functions is then introduced and is sufficiently flexible to include common choices for severity distribution considered as, or understood to be, heavy-tailed such as the log-normal, Pareto, Burr and Weibull ($\tau < 1$) distributions.

2.1 Exponential Distribution as Reference

Let the right tail of the distribution, $\overline{F}(x)$, be defined as:

$$\overline{F}(x) = P(X > x).$$

One way to define a light-tailed distribution could be to stipulate a sufficiently fast decaying right tail of the severity distribution. The obvious question then immediately arises - faster than what? One idea is to use the exponential distribution as a reference. For example, if

$$\limsup_{x\to\infty} \frac{\overline{F}(x)}{e^{-\lambda x}} < \infty, \quad \text{for some } \lambda \geq 0, \tag{2.1}$$

F could be called *light-tailed*, and if

$$\liminf_{x\to\infty} \frac{\overline{F}(x)}{e^{-\lambda x}} > 0, \quad \text{for all } \lambda \geq 0, \tag{2.2}$$

F could be called *heavy-tailed*.

In Example 2.1, it is shown that both the Exponential and Gamma distributions are considered light-tailed, whereas the Pareto distribution is considered heavy-tailed.

Example 2.1.

1. Exponential distribution, $\overline{F}(x) = e^{-\gamma}$. By choosing $\gamma = \lambda$ in (2.1) above, it is seen

$$\limsup_{x\to\infty} \frac{e^{-\gamma x}}{e^{-\gamma x}} = 1 < \infty.$$

2. Gamma distribution, $\overline{F}(x) = \int_x^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\beta t}\, dt$. An application of l'Hospital's rule yields that

$$\limsup_{x\to\infty} \frac{\overline{F}(x)}{e^{-\lambda x}} < \infty \quad \text{for } \lambda < \beta.$$

3. Pareto distribution, $\overline{F}(x) = \left(\frac{\sigma+x}{\sigma}\right)^{-\alpha}$. For sufficiently large x, the influence of σ on $\overline{F}(x)$ is negligible, i.e. $\overline{F}(x) \sim x^{-\alpha}$ and it suffices to study $x^{-\alpha}/e^{-\lambda x}$. Now, since the exponential function decays faster to 0 than any polynomial, for any choice of λ and α, (2.2) yields

$$\liminf_{x\to\infty} \frac{\overline{F}(x)}{e^{-\lambda x}} > 0.$$

Another way of determining whether or not a distribution could be considered light-tailed or heavy-tailed could be by studying its mean excess function.

Definition 2.1 (Mean excess function, p. 86 in Mikosch [20]). *Let X be a non-negative random variable with finite mean and distribution function $F_X(x)$. Let further $x_l = inf\{x : F_X(x) > 0\}$ and $x_u = sup\{x : F_X(x) < 1\}$. Then, the mean excess function is defined as*

$$e_F(u) = E(X - u|X > u), \quad u \in (x_l, x_u).$$

If $\lim_{u\to\infty} e_F(u) = \infty$, then $F_X(x)$ is heavy-tailed; if $\lim_{u\to\infty} e_F(u) < \infty$, then $F_X(x)$ is light-tailed.

For the purpose of the present text, the interval (x_l, x_u) is understood to be $[0, \infty)$. In an insurance context, $e_F(u)$ can be interpreted as the expected claims cost in the unlimited layer (u, ∞). A convenient way of expressing (2.1) is in the form

$$e_F(u) = \frac{1}{\overline{F}(u)} \int_u^\infty \overline{F}(x)dx, \quad u \in [0, \infty). \tag{2.3}$$

(2.3) is derived through the following steps:

$$\begin{aligned} E(X - u|X > u) &= E(X|X > u) - u \\ &= \frac{1}{\overline{F}(u)} \int_u^\infty f(x)dx - u \\ &= \frac{1}{\overline{F}(u)} \left(u\overline{F}(u) + \int_u^\infty \overline{F}(x)dx \right) - u \\ &= \frac{1}{\overline{F}(u)} \int_u^\infty \overline{F}(x)dx. \end{aligned}$$

The third equality is arrived at by using integration by parts and using $\lim_{x\to\infty} x\overline{F}(x) = 0$. The mean excess function for the Exponential, Gamma and Pareto distributions, respectively, are given below.

Example 2.2.

1. Exponential distribution, Exp(γ).

$$e_F(u) = \frac{1}{e^{-\lambda u}} \int_u^\infty e^{-\lambda x} dx = \frac{1}{\lambda},$$

i.e. $e_F(u)$ does not depend on u. In particular, it does not increase in u.

2. Gamma distribution. The calculations are omitted here since they require some space. In Mikosch [20] p. 90, it is given as

$$e_F(u) = \frac{1}{\beta}\left(1 + \frac{\alpha - 1}{\beta u} + o\left(\frac{1}{u}\right)\right).$$

Please note that $e_F(u) \to \frac{1}{\beta}$ as $u \to \infty$.

3. Pareto distribution, Pa(α, σ) and $\alpha > 1$.

$$\begin{aligned} e_F(u) &= \left(\frac{\sigma + u}{\sigma}\right)^{-\alpha} \int_u^\infty \left(\frac{\sigma + x}{\sigma}\right)^{-\alpha} dx \\ &= \left(\frac{\sigma + u}{\sigma}\right)^{-\alpha} \left[- \frac{\sigma}{\alpha - 1}\left(\frac{\sigma + x}{\sigma}\right)^{-(\alpha-1)}\right]_u^\infty \\ &= \left(\frac{\sigma + u}{\sigma}\right)^{-\alpha} \left(\frac{\sigma}{\alpha - 1}\left(\frac{\sigma + u}{\sigma}\right)^{-(\alpha-1)}\right) \\ &= \frac{\sigma + u}{\alpha - 1} \\ &= EX + \frac{u}{\alpha - 1} \end{aligned}$$

Please note that $e_F(u)$ depends on u and is increasing in u.

Thus, the mean excess function for the Gamma distribution is decreasing in u, whereas it is increasing in u for the Pareto distribution with the Exponential distribution constitutes an exception as its mean excess function converges to a positive constant. Apart from being consistent with (2.1) and (2.2), it illustrates an important difference between light-tailed distributions and heavy-tailed distributions: in the heavy-tailed case, the expected claims cost in the layer does not decrease by increasing the deductible u - it actually increases. In the light-tailed case, increasing the deductible, decreases the expected value of the claims cost in the layer.

2.2 Regular Variation

The family of distribution functions whose right tail is regularly varying is one candidate for classifying a distribution as heavy-tailed. One simple reason for this is that members of the family, such as the Pareto distribution, the Burr distribution and log-gamma distribution, all usually considered to be heavy-tailed, have been seen to relatively well fit observed claims data. As remarked upon in Mikosch [20], regular variation can on a more general level be described as a small deviation from exact power law [2] behaviour and since these laws have been observed to reasonably well describe various social or natural phenomena, it becomes of interest to study regularly varying functions. For example, the exceedances of a high threshold by i.i.d. data can be described by a power law behaviour. In terms of mathematical properties, regularly varying distributions are closed under summation (cf. Theorem 2.1 and Corollary 2.1) and also possess the property that the right tail of the distribution of $S_n = X_1 + \cdots + X_n$ is determined by the right tail of the distribution of $\max(X_1, \ldots, X_n)$ (cf. Equation 2.4). The closure property implies that the heavy-tail of the individual $X_i : s$ cannot be diversified away in the sense of the Central Limit Theorem.

For a comprehensive treatment of regular variation, its theoretical properties and various applications, please see Bingham et al. [3]. For a short note on the relationship between regular variation and probability theory, please see Bingham [2].

Definition 2.2 (Slowly varying function, p. 99 in Mikosch [20])**.** *Let a function $L(x)$ on $(0, \infty)$ be called slowly varying function (at infinity) if $\lim_{x\to\infty} \frac{L(cx)}{L(x)} = 1$, for all $c \geq 0$.*

Definition 2.3 (Regularly varying functions and and regularly varying random variable, p. 99 in Mikosch [20])**.** *Let $L(x)$ be as in Definition 2.2. Then*

1. *For any $\delta \in \mathbb{R}$, the function*
$$f(x) = x^{\delta} L(x), \quad x \geq 0,$$
is said to be regularly varying with index δ.

2. *A positive random variable X and its distribution are said to be regularly varying with tail index $\alpha \geq 0$ if*
$$P(X > x) = L(x)x^{-\alpha}, \quad x \geq 0.$$

To illustrate Definition 2.3 an example is provided below.

Example 2.3. By studying the right tail of the Exponential and Gamma distributions, respectively, it is seen that they do not admit the representation in Definition 2.3 above. The case is different with the Pareto distribution, $P(X > x) = (\frac{\sigma+x}{\sigma})^{-\alpha}$, as

$$P(X > x) = \left(\frac{\sigma + x}{\sigma}\right)^{-\alpha} = x^{-\alpha}\left(\frac{1 + \sigma/x}{\sigma}\right)^{-\alpha},$$

and, with $L(x) = \left(\frac{1+\sigma/x}{\sigma}\right)^{-\alpha}$,

$$\begin{aligned}
\lim_{x\to\infty} \frac{L(cx)}{L(x)} &= \lim_{x\to\infty} \left(\frac{1 + \sigma/cx}{\sigma}\right)^{-\alpha} \Big/ \left(\frac{1 + \sigma/x}{\sigma}\right)^{-\alpha} \\
&= \lim_{x\to\infty} \left(\frac{1 + \sigma/cx}{1 + \sigma/x}\right)^{-\alpha} \\
&= 1
\end{aligned}$$

The right tail of the Pareto distribution is thus seen to be regularly varying with tail index α.

[2]The power law (also called the scaling law) states that a relative change in one quantity results in a proportional relative change in another. Mathematically, it defined as $f(x) = ax^k$ where a and k are constants. An introduction to power laws and their applicability can be found in [24].

A useful concept for the sequel will be the *integrated tail distribution.*

Definition 2.4 (Integrated Tail Distribution, Mikosch [19], p. 43)**.** *The integrated tail distribution, $F_{X,I}(u)$, of $F_X(x)$, is defined as*

$$\overline{F}_{X,I}(u) = \frac{1}{EX}\int_0^u \overline{F}_{I,X}(x)\,\mathrm{d}x,$$

for $u > 0$.

Example 2.4 shows that the integrated tail distribution of the Pareto distribution is also regularly varying.

Example 2.4. Let $\overline{F}_X(x) = (\frac{\sigma+x}{\sigma})^{-\alpha}$ with $EX = \frac{\sigma}{\alpha-1}$. Then, $F_{I,X}(x)$ is given by

$$\begin{aligned}
F_{X,I}(u) &= \frac{1}{EX}\int_0^u \overline{F}_X(x)dx \\
&= \frac{\alpha-1}{\sigma}\int_0^u \left(\frac{\sigma+x}{\sigma}\right)^{-\alpha} dx \\
&= \frac{\alpha-1}{\sigma}\left[-\frac{\sigma}{\alpha-1}\left(\frac{\sigma+x}{\sigma}\right)^{-(\alpha-1)}\right]_0^u \\
&= 1 - \left(\frac{\sigma+u}{\sigma}\right)^{-(\alpha-1)}.
\end{aligned}$$

Hence, $\overline{F}_{I,X}(x) = (\frac{\sigma+x}{\sigma})^{-(\alpha-1)}$ and it is seen that $\overline{F}_{I,X}(x)$ is regularly varying with tail index $\alpha - 1$.

Theorem 2.1 states that heavy tails, in the sense of regularly varying tails, cannot be diversified away, in the sense of the Central Limit Theorem, by aggregating independent claim sizes. Here, $o(1) \to 0$ as $x \to \infty$.

Theorem 2.1 (Closure property of regularly varying random variables, p. 101 in Mikosch [20])**.** *Assume X_1 and X_2 are independent regularly varying random variables with the same tail index $\alpha > 0$, i.e.*

$$P(X_i > x) = L_i(x)x^{-\alpha}, \quad x \geq 0.$$

for possibly different slowly varying functions L_i. Then $X_1 + X_2$ is regularly varying with the same tail index, i.e.

$$P(X_1 + X_2 > x) = x^{-\alpha}(L_1(x) + L_2(x))(1 + o(1)), \quad x \geq 0.$$

as $x \to \infty$.

An important corollary to Theorem 2.1 is Corollary 2.1.

Corollary 2.1 (p. 102 in Mikosch [20])**.** *Assume $X_1, ..., X_n$ are n i.i.d. regularly varying random variables with tail index $\alpha > 0$ and with distribution function F. Then, $S_n = \sum_{i=1}^n X_i$ is regularly varying with tail index $\alpha > 0$ and, for large x, the following approximation is valid*

$$P(S_n > x) = n\overline{F}(x)(1 + o(1)).$$

Please note that Theorem 2.1 and Corollary 2.1 are asymptotic results. The implication of Theorem 2.1 and Corollary 2.1 is that the tail of the distribution of $S_n = X_1 + \cdots + X_n$ does not get "averaged out" or "diversified away" when the distribution function, F, is regularly varying. It thus shows that the Central Limit Theorem is dangerous to use in this case since it underestimates the probability of very large losses. In contrast, when considering a severity distribution that does not exhibit this property, e.g. the Exponential distribution or the Gamma distribution, the Central Limit Theorem should still be able to give a fairly accurate description of the right tail of S_n. The below example illustrates the above reasoning for two portfolios with different number of risks, n.

Example 2.5. In this example the suitability of the Central Limit Theorem is investigated for two portfolios of risks, $n = 20$ and $n = 100$, respectively. Figure 1 and Table 1 show to what extent the Central Limit Theorem is able to approximate the distribution of S_n for each choice of severity distribution for the individual X_i.

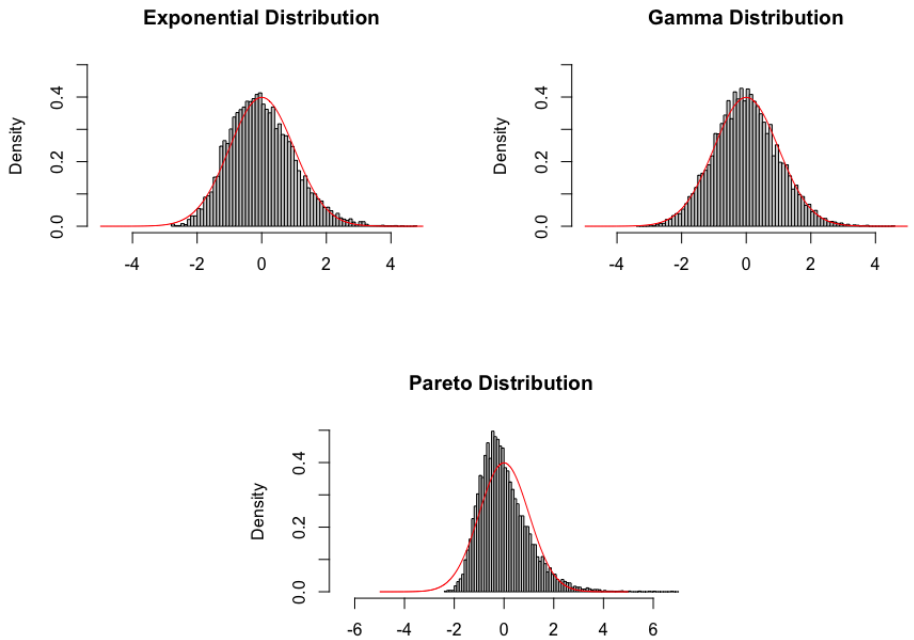


Figure 1: Histograms illustrating the approximation given by Central Limit Theorem for a portfolio of 20 risks, i.e. $S_n = \sum_{i=1}^{n} X_i$ with $n = 20$. The top-left exhibit shows the approximation for the Exponential distribution, the top-right exhibit the Gamma distribution (top right), and the bottom one the approximation for the Pareto distribution. The red line in each histogram is the density function of the standard Normal distribution.

Distribution/Quantile	90th	95th	99th	99.5th	99.7th	99.9th
Standard Normal distribution	1.2816	1.6449	2.3263	2.5758	2.7478	3.0902
$S_n = \sum_{i=1}^{n} X_i, X_i \sim \text{Exp}(1)$	1.3216	1.7693	2.6405	3.0065	3.1627	3.6394
$S_n = \sum_{i=1}^{n} X_i, X_i \sim \Gamma(5, 2)$	1.2914	1.6813	2.4363	2.7694	2.9583	3.3976
$S_n = \sum_{i=1}^{n} X_i, X_i \sim \text{Pareto}(4, 4)$	1.2898	1.8263	3.0576	3.5483	3.8371	4.7701

Table 1: A comparison of the right tail of S_n for $n = 20$ with the quantiles of the standard Normal distribution to see the quality of approximation of the Central Limit Theorem.

By studying Figure 1, the Central Limit Theorem does seem to provide an acceptable approximation in the bulk of the distribution in case of the Exponential and Gamma distributions and also in the right tail; for the Pareto distribution, the approximation, on the other hand, is poor even in the bulk. When considering the right tail, as shown in Table 1, the approximation becomes worse the more extreme the quantile under consideration, especially for the Pareto distribution.

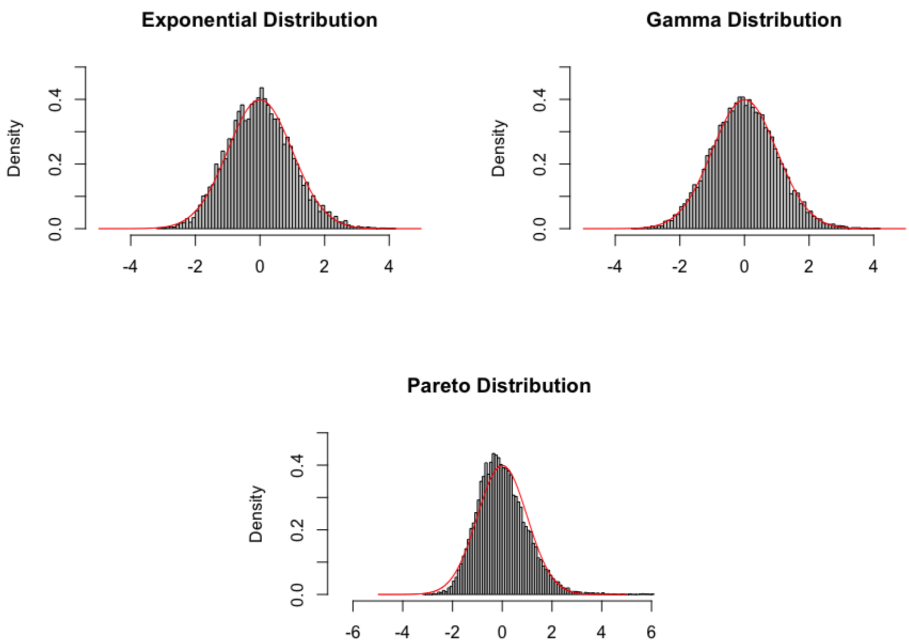


Figure 2: Histograms illustrating the approximation given by Central Limit Theorem for a portfolio of 100 risks, i.e. $S_n = \sum_{i=1}^{n} X_i$ with $n = 100$. The top-left exhibit shows the approximation for the Exponential distribution, the top-right exhibit the Gamma distribution (top right), and the bottom one the approximation for the Pareto distribution. The red line in each histogram is the density function of the standard Normal distribution..

Distribution/Quantile	90th	95th	99th	99.5th	99.7th	99.9th
Standard Normal distribution	1.2816	1.6449	2.3263	2.5758	2.7478	3.0902
$S_n = \sum_{i=1}^n X_i, X_i \sim \text{Exp}(1)$	1.2797	1.7026	2.5055	2.7482	3.0313	3.4771
$S_n = \sum_{i=1}^n X_i, X_i \sim \Gamma(5, 2)$	1.2826	1.6757	2.4297	2.7684	2.9437	3.3219
$S_n = \sum_{i=1}^n X_i, X_i \sim \text{Pareto}(4, 4)$	1.2885	1.7207	2.6519	3.1668	3.6176	4.2764

Table 2: A comparison of the right tail of S_n for $n = 100$ with the quantiles of the standard Normal distribution to see the quality of approximation of the Central Limit Theorem.

In Figure 2 above, again the Central Limit Theorem does seem to provide an acceptable approximation in the bulk of the distribution for each severity distribution. When considering the right tail of the distribution in Table 2, the approximation is better but still fails to account for the heavy tail of the Pareto distribution. The approximation in the tail for the Exponential and Gamma distributions is also better although it does not entirely manage to capture the tail behaviour in each case.

Example 2.5 above illustrates the danger of relying on the Central Limit Theorem in an indiscriminate manner: attention must be paid to the individual severity distribution and whether or not it is light-tailed or heavy-tailed since the tail behaviour of the aggregate severity distribution, S_n, is fundamentally different in each case. Even increasing the portfolio size in case of the Pareto distribution does not help since still the probability for very large lasses would be underestimated using the Central Limit Theorem.

One further property can be shown if X_i has a regularly varying right tail. Let $M_n = \max(X_1, \ldots, X_n)$ denote the partial maximum. Then, for $n \geq 2$ and as $x \to \infty$ (see Mikosch [20], p. 102)

$$\begin{aligned}
P(M_n > x) &= P(X_1 > x) + P(X_1 \leq x, X_2 > x) + \cdots + P(X_1 \leq x, X_2 \leq x, \ldots, X_n > x) \\
&= P(X_1 > x) + P(X_1 \leq x)P(X_2 > x) + \cdots + P(X_1 \leq x)P(X_2 \leq x) \cdots P(X_n > x) \\
&= \overline{F}(x) + F(x)\overline{F}(x) + \cdots + F^{n-1}(x)\overline{F}(x) \\
&= \overline{F}(x)(1 + F(x) + \cdots + F^{n-1}(x)) \\
&= \overline{F}(x) \sum_{k=0}^{n-1} F^k(x) \\
&= n\overline{F}(x)(1 + o(1))
\end{aligned}$$

Using 2.2 , the statement of Theorem 2.1 can be reformulated as if X_i is regularly varying with tail index $\alpha > 0$, then

$$\lim_{x \to \infty} \frac{P(S_n > x)}{P(M_n > x)} = 1 \tag{2.4}$$

for $n \geq 2$. (2.4) implies that, under the assumption of regular variation, the distribution of the tail of S_n is essentially determined by the tail of the distribution of M_n. This is yet another way of thinking of what properties a heavy-tailed distribution should exhibit. For reference, please see Mikosch [20], p. 102.

2.3 Subexponential Distributions

The subexponential class of distributions is another candidate for classifying a severity distribution as heavy-tailed. The class derives it name from one of its properties, namely that the right tail of a subexponential distribution decreases more slowly than any exponential tail; cf. p. 1 in

Goldie and Klüppelberg [11]. It contains the class of regularly varying distributions in Section 2.2 as a sub-class and is heavily relied upon when fitting distributions to actual claims data (cf. Mikosch [20], Section 3.2.4.). Similar to the class of regularly varying function, it is closed under summation and the behaviour of the right tail of the distribution of the sum, $S_n = X_1, \cdots, X_n$, $\{X_n,\, n \geq 1\}$, an i.i.d. sequence with each X_i being subexponential, is determined by the distribution of the right tail of $\max(X_1, \ldots, X_n)$. In addition, it also possesses the interesting property that S_n is large precisely due the first k terms of the sum being large, see Embrechts et al. [8], Chapter 1.

For an introduction to subexponential distributions and their properties, including ruin theory, please see any of Goldie and Klüppelberg [11], Klüppelberg [15], Klüppelberg [16] or Teugels [26]. For their application in an non-life insurance context, please see e.g. Embrechts et al. [8] or Klüppelberg and Mikosch [17].

Definition 2.5 (Subexponential distribution, p. 103 in Mikosch [20])**.** *The positive random variable X with unbounded support and its distribution are said to be subexponential if for a sequence of i.i.d. random variables $(X_{I,n})_{n\in\mathbb{N}}$ with $n \geq 2$, the following relation holds*

$$P(S_n > x) = P(M_n > x)(1 + o(1)) \tag{2.5}$$

for large x.

By using $P(M_n > x) = n\overline{F}(x)(1 + o(1))$ from the last section, the defining property (2.5) above can be expressed as

$$\lim_{x\to\infty} \frac{P(S_n > x)}{\overline{F}(x)} = n \tag{2.6}$$

for $n \geq 2$. For an example of a subexponential distribution, by comparing (2.4) with Definition 2.5, note that every distribution function, F, with a regularly varying right tail with tail index $\alpha > 0$ is also subexponential. In particular, the Pareto distribution is subexponential.

An illustration of (2.6) is given in Figure 3 in case of the Pareto distribution.

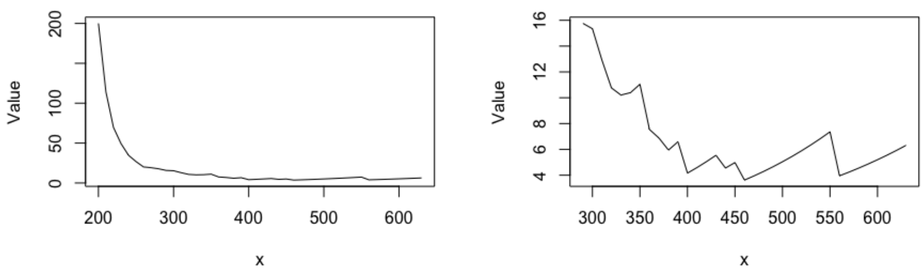


Figure 3: Illustration of (2.6) when $n = 100$ in case of the Pareto distribution, Pa$(4, 4)$. Note the jagged shape of the line in the right-hand graph due to $P(S_n > x)$ being approximated numerically. Note further the large value of x required to obtain a relatively good approximation of (2.6). Here,

The Exponential distribution, on the other hand, is not subexponential, as is shown in Example 2.6.

Example 2.6. If $X \sim \text{Exp}(\gamma)$, then, $S_n \sim \Gamma(n, \gamma)$. Also, it holds

$$\begin{aligned} P(M_n > x) &= 1 - P(M_n \leq x) \\ &= 1 - [F(x)]^n \\ &= 1 - (1 - e^{\gamma x})^n \end{aligned}$$

since the $M_n \leq x$ if all $X_i \leq x$, $i = 1, \ldots, n$. By using l'Hospital's rule, (2.6) can be evaluated and it is seen that

$$\lim_{x \to \infty} \frac{P(S_n > x)}{P(M_n > x)} \neq 1$$

for any $n \geq 2$.

In Theorem 2.2, three basic properties of subexponential distributions are listed.

Theorem 2.2 (Basic properties of subexponential distributions, pp. 103 - 104 in Mikosch [20])**.**

1. *If F is subexponential, then for any $y > 0$,*

$$\lim_{x \to \infty} \frac{\overline{F}(x - y))}{\overline{F}(x)} = 1 \tag{2.7}$$

2. *If (2.7) holds above, then for all $\epsilon > 0$,*

$$e^{\epsilon x}\overline{F}(x) \to \infty, \quad x \to \infty. \tag{2.8}$$

3. *If F is subexponential, then, given $\epsilon > 0$, there exists a finite constant K so that for all $n \geq 2$,*

$$\frac{P(S_n > x)}{\overline{F}(x)} \leq K(1 + \epsilon)^n, \quad x \geq 0. \tag{2.9}$$

As pointed out in Mikosch [20] pp. 104-105, (2.7), means that the tails $P(X > x)$ and $P(X > x + y)$ are not significantly different if x is sufficiently large, for a fixed $y > 0$. Hence, if $x \to \infty$, then

$$\begin{aligned} \frac{\overline{F}(x - y)}{\overline{F}(x)} &= \frac{P(X > x + y)}{P(X > x)} \\ &= \frac{P(X > x + y, X > x)}{P(X > x)} \\ &= P(X > x + y \mid X > x) \to 1. \end{aligned} \tag{2.10}$$

Thus, once X has exceeded a large threshold, it is very likely it will exceed an even larger threshold. This can be taken as yet another definition of a heavy-tailed distribution. To illustrate (2.10), consider the below example.

Example 2.7.

1. Exponential distribution, $\text{Exp}(\gamma)$. By using (2.7),

$$\lim_{x \to \infty} \frac{e^{-\gamma(x+y)}}{e^{-\gamma x}} = e^{-\gamma y} < 1$$

2. Pareto distribution, Pareto(α, σ). Again, by using (2.7)

$$\lim_{x\to\infty} \left(\frac{1 + \frac{x+y}{\sigma}}{1 + \frac{x}{\sigma}} \right)^{-\alpha} = 1$$

The second property, (2.8), is what motivates the name *subexponential* since the right tail of the distribution decays slower to 0 than any exponential function. The same property also says that the moment generating function for $h > 0$ does not exist for subexponential distributions. Indeed,

$$\begin{aligned} E(e^{hX}) &= \int_0^\infty P(e^{hX} > y)dy = \int_0^\infty P(X > \log(y)/h)dy \\ &= h \int_{-\infty}^\infty e^{hX} P(X > x)dx = \infty \end{aligned} \tag{2.11}$$

as $x \to \infty$, where the third equal sign is obtained by the substitution $x = \log(y)/y$. The fact that the moment generating function does not exist for subexponential distributions will be useful in Chapter 3 on the Cramér-Lundberg model.

Another interesting property of subexponential distributions is the following, which says that S_n is large due to precisely the sum of the first k terms being large since, for $1 \leq k \leq n$,

$$\begin{aligned} \lim_{x\to\infty} P(S_k > x \mid S_n > x) &= \lim_{x\to\infty} \frac{P(S_k > x, S_n > x)}{P(S_n > x)} \\ &= \lim_{x\to\infty} \frac{P(S_k > x)}{\overline{F}(x)} \frac{\overline{F}(x)}{P(S_n > x)} \\ &= \lim_{x\to\infty} \frac{\frac{P(S_k > x)}{\overline{F}(x)}}{\frac{P(S_n > x)}{\overline{F}(x)}} \\ &= \frac{k}{n}. \end{aligned} \tag{2.12}$$

In particular, $k = 1$ says that S_n is large due to one term being large.

3 Ruin Theory

This chapter outlines the basics of ruin theory and starts with defining the well-known homogeneous Poisson process along with the remarkable order statistics property of the Poisson process, the latter being very useful for simulation purposes, in Section 3.1. The chapter then proceeds with defining the risk process, $U(t)$ and the corresponding ruin probability, $\psi(u)$, in the setting of the famous Cramér-Lundberg model, see Definition 3.4, and a regularity condition in form of the Net Profit Condition, Definition 3.6, and then proceeds to distinguish between the light-tailed case and the heavy-tailed case in terms of the behaviour of the ruin probability, $\psi(u)$, where u is the amount of initial capital, in Section 3.2.1 and Section 3.2.2. In the light-tailed case, Lundberg's exponential bound, Theorem 3.2, shows that there exists an upper bound for the ruin probability, and Cramér's theorem, Theorem 3.3, gives the exact asymptotics for the ruin probability. In the heavy-tailed case, Theorem 3.5 shows that these asymptotics is entirely different with the decay being much slower. In both instances, the asymptotics should be understood in the sense of how the ruin probability, $\psi(u)$, behaves as the initial capital, u, tends to infinity.

The literature on ruin theory is vast and there exist several works worthwhile consulting. For further reference and reading, please see any of Bühlmann [4], Embrechts et al. [8], Grandell [12], Rolski [21] or Rolski et al.[22].

3.1 Homogeneous Poisson Process

Definition 3.1 (Counting process, p. 297 in Ross [23]). *A stochastic process, $N(t), t \geq 0$, is a counting process if $N(t)$ represents the number of events by time t.*

As stated in Ross [23], the counting process $\{N(t), t \geq 0\}$ satisfies the following criteria:

(i) $N(t) \geq 0$

(ii) $N(t)$ is integer valued.

(iii) If $s < t$, then $N(s) \leq N(t)$

(iv) For $s < t$, $N(t) - N(s)$ equals the number of events that occur in the interval $(s, t]$

In addition, $N(t)$ is said to have independent increments if the numbers of events in disjoint time intervals are independent. This means that, for $0 < s < t$, the number of events in the interval $(s, t]$, $N(t) - N(s)$, is independent of the number of events, $N(s)$, in the interval $(0, s]$.

The Poisson process, see Definition 3.2, is an example of a counting process typically used in non-life insurance to model the arrival of claims.

Definition 3.2 (Homogeneous Poisson process with intensity λ, p. 304 in [23]). *The counting process $\{N(t), t \geq 0\}$ is said to be a homogeneous Poisson process with intensity λ, $\lambda > 0$, if*

(i) $N(0) = 0$

(ii) The process has independent increments, i.e. the number of events in disjoint intervals are independent.

(iii) The number of events in any interval of length t is Poisson distributed with mean value λt.

A Poisson process is thus a counting process with an additional structure imposed. Although not always the most realistic model choice it possesses a number of analytical properties, which makes it tractable to use as a benchmark for modelling the claims arrival process, see e.g. Mikosch [20], pp. 32-38 for a discussion in the context of the Danish Fire Insurance Data 1980-1990.

In Chapter 3, the notion of *inter-arrival times* will be used. Definition 3.3 introduces this concept in the context of a renewal counting process.

Definition 3.3 (Renewal Counting Process, p. 53 in Mikosch [20]). *Let $\{W_n, n \geq 1$ be an i.i.d. sequence of positive random variables. Then,*

$$T_0 = 0, \quad T_n = W_1 + \cdots + W_n$$

is said to be a renewal sequence and the counting process, and

$$N(t) = \#\{i \geq 1 : T_i < t\}, \quad t \geq 0.$$

is the corresponding renewal (counting) process.

The Poisson process is an example of a renewal counting process (Mikosch [20], p. 53). The sequences $\{W_n = T_n - T_{n-1}, n \geq 1$ and $\{T_n, n \geq 1$ are referred to as the inter-arrival and arrival times, respectively, of the renewal process $N(t)$. The inter-arrival time is the time elapsed between two successive events of the renewal sequence. In a non-life insurance context, the T_n represent the claim arrival times and W_n the time elapsed between the n:th and $(n-1)$:th claims. For the Poisson process in Definition 3.2, the sequence of inter-arrival times are exponentially distributed with parameter λ (see [23] for further details).

Example 3.1. Let $0 \leq T_1 \leq T_2 \leq \dots$ denote the event arrival times of the Poisson process in Definition 3.2 and let $W_n := T_n - T_{n-1}, n \geq 1$ with $T_0 = 0$, denote the inter-arrival times of the process. Now, consider the event $\{T_1 > t\}$, which implies that $N(t) = 0$, i.e. no events have occurred in $[0, t]$. By using the properties of the Poisson process, it holds

$$P(W_1 > t) = P(N(t) = 0) = e^{-\lambda t},$$

which means that W_1 is $\text{Exp}(\lambda)$ distributed, i.e. considered from $t = 0$ the time until the first event is $\text{Exp}(\lambda)$ distributed.

Moreover, for W_n and W_{n-1} and $0 < s < t$:

$$\begin{aligned} P(W_n > t \,|W_{n-1} = s) &= P(\text{no events in } (s, s+t]|W_{n-1} = s) \\ &= P(\text{no events in } (s, s+t]) \\ &= e^{-\lambda t}, \end{aligned}$$

which, again, means that, seen from time s, the time until the next event is $\text{Exp}(\lambda)$ distributed.

Before closing this section, the order statistics property of the Poisson process is stated in the form of a theorem below. This is a remarkable property and at the same time one of the characterising properties of the process which will be very useful when simulating. For reference, please see Mikosch [20], Section 2.1.6, for the proof of the theorem and more background on order statistics.

Theorem 3.1 (Order statistics property of the homogeneous Poisson process, p. 24 in Mikosch [20])**.** *Consider the homogeneous Poisson process of Definition 3.2, $\{N(t), t \geq 0\}$, with continuous a.e. positive intensity function λ and arrival times $0 < T_1 < T_2 \cdots < T_n$ a.s. Then, the conditional distribution of the vector $(T_1, \dots, T_n)$ given $\{N(t) = n\}$ is the distribution of the ordered sample $(X_{(1)}, \dots, X_{(n)})$ of an i.i.d. sample $X_1, \dots, X_n$ with common probability density $\frac{\lambda}{t}$ on $0 < x \leq t$:*

$$(T_1, \dots, T_n \,|N(t) = n) \stackrel{\text{d}}{=} (X_{(1)}, \dots, X_{(n)})$$

In other words, the left-hand vector has the conditional density

$$\begin{aligned} f_{T_1,\dots,T_n}(x_1, \dots, x_n \,|N(t) = n) &= \frac{n!}{(\lambda t)^n}\lambda^n \\ &= n!\, t^{-n} \end{aligned}$$

for $0 < x_1 \cdots < x_n < t$.

The joint conditional density of the arrival times of the homogeneous Poisson process is thus equal to the joint density of a uniform ordered sample $U_{(1)} <, \dots, < U_{(n)}$ of an i.i.d. sample $U_1, \dots, U_n$, $U_i \sim U(0, t)$. Thus, given there are n arrivals in the interval $[0, t]$, these arrivals constitute the points of a uniform ordered sample in $(0, t)$.

3.2 Risk Process and the Cramér-Lundberg Model

The object of interest in this chapter is the so-called *surplus* or *risk process* defined according to

$$U(t) = u + p(t) - S(t), \quad t \geq 0.$$

where,

- $u \geq 0$ is the initial capital of the insurer at $t = 0$.
- $p(t)$ is the continuous premium income of the portfolio. In what follows, $p(t)$ is assumed to be linear and deterministic, i.e. $p(t) = ct$, where $c \geq 0$ is the premium rate of the portfolio.
- $S(t)$ is the total claim amount process of the portfolio, see Definition 3.4.

$U(t)$ can be thought of as the insurer's surplus at time $t \geq 0$.

The Cramér-Lundberg model is defined in Definition 3.4 below.

Definition 3.4 (Cramér-Lundberg model, p. 12 in Mikosch [20])**.** *In the Cramér-Lundberg model, the following assumptions hold for the total claim amount process $S(t)$:*

$$S(t) = \sum_{i=1}^{N(t)} X_i, \quad t \geq 0:$$

- *Claims happen at the arrival times $0 \leq T_1 \leq T_2 \leq \cdots$ of a homogeneous Poisson process $N(t) = \#\{i \geq 1 : T_i \leq t$, $t \geq 0$ with intensity λ.*
- *The* i*:th claim arriving at T_i causes the claim size causes the claim size X_i. The sequence (X_i) constitutes an i.i.d. sequence of non-negative random variables.*
- *The sequences (T_i) and (X_i) are independent. In particular, $N(t)$ and (X_i) are independent.*

In ruin theory, one is concerned with the probability that the insurer's surplus, $U(t)$, at some future time point is negative. The event $U(t) < 0$ is called the *ruin* of the insurance company.

Definition 3.5 (Ruin, Ruin Time, Ruin Probability, pp. 152-153 in Mikosch [20])**.** *The event that $U(t)$ falls below null is called* ruin:

$$\text{Ruin} = \{U(t) < 0 \, for \, some \, t > 0\}$$

The time T when $U(t)$ falls below null for the first time is called ruin time:

$$T = \inf\{t > 0 : U(t) < 0\}$$

The probability of ruin is then given by

$$\psi(u) = P(\text{Ruin} \mid U(0) = u) = P(T < \infty), \quad u > 0.$$

By construction of the risk process, $U(t)$, ruin can only occur at times $t = T_n$ $t \geq 1$, i.e. when a claim arrives, since $U(t)$ linearly increases in the intervals (T_n, T_{n+1}). The sequence $(U(T_n))_{n \geq 1}$ is called the *skeleton process* of the risk process, $U(t)$, or, alternatively, as in Wurtich [27] p. 133, the switch to operational time. Regardless of its name, this transformation makes it possible to express ruin in terms of the inter-arrival times, W_n (see Definition 3.2 and Definition 3.3, the claim sizes X_n and the premium rate c.

$$\begin{aligned}\text{Ruin} &= \Big\{\inf_{t>0} U(t) < 0\Big\} = \Big\{\inf_{n\geq 1} U(T_n) < 0\Big\} \\ &= \Big\{\inf_{n\geq 1} [u + p(T_n) - S(T_n)] < 0\Big\} \\ &= \Big\{\inf_{n\geq 1} \Big[u + p(T_n) - \sum_{i=1}^{n} X_i\Big] < 0\Big\}.\end{aligned}$$

Now, write

$$Z_n = X_n - cW_n, \quad S_n = Z_1 + \cdots + Z_n \quad n \geq 1, \quad S_0 = 0,$$

then the ruin probability, $\psi(u)$, can be formulated alternatively as

$$\psi(u) = P\Big(\inf_{n\geq 1}(-S_n) < -u\Big) = P\Big(\sup_{n\geq 1} S_n > u\Big). \tag{3.1}$$

Much effort has been expended on studying $\lim_{u\to\infty}\psi(u)$, the asymptotic behaviour of the ruin probability as the amount of initial capital tends to infinity (see e.g. Embrechts et al [8], Mikosch [20], Chapter 4, or any of the references listed in Chapter 1. It will be apparent from Section 3.2.1 and Section 3.2.2 that this behaviour is fundamentally different depending on whether or not the severity distribution for X_i is light-tailed or heavy-tailed. Before proceeding to this, a regularity condition, Definition 3.6, is needed on the process Z_n in order to avoid studying the case in which ruins occurs with probability 1 a.s., regardless of the amount of initial capital, u, see e.g. Spitzer [25].

Definition 3.6 (Net Profit Condition, p. 156 in Mikosch [20]). *The sequence Z_n is said to satisfy the Net Profit Condition (NPC) if*

$$EZ_1 = EX_1 - cEW_1 < 0,$$

i.e. the premium rate c satisfies $c = \frac{EX_1}{EW_1} > 0$.

The NPC can also be expressed in terms of a safety loading, $\rho > 0$, which implies a premium rate, c according to:

$$c = (1+\rho)\frac{EX_1}{EW_1}. \tag{3.2}$$

3.2.1 Light-tailed Case

In this section, the famous Lundberg upper bound on the ruin probability is formulated as well as Cramér's equally famous theorem on the exact asymptotics for the ruin probability. Both results are valid in the light-tailed case in the sense that the moment generating function exists in some neighbourhood around the origin of X_i. The existence of the moment generating function for the claims, X_1, assumes that the right tail of the severity distribution decays exponentially fast. The assumption of an exponentially decaying right tail excludes in particular subexponential severity distributions as shown by (2.11). Therefore, for heavy-tailed distributions such as e.g. the Burr, Pareto and Weibull ($\tau < 1$) distributions, neither the Lundberg bound nor Cramér's theorem are applicable. In what follows, the assumptions of the Cramér-Lundberg model and NPC are always valid.

Definition 3.7 (Adjustment or Lundberg coefficient, p. 158 in Mikosch [20])**.** *Assume that the moment generating function of Z_1 exists in some neighbourhood $(-h_0, h_0), h_0 \geq 0$ of the origin. If a unique solution r exists to the equation*

$$M_{Z_1}(h) = Ee^{h(X_1 - cW_1)} = 1 \tag{3.3}$$

exits it is called the adjustment *or* Lundberg coefficient*.*

(3.3) will be referred to as the Lundberg equation. Two examples illustrating the usage of the Definition 3.7 are given below in case of the Exponential distribution and the Gamma distribution, respectively.

Example 3.2 (Exponential distribution)**.** Assume the Cramér-Lundberg model with claims sizes i.i.d. Exp(γ) and $N(t) \sim \text{Po}(\lambda t)$. The latter assumption means that inter-arrival times, W_i, are Exp(λ) random variables. Now, since

$$M_{X_1}(h) = \frac{\gamma}{\gamma - h}, \quad h < \gamma$$

and

$$M_{cW_1}(-h) = \frac{\lambda}{\lambda + ch}, \quad h > -\frac{\lambda}{c},$$

(3.3) takes the form

$$M_{Z_1}(h) = \frac{\gamma}{\gamma - h}\frac{\lambda}{\lambda + ch} = 1, \quad -\frac{\lambda}{c} < h < \gamma,$$

which can be solved analytically and gives the solution, r, as

$$r = \gamma - \frac{\lambda}{c} > 0.$$

By using (3.2) the premium rate, c, can be written as

$$c = \frac{EX_1}{EW_1}(1 + \rho)$$

and thus the adjustment coefficient, r, can be expressed as

$$r = \gamma\frac{\rho}{1 + \rho}.$$

In Figure 4 below, $M_{Z_1}(h)$ is illustrated for $\gamma = 0.5$, $\lambda = 1$ and $\rho = 0.05$, which yields $r \approx 0.024$.

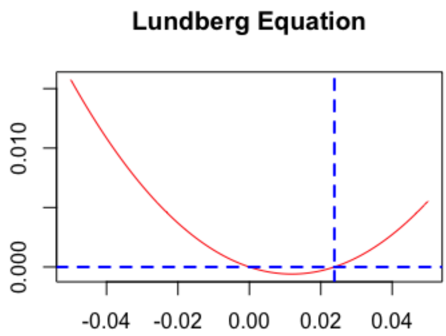


Figure 4: Illustration of the solution to the (3.3) in case of the Exponential distribution, $\text{Exp}(\gamma)$. For the mathematical expression of $M_{Z_1}(h)$ in this case, see Example 3.2. The red, solid line shows $M_{Z_1}(h)$ and the blue dashed line the analytical solution, $r = \gamma\frac{\rho}{1+\rho}$. With $\gamma = 0.5$, $\lambda = 1$ and $\rho = 0.05$, the numerical value of r is approximately 0.024.

Example 3.3 below determines the adjustment coefficient in case of the Gamma distribution.

Example 3.3 (Gamma distribution)**.** Again, assume the Cramér-Lundberg model but this time with claims i.i.d. $\Gamma(\alpha, \beta)$ and $N(t) \sim \text{Po}(\lambda t)$ as before. Thus, $M_{cW_1}(-h)$ is unchanged and $M_{X_1}(h)$ is given by

$$M_{X_1}(h) = \left(\frac{\beta}{\beta - h}\right)^{\alpha}, \quad h < \beta.$$

Equation (3.3) is thus given by

$$M_{Z_1}(h) = \left(\frac{\beta}{\beta - h}\right)^{\alpha} \frac{\lambda}{\lambda + ch} = 1, \quad -\frac{\lambda}{c} < h < \beta,$$

which is here solved numerically. In Figure (5) below, $M_{Z_1}(h)$ is illustrated for $\alpha = 5$, $\beta = 2$ and $\rho = 0.05$, which yields $r \approx 0.0160$.

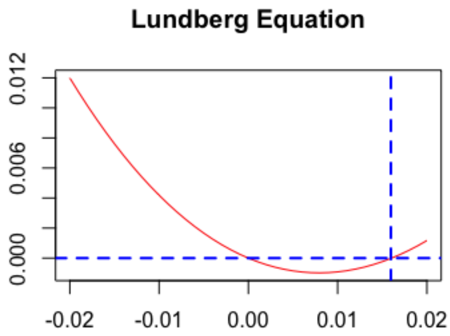


Figure 5: Illustration of the solution to the (3.3) in case of the Gamma distribution, $\Gamma(\alpha, \beta)$). For the mathematical expression of $M_{Z_1}(h)$ in this case, see Example 3.3. The red, solid line shows $M_{Z_1}(h)$ and the blue dashed line the numerical solution. With $\alpha = 5$, $\beta = 2$ and $\rho = 0.05$, r is estimated to 0.0160.

After these two examples, the Lundberg Inequality is stated below.

Theorem 3.2 (The Lundberg inequality, p. 159 in Mikosch [20])**.** *Assume the renewal model with NPC satisfied and assume further the existence of the adjustment coefficient, r. Then the following inequality holds for all* $u \geq 0$*:*

$$\psi(u) \leq e^{-ru}. \tag{3.4}$$

The ruin probability, $\psi(u)$, is thus exponentially bounded, the size of which depends on the amount on initial capital, u, and the value of the adjustment coefficient, r. This means that the ruin probability can thus be made arbitrarily small for a sufficiently large value of the initial capital, as shown in Example 3.4 below.

Example 3.4 (Lundberg inequality for Exp(γ) claim sizes)**.** From Theorem 3.2 and Example 3.2, the Lundberg inequality, (3.4), is

$$\begin{aligned}\psi(u) &\leq e^{-(\gamma - \frac{\lambda}{c})u} \\ &= e^{-\gamma \frac{\rho}{1+\rho} u}.\end{aligned}$$

It is clear that $\psi(u)$ can be made arbitrarily small if the initial capital is sufficiently large. The same observation holds for the safety loading ρ since $\frac{\rho}{1+\rho}$ is close to 1 if ρ is large enough. From this follows also that the upper bound does not change significantly for large ρ. Lastly, by noting that $\gamma = \frac{1}{\frac{1}{\gamma}} = (EX_1)^{-1}$ it is seen that the smaller the expected claim size, the smaller the ruin probability.

Cramér's theorem on the exact asymptotics of the ruin probability is given in Theorem 3.3 below. The theorem is quite astonishing insofar as it gives exact asymptotics for a complex object such as $\psi(u)$.

Theorem 3.3 (Cramér's ruin bound, p. 162 in Mikosch [20])**.** *Consider the Cramér-Lundberg model with NPC satisfied. In addition, assume that the claim size distribution function* F_{X_i} *has* *a density, that the moment generating function of X_i exits in some neighbourhood $(-h_0, h_0)$ of the origin and that the adjustment coefficient, r, exits and lies in $(0, h_0)$. Then there exists a constant $C > 0$ such that*

$$\lim_{u\to\infty} e^{ru}\psi(u) = C.$$

By using Theorem 3.3 in conjunction with Theorem 3.4, it is possible to obtain equality in Theorem 3.2 in case of the Exponential distribution. Theorem 3.4 states that the *non-ruin probability*, $\phi(u) = 1 - \psi(u)$, can be represented as a compound geometric probability.

Theorem 3.4 (Representation of the non-ruin probability as a compound geometric probability, p. 173 in Mikosch [20])**.** *Assume the Cramér-Lundberg model with $EX_1 < \infty$ and NPC. In addition, assume the claims X_i have a probability density function and let $(X_{I,n})$ be a sequence of independent and identically distributed random variables with distribution function $F_{X_i,I}$. Then, the non-ruin probability is given by*

$$\varphi(u) = \frac{\rho}{1+\rho}\left[1 + \sum_{n=1}^{\infty}(1+\rho)^{-n}P(X_{I,1} + \cdots + X_{I,n} \le u)\right]. \tag{3.5}$$

In most cases (3.5) cannot be evaluated explicitly. However, one exception to this is when (X_n) $\sim$ Exp(γ) in which case $(X_{I,n})$ also is Exp(γ) distributed. To see this, note that

$$\begin{aligned} F_{X,I}(u) &= \frac{1}{EX}\int_0^u \overline{F}_X(x)dx \\ &= \gamma\int_0^u e^{-\gamma x}dx \\ &= \gamma\left[-\frac{1}{\gamma}e^{-\gamma x}\right]_0^u \\ &= 1 - e^{-\gamma u} \end{aligned}$$

i.e. $(X_{I,n})$ is indeed Exp(γ).

Now, using the fact that $X_{I,1}+\cdots+X_{I,n} \sim \Gamma(n,\gamma)$ (by for example using the moment generating function), it holds

$$
\begin{aligned}
\varphi(u) &= \frac{\rho}{1+\rho}\left[1+\sum_{n=1}^{\infty}(1+\rho)^{-n}P(X_{I,1}+\cdots+X_{I,n}\le u)\right]\\
&= \frac{\rho}{1+\rho}\left[1+\sum_{n=1}^{\infty}(1+\rho)^{-n}\int_0^u \frac{\gamma^n}{\Gamma(n)}t^{n-1}e^{-\gamma t}dt\right]\\
&= \frac{\rho}{1+\rho}\left[1+\int_0^u \sum_{n=1}^{\infty}\Big(\frac{\gamma}{1+\rho}t\Big)^n \frac{1}{\Gamma(n)}\frac{1}{t}e^{-\gamma t}dt\right]\\
&= \frac{\rho}{1+\rho}\left[1+\int_0^u \sum_{n=0}^{\infty}n\Big(\frac{\gamma}{1+\rho}t\Big)^n \frac{1}{n!}e^{-\frac{\gamma}{1+\rho}t}\frac{1}{t}e^{-\gamma\frac{\rho}{1+\rho}t}dt\right]\\
&= \frac{\rho}{1+\rho}\left[1+\int_0^u \frac{\gamma}{1+\rho}t\frac{1}{t}e^{-\gamma\frac{\rho}{1+\rho}t}dt\right]\\
&= \frac{\rho}{1+\rho}\left[1+\frac{\gamma}{1+\rho}\int_0^u e^{-\gamma\frac{\rho}{1+\rho}t}dt\right]\\
&= \frac{\rho}{1+\rho}\left[1+\frac{1}{\rho}(1-e^{-\gamma\frac{\rho}{1+\rho}u})\right]\\
&= 1-\frac{1}{1+\rho}e^{-\gamma\frac{\rho}{1+\rho}u}
\end{aligned}
$$

Now, using that $\psi(u)=1-\varphi(u)$, it holds

$$\psi(u)=\frac{1}{1+\rho}e^{-\gamma\frac{\rho}{1+\rho}}$$

as the explicit expression for the ruin probability in case of $(X_{I,n})_{n\in\mathbb{N}}$ being i.i.d. Exp(γ). When comparing the above expression with the upper bound given by Example 3.4, it seen that

$$\psi(u)=\frac{1}{1+\rho}e^{-\gamma\frac{\rho}{1+\rho}u}\le e^{-\gamma\frac{\rho}{1+\rho}u},$$

i.e. the bound was exact bar the constant $\frac{1}{1+\rho}$.

In the next section, the heavy-tailed case is presented and it will be evident that the asymptotics are quite different than as presented in Theorem 3.3.

3.2.2 Heavy-tailed Case

In this section, an important result for the asymptotics of the ruin probability $\psi(u)$ is presented in case of heavy-tailed distributions, whereby heavy-tailed is here understood as the severity distribution being subexponential. Theorem 3.5 below is the equivalent of Cramér's ruin bound in the light-tailed case. It is worth noticing that Theorem 3.5 assumes that the integrated tail distribution of the severity distribution, $\overline{F}_{X,I}$, is subexponential rather than the severity distribution itself.

Theorem 3.5 (Ruin Probability when the integrated claim size distribution is subexponential, pp. 174-175 in Mikosch [20])**.** *Assume the Cramér-Lundberg model with $EX_1<\infty$ and NPC. In* *addition, assume that the claim sizes X_i have a density and that the integrated tail distribution, $F_{X_1,I}(x)$, is subexponential. Then the ruin probability $\psi(u)$ satisfies the asymptotic relationship*

$$\lim_{u\to\infty} \frac{\psi(u)}{\overline{F}_{X_1,I}(u)} = \rho^{-1}.$$

Proof. The key is using the representation in Theorem 3.5, which for the ruin probability $\psi(u) = 1 - \varphi(u)$ is given by

$$\lim_{u\to\infty} \frac{\psi(u)}{\overline{F}_{X,I}(u)} = \lim_{u\to\infty} \frac{\rho}{1+\rho} \sum_{n=1}^{\infty} (1+\rho)^{-n} \frac{P(X_1 + \cdots + X_n > u)}{\overline{F}_{X,I}(u)}$$

Now, if the limit can be moved inside of the summation, then it will be possible to use the subexponential property of $\overline{F}_{X,I}(u)$, i.e

$$\lim_{u\to\infty} \frac{P(X_1 + \cdots + X_n > u)}{\overline{F}_{X,I}(u)} = n,$$

and it would follow

$$\begin{aligned}
\lim_{u\to\infty} \frac{\psi(u)}{\overline{F}_{X,I}(u)} &= \lim_{u\to\infty} \frac{\rho}{1+\rho} \sum_{n=1}^{\infty} (1+\rho)^{-n} \frac{P(X_1 + \cdots + X_n > u)}{\overline{F}_{X,I}(u)} \\
&= \frac{\rho}{1+\rho} \sum_{n=1}^{\infty} (1+\rho)^{-n} \lim_{u\to\infty} \frac{P(X_1 + \cdots + X_n > u)}{\overline{F}_{X,I}(u)} \\
&= \frac{\rho}{1+\rho} \sum_{n=1}^{\infty} (1+\rho)^{-n} n \\
&= \frac{\rho}{1+\rho} \frac{(1+\rho)}{(1-(1+\rho))^2} \\
&= \rho^{-1}
\end{aligned}$$

The justification for the interchange of the limit and the sum is given by Lebesque's dominated convergence theorem since

$$\frac{P(X_1 + \cdots + X_n > u)}{\overline{F}_{X,I}(u)} \le K(1+\epsilon)^n \quad \text{for any } \epsilon > 0$$

according to (2.9) and thus

$$\frac{\rho}{1+\rho} \sum_{n=1}^{\infty} (1+\rho)^{-n} \frac{P(X_1 + \cdots + X_n > u)}{\overline{F}_{X,I}(u)} \le \frac{\rho}{1+\rho} \sum_{n=1}^{\infty} (1+\rho)^{-n} K(1+\epsilon)^n < \infty.$$

Choose $\epsilon < \rho$ and the result follows. □

More details and discussion on the interpretation of Theorem 3.5 can be found in Embrechts and Veraverbeke [7].

An example of a distribution that satisfies the conditions in Theorem 3.5 is the Pareto distribution. Indeed, Example 2.4 showed that $\overline{F}_{I,X}(x)$ is regularly varying and by (2.4) and Definition 2.5 it follows it is also subexponential. As mentioned in Mikosch [20], p. 176, it is not straightforward to verify that the integrated tail distribution is subexponential. However, for the usual choices of heavy-tailed severity distributions, such as the log-normal and Weibull ($\tau < 1$) distributions, the integrated tail distribution is in both instances indeed subexponential. There exists, however, one case where subexponentiality can be verified directly: if the distribution function is regularly varying with tail index $\alpha > 1$, then the integrated tail distribution is regularly varying with tail index $\alpha - 1$ and thus subexponential (cf. Mikosch [20], p. 176). This is a consequence of Karamata's theorem (see Appendix). By comparing Theorem 3.5 with Cramér's ruin bound, Theorem 3.3, is seen that the ruin probability in the heavy-tailed case is essentially of the same order as $\overline{F}_{X,I}(u)$, which is non-negligible for large values of initial capital, u (cf. Corollary 2.1 and the basic properties of subxponential distributions in Section 2.3. In contrast, the ruin probability can be made arbitrarily small in the light-tailed case. The implication is that portfolios with heavy-tailed claims are dangerous due to the largest claim having a significant impact on the overall behaviour over the long-term horizon (cf. (2.12) and that ruin occurs spontaneously. For a theoretical explanation of this phenomena, please see Embrechts et al. [8]. In the light-tailed case, for ruin to occur, it is not due any one claim being large but rather the mass of claims affecting the portfolio. This fact will be evident in the next section with numerical results.

3.3 Numerical Examples

In this section, numerical results are presented illustrating the concepts in this and the previous chapter. The section contains results for the risk process, $U(t)$, defined in Section 3.2, and the corresponding ruin probability $\psi(u)$, Equation 3.1, both in terms of how $\psi(u)$ depends on the amount if initial capital u and the time horizon t.

3.3.1 Simulation Methodology

To simulate the risk process, $U(t)$, over the interval $[0, t]$, the order statistics property of the homogeneous Poisson process in Theorem 3.1 is utilised along with the fact that the number of arrivals, $N(t)$, over the interval $[0, t]$ is Po(λt) distributed. This means that if $N(t) = n$, the distribution of the arrival times [3] $0 < T_1 < T_2 < \cdots, T_n < t$ constitutes an uniform ordered sample $U_{(1)} <, \ldots, < U_{(n)}$ of an i.i.d. sample $U_1, \ldots, U_n$, $U_i \sim U(0, t)$. With the arrival times known, it is an easy task to construct the corresponding Poisson process, the total claim amount process, $S(t)$, the premium income of the portfolio, $p(t) = ct$, and, finally, the risk process, $U(t)$. In order to derive a numerical estimate of the ruin probability, $\hat{\psi}(u)$ in Section 3.3.2 and then in Chapter 4, the algorithm in Figure 6 is repeated many times and $\hat{\psi}(u)$ calculated as the number of paths where $U(t) < 0$, $t > 0$ divided by the total simulated paths.

Figure 6 is an illustration of one sample path of the risk process with the following assumptions:

- Claims arrive according to a homogeneous Poisson process with $\lambda = 1$.
- The claim size distribution is Exp(γ) with $\gamma = 0.5$.
- The safety loading, ρ, is 0.05.
- The Net Profit Condition of Definition 3.6 is satisfied with $c = (1+\rho)\frac{EX_1}{EW_1} = (1+0.05)\frac{1}{1/2} = 2.1$.
- The process is simulated over $[0, 10]$.
- Initial capital, u, is equal to 15.

[3]Please note that the arrival times here represent the switch to operational time referred to in Section 3.2.

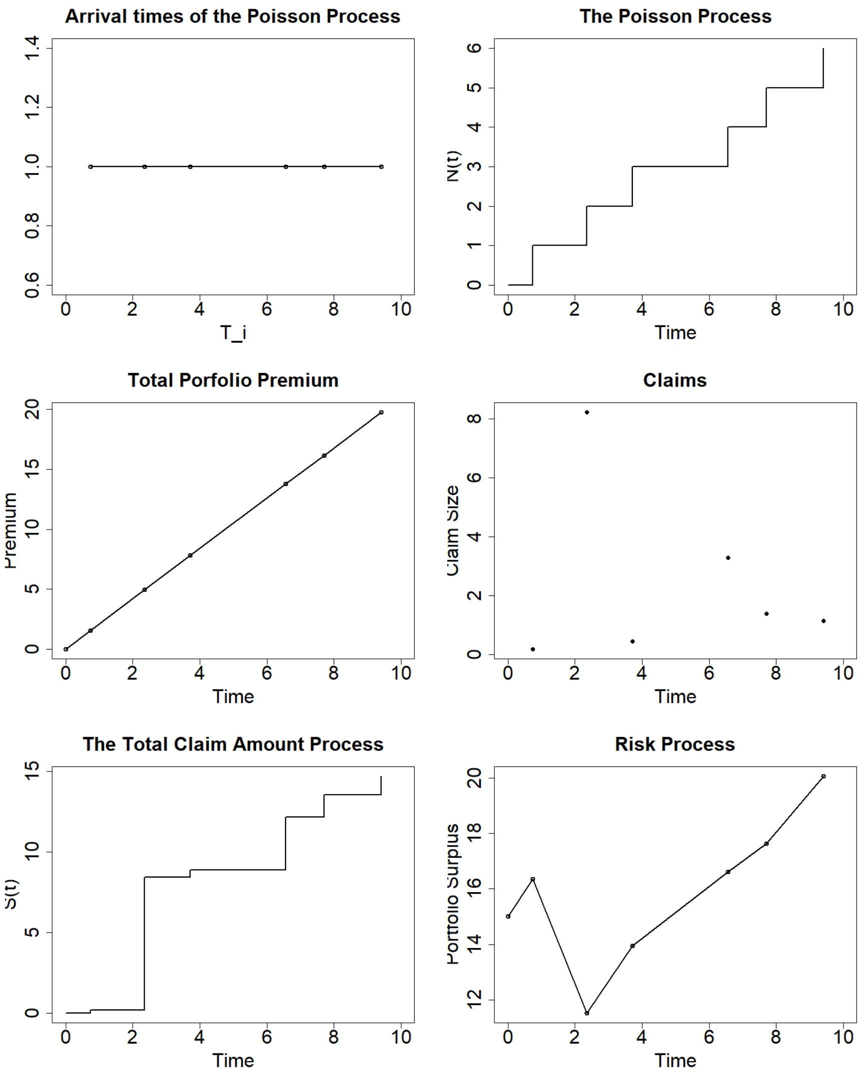


Figure 6: Illustration of the methodology for simulating one sample path of the risk process, $U(t)$, with the total claim amount process, $S(t)$, specified as the Cramér-Lundberg model, over $[0, 10]$. **Top row**: the left-hand exhibit illustrates the claim arrival times of the Poisson process and the right-hand exhibit illustrates the corresponding Poisson process. Here, thirteen claims were registered over $[0, 10]$. **Middle row**: the left-hand exhibit shows the amount of premium collected at time points corresponding to the claim arrival times and the right-hand exhibit illustrates the claim sizes at each arrival time. **Bottom row**: The left-hand exhibit illustrates the total claim amount process corresponding to the claims and claim arrival times, and the right-hand exhibit illustrates the resulting risk process.

3.3.2 Illustrations

Figures 7 below illustrate the risk process, $U(t)$, $0 \leq t \leq T$, for $T = 100$. In subsequent figures and tables, results are presented for $T = 100$ and $T = 1000$ with the reason being solely to illustrate the temporal aspect of $U(t)$.

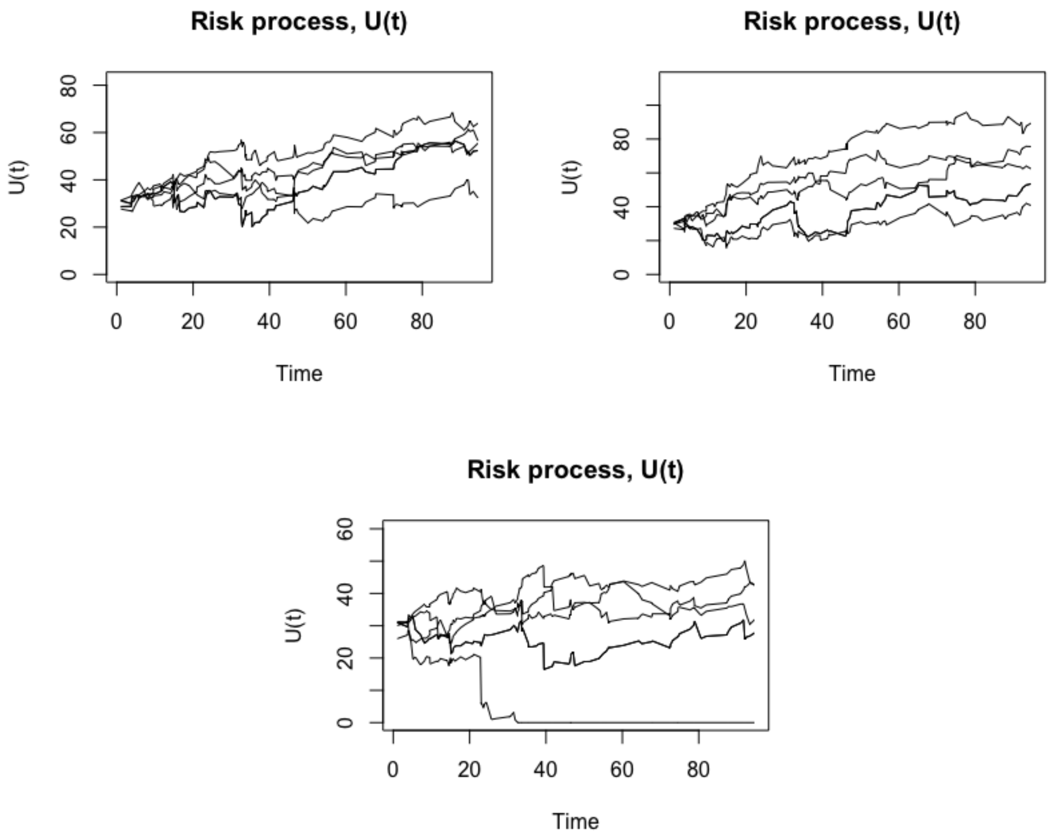
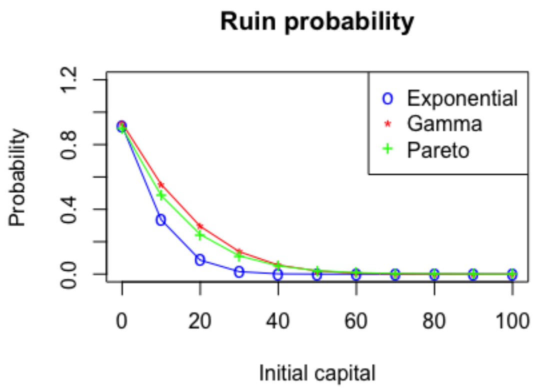


Figure 7: Illustration of some sample paths of the risk process, $U(t)$, for three different choices of severity distribution with NPC satisfied in each case. Note that NPC implies a different premium rate, c, as specified in Definition 3.6 and the subsequent expression (3.2), $c = (1+\rho)\frac{EX_1}{EW_1}$. Here, $EW_1 = \frac{1}{\lambda}$ was assumed identical for all three instances with $\lambda = 1$. The top-left figure shows $U(t)$ in case of the Exponential distribution, Exp(1); the top-right figure $U(t)$ in case of the Gamma distribution, $\Gamma(5, 2)$; and the bottom figure $U(t)$ in case of the Pareto distribution, Pa(4,4). In all three instances, $\rho = 0.05$ and $T = 100$. Ruin does only occur in case of the Pareto distribution.

By studying Figure 7, it is seen that ruin does not occur for neither the Exponential case nor the Gamma case, whereas ruin does occur in the Pareto case for one sample path. Noteworthy in the Pareto case are also the large downward jumps in the trajectories, which means the prevalence of a large claim and that ruin might just be around the corner. In contrast, these large downward jumps are not present in either the Exponential case of the Gamma case. The implications on a portfolio of risks in the presence of a heavy-tailed severity distributions such as the Pareto distribution will be investigated in Chapter 4.

Figure 8 below depicts how the ruin probability depends on the initial capital, u, over the interval $0 \leq t \leq T$ for $T = 100$ and $T = 1000$, respectively.



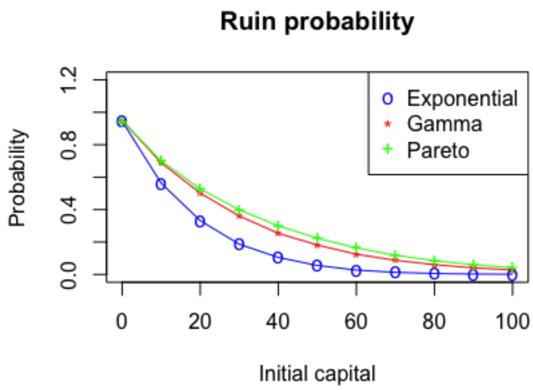


Figure 8: Illustration of the ruin probability, $\psi(u)$, for $u = 10, 20, \ldots, 100$ with $T = 100$ (upper center) and $T = 1000$ (bottom center) for the Exponential distribution, Exp(1), (blue line), Gamma distribution, $\Gamma(5, 2)$, (red line), and Pareto distribution, Pa(4,4), (green line). Note that NPC implies a different premium rate, c, as specified in Definition 3.6 and the subsequent expression (3.2), $c = (1+\rho)\frac{EX_1}{EW_1}$ for each choice of severity distribution. Here, $EW_1 = \frac{1}{\lambda}$ was assumed identical for all three instances with $\lambda = 1$. In all three instances, $\rho = 0.05$.

From Figure 8 and Tables 3 and 4, it is clear that increasing the initial capital decreases the ruin probability, for fixed T, and increasing T increases the ruin probability. In both instances, the ruin probability is lowest in the case of the Exponential distribution (blue line) and highest for the Pareto distribution (green line) as u increases.

Distribution/Initial capital	0	10	50	70	100
$X_i \sim \text{Exp}(1)$	0.9138	0.3353	0.000	0.000	0.000
$X_i \sim \Gamma(5, 2)$	0.9309	0.5538	0.0198	0.0015	0.000
$X_i \sim \text{Pareto}(4, 4)$	0.9040	0.4850	0.0215	0.0041	0.0004

Table 3: Illustration of how the ruin probability depends on the level of initial capital with $T = 100$ supplementing Figure 8.

Distribution/Initial capital	0	10	50	70	100
$X_i \sim \text{Exp}(1)$	0.9453	0.5625	0.0556	0.0131	0.0013
$X_i \sim \Gamma(5, 2)$	0.9507	0.6886	0.1814	0.0871	0.0279
$X_i \sim \text{Pareto}(4, 4)$	0.9459	0.7002	0.2233	0.1178	0.0419

Table 4: Illustration of how the ruin probability depends on the level of initial capital with $T = 1000$ supplementing Figure 8.

A final observation based on the results in Tables 3 and 4 is the non-negligible ruin probability of 0.028 in the Gamma case and 0.042 in the Pareto case, respectively, when $T = 1000$.

Figure 9 illustrates how the $\psi(u)$ behaves as t increases for $u = 0, 10, 50$ when $T = 100$.

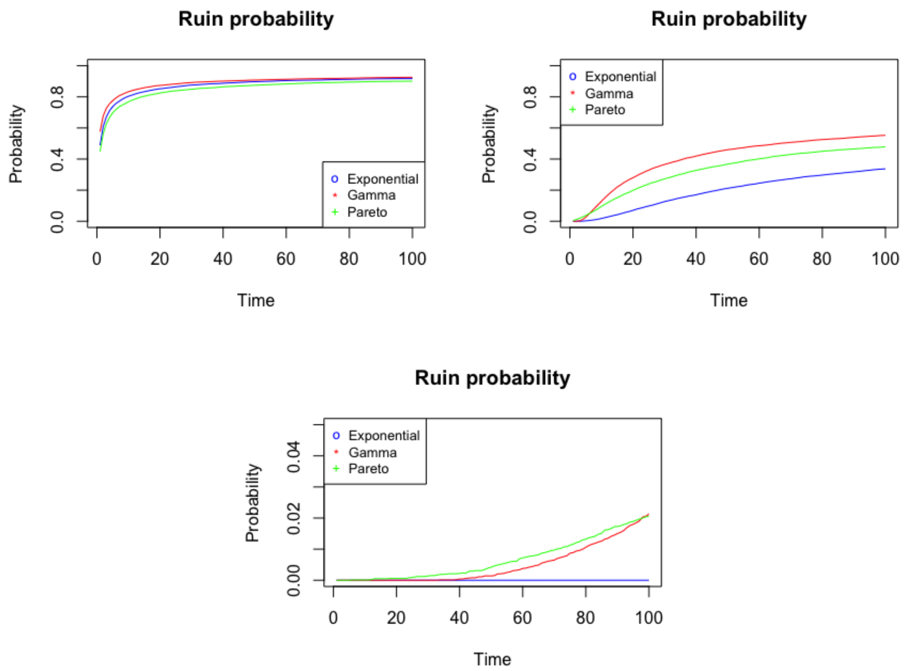


Figure 9: Illustration of the ruin probability, $\psi(u)$, for $u = 0$ (top-left), $u = 10$ (top-right) and $u = 50$ (bottom), on $0 \le t \le 100$. In each exhibit, the blue line represents the Exponential distribution, Exp(1), the red line the Gamma distribution, $\Gamma(5, 2)$, and the green line the Pareto distribution, Pa(4,4). Note that NPC implies a different premium rate, c, as specified in Definition 3.6 and the subsequent expression (3.2), $c = (1 + \rho)\frac{EX_1}{EW_1}$, for each choice of severity distribution. Here, $\rho = 0.05$ and $EW_1 = \frac{1}{\lambda}$ was assumed identical for all three instances with $\lambda = 1$.

Figure 10 illustrates how the $\psi(u)$ behaves as t increases for three different values of $u = 0, 10, 50$ when $T = 1000$.

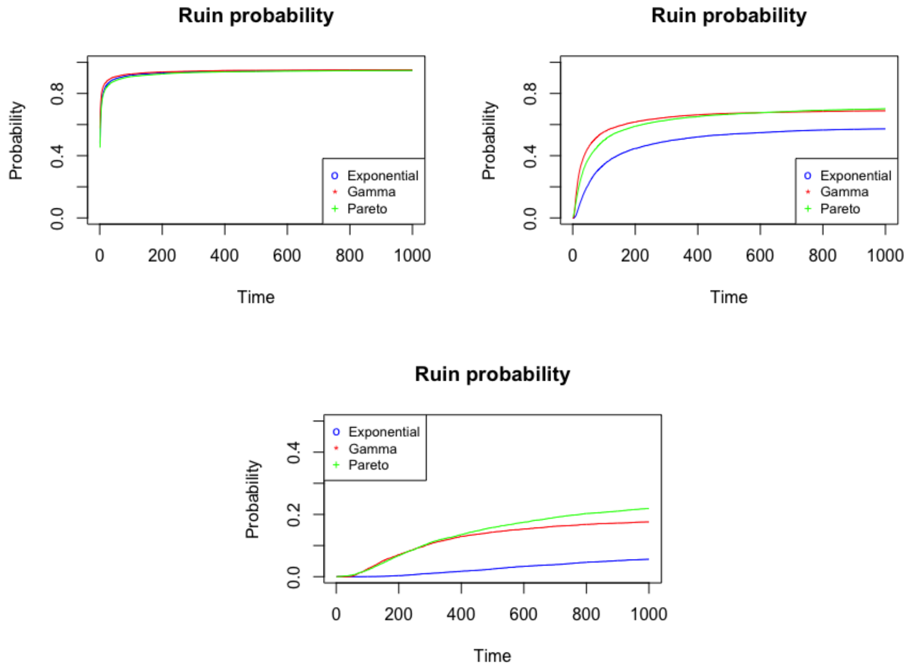


Figure 10: Illustration of the ruin probability, $\psi(u)$, for $u = 0$ (top-left), $u = 10$ (top-right) and $u = 50$ (bottom), on $0 \leq t \leq 1000$. In each exhibit, the blue line represents the Exponential distribution, Exp(1), the red line the Gamma distribution, $\Gamma(5, 2)$, and the green line the Pareto distribution, Pa(4,4). Note that NPC implies a different premium rate, c, as specified in Definition 3.6 and the subsequent expression (3.2), $c = (1+\rho)\frac{EX_1}{EW_1}$, for each choice of severity distribution. Here, $\rho = 0.05$ and $EW_1 = \frac{1}{\lambda}$ was assumed identical for all three instances with $\lambda = 1$.

By studying Figures 9 and 10, it is easy to observe that, for a given level of initial capital u, $\psi(u)$ increases as t increases. An equally easy and intuitive observation is $\psi(t; u_2) \leq \psi(t; u_1)$ if $u_1 \leq u_2$, i.e. increasing the initial capital decreases the ruin probability and this holds as t increases. The results presented thus far above seem to a large extent intuitive: increasing the initial capital decreases the ruin probability and increasing the time horizon increases the ruin probability. The behaviour of $U(t)$ also seems to be different depending on whether the severity distribution is light-tailed or heavy-tailed in the sense of ruin being more likely in the latter case and, should it occur in the heavy-tailed case, it is due to a single large claim rather than a mass of claims.

4 Case Study: Heavy-tailed Portfolio Dynamics

The aim of this chapter is to illustrate the portfolio behaviour, defined in terms of ruin probability, in the presence of heavy-tails and available mitigating measures. Here, mitigating measures should be understood as those available to reduce the impact on the portfolio ruin probability. Results will this be presented in the presence and without mitigating measures. The results have been obtained by using the simulation methodology outlined in Section 3.3.1, where the methodology for one sample path was illustrated.

4.1 Assumptions

Consider a stylised insurance company with a limited portfolio of risks, namely three lines of business with the following assumptions:

- Line of business 1: The Cramér-Lundberg model with claims arriving according to a Poisson process, $N_1(t)$, with homogeneous intensity λ, and with severity distribution $\text{Exp}(\gamma)$. For sake of simplicity, assume $\lambda = \gamma = 1$.
- Line of business 2: The Cramér-Lundberg model with claims arriving according to a Poisson process, $N_2(t)$, with homogeneous intensity 0.8, and with severity distribution $\Gamma(2, 5)$.
- Line of business 3: The Cramér-Lundberg model with claims arriving according to a Poisson process, $N_3(t)$, with homogeneous intensity 0.2, and with severity distribution $\text{Pareto}(2, 10)$ [4].

Assume further a safety loading, $\rho = 0.05$, for all three lines of business, a time horizon of $T = 100$. With these assumptions, the ruin probability for each individual line of business as well as for the portfolio is presented below for $u = 100, 110, \ldots, 200$. For sake of simplicity, the initial capital is evenly distributed to each line of business. In view of the Pareto distribution being heavy-tailed and thus having a significant influence over the portfolio ruin probability, the assumption of an evenly distributed initial capital per line of business may not be a reasonable one. Lastly, independence is assumed between all three risks in the portfolio and that an acceptable threshold for the portfolio ruin probability is 0.5%.

If the surplus of the individual lines of business at time $t > 0$ is denoted $U_i(t)$, then the ruin of the portfolio is defined when the portfolio surplus, $U(t) = U_1(t) + U_2(t) + U_3(t)$ is below null for any $t > 0$.

For further reference on the behaviour of the ruin probability of a portfolio of an insurance company with lines of businesses considered heavy-tailed, please see e.g. Hult and Lindskog [13]. This article also analyses the impact of rules for transfer of capital between the different lines of business on the ruin probability and draw conclusions about possible benefits from diversification in the portfolio. In Hult and Lindskog [14], the analysis is extended to study the asymptotic decay of finite time ruin probabilities for an insurance company that faces heavy-tailed claims and uses predictable investment strategies defined as investments in risky assets. Both of these papers contain further references on the topic.

4.2 Without Mitigating Measures

Table 5 and Figure 11 below exhibits each the ruin probability for the portfolio and for the individual lines of business. It is evident that the heavy tail of the Pareto distribution has a large influence on the ruin probability of the portfolio, and the lighter tails of the Exponential and Gamma distributions have insignificant influence on the ruin probability as the initial capital increases. There is also a certain amount of diversification in the portfolio as the ruin probability of the portfolio is not quite as severe as for the Line of business 3.

[4] Please note the choice $\alpha = 2$ implies EX^2 does not exist and hence an infinite variance. However, in the present text, this small(!) problem is overlooked for the purpose of illustration.

Distribution/Initial capital	100	120	140	160	180	200
$X_i \sim \text{Exp}(1)$	0.00737	0.00162	0.00034	0.00007	0.00002	0.00000
$X_i \sim \Gamma(5, 2)$	0.10867	0.06002	0.03178	0.01590	0.00692	0.00319
$X_i \sim \text{Pareto}(2, 10)$	0.42588	0.38980	0.35155	0.31982	0.29408	0.26724
Portfolio	0.16345	0.13055	0.10244	0.08267	0.06756	0.05576

Table 5: Illustration of how the ruin probability depends on the level of initial capital with $T = 100$. The initial capital in each column has been allocated evenly to all three lines of business.

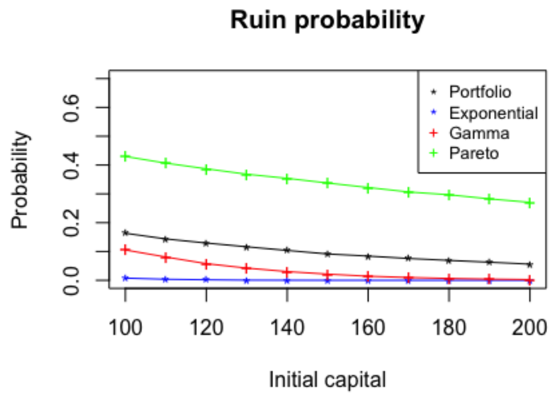


Figure 11: Illustration of the portfolio ruin probability together with the ruin probability for the individual line of business.

Figure 12 below exhibits the distribution of the portfolio surplus, $U(t)$, at $t = 25, 50, 75, 100$, in the interval $(-500, 450)$ with initial capital equal to 200. The heavy left tail of the distribution, which becomes more pronounced as t increases, shows the influence of the Pareto distribution on the portfolio surplus.

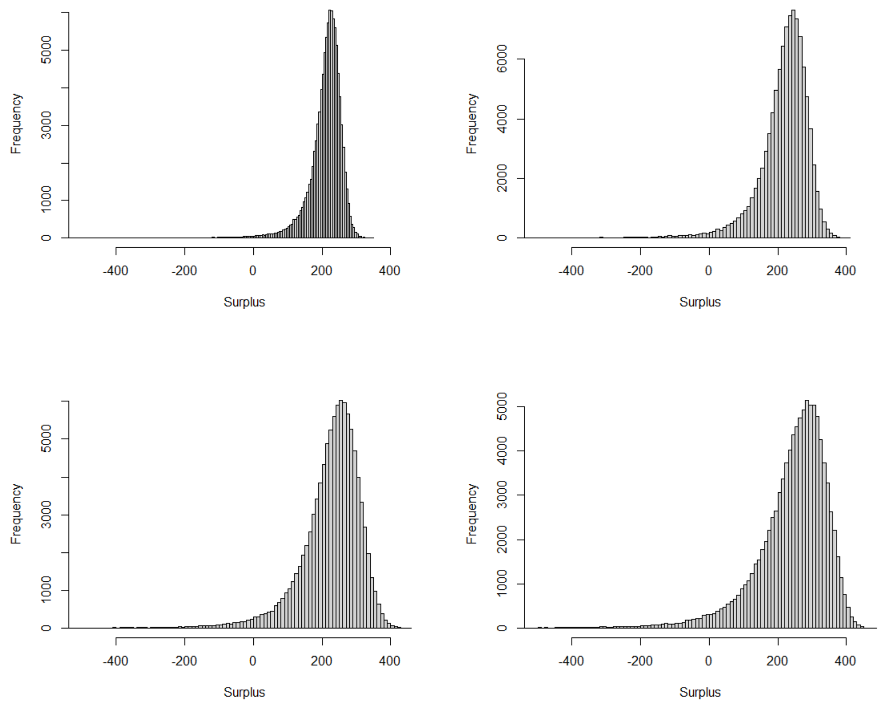


Figure 12: Illustration of the portfolio surplus, $U(t)$, at $t = 25, 50, 75, 100$ and 100 for values $(-200, 400)$ with initial capital equal to 200.

Table 6 exhibits a subset of quantiles for the left tail of the distribution along with the mean, standard deviation coefficient of variation (CoV), and minimum and maximum of the distribution,

Time	25	50	75	100
0.1th	-474.56	-742.65	-993.89	-1192.69
0.5th	-94.00	-221.09	-317.35	-401.91
1th	-8.62	-96.18	-156.90	-223.05
2.5th	74.62	19.94	-20.03	-56.52
10th	153.61	132.09	116.08	107.67
Mean	207.00	213.39	219.44	234.83
Standard deviation	73.36	105.72	127.84	149.03
CoV	0.35	0.50	0.58	0.63
Min	-5750.16	-8176.96	-8168.95	-8559.23
Max	345.56	407.10	453.25	485.68

Table 6: The quantiles of the portfolio surplus, $U(t)$, at $t = 25, 50, 75$ and 100 for values along with the mean, standard deviation, coefficient of variation, minimum and maximum. Initial capital equal to 200.

Table 6 yet again illustrates the influence of the heavy tail of the Pareto distribution on the distribution of the portfolio surplus. It is also apparent how the distribution widens as t increases. Results for each individual lines of business are found in the Appendix.

4.3 With Mitigating Measures

The question at hand is to determine available measures to the insurance company to decrease the ruin probability on a portfolio level to an acceptable level, say 0.5 %, over the specified time horizon. On a first glance, these could include:

1. Increase the level of capital, u, by for example raising capital from capital markets or decreasing or withholding payment of dividends to shareholders.

2. Increase the safety loading, ρ, for each line of business. In effect, this means increasing premiums charged to its customers in return for offering insurance to the customers.

3. Purchasing reinsurance coverage. This could be done in numerous ways but here a standard excess-of-loss coverage will only be investigated The aim is to limit the impact of large claims on the company's surplus.

4. Increase the size of the portfolio by adding risks assumed independent of already existing risks in the portfolio in order to benefit from increased diversification.

As already noted and evident from Theorem 3.3 and Theorem 3.5, increasing the level of initial capital will decrease the ruin probability below any specified level. However, circumstances might not allow more capital to be raised or it may be prohibitively expensive to do so to. Withholding or decreasing dividend payments is more feasible although it may not be looked upon favourbly by e.g. shareholders or capital markets. The second alternative, increasing premiums charged to customers, might be difficult due to market competition or it might be necessary to raise premiums to such a large extent to achieve the desired reduction in the ruin probability that customers will leave. Purchasing reinsurance coverage is possible to the extent permitted by reinsurance markets, whose available capacity varies over time. The last alternative, increasing the portfolio size with light-tailed risks, is certainly an option but it is doubtful if enough premium income is generated to offset severe claims from the Pareto distribution. If the severity distribution has a right tail, which is regularly varying, by Theorem 2.1 and Corollary 2.1, it follows that the tail can not be diversified away.

4.3.1 Increasing the Safety Loading

This could either be done for all lines of business in the portfolio or solely for Line of business 3. However, given the heavy tail of the Pareto distribution and its effect on the tail behaviour of portfolio (as shown in Table 5), it is doubtful enough premiums will be collected to pay for the large the claim should it occur. To illustrate, assume premiums are increased by 20 % each for Lines of business 1 and 2, and doubled for Line of business 3. The results are presented in Table 7 and Figure 13 below for the portfolio.

Portfolio/Initial capital	100	120	140	160	180	200
Base	0.16345	0.13055	0.10244	0.08267	0.06756	0.05576
Alternative 2	0.05458	0.04455	0.03590	0.03095	0.02585	0.02264

Table 7: Illustration of how the ruin probability depends on the level of initial capital for the base portfolio and alternative portfolio with premiums increased (Alternative 2).

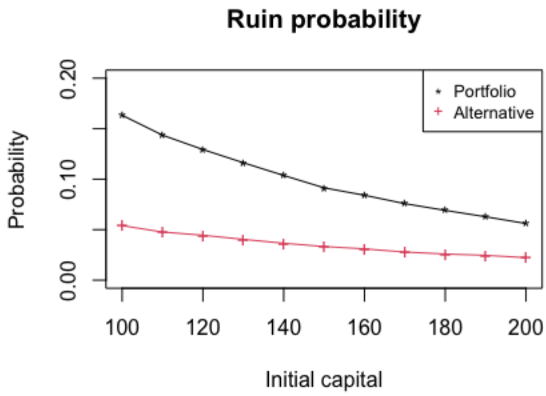


Figure 13: Illustration of how the ruin probability depends on the level of initial capital at $T = 100$ for the base portfolio (black line) and the alternative portfolio with premiums increased (red line).

As can be expected, the ruin probability decreases if premiums are increased and the decrease is, in relative terms, higher for smaller values of the initial capital, albeit not by much. In spite of this, the ruin probability is still considered unacceptably high at ca. 2.2 % even for an initial capital of 200, implying insufficient capital to absorb the very large claims from the Pareto distribution. The relative flatness in the red line in Figure 13 indicates decreasing marginal effects on the ruin probability as the claims causing ruin for large values of initial capital are indeed very severe.

4.3.2 Reinsurance Coverage

For sake of simplicity, assume, for now, unlimited type of coverage is offered by the reinsurance company in the sense that it does not cap losses exceeding the deductible d. Assume further that coverage is only purchased for the Pareto line of business as this is the most capital expensive one. Now, from the insurance company's perspective, the surplus process assumes the form

$$\tilde{U}(t) = u + \tilde{c}t + \sum_{n=1}^{N_3(t)} \min(X_n, d)$$

for this line of business, where $\tilde{c}$ is determined according to NPC:

$$\tilde{c} = (1+\rho)\frac{E[\min(X_1, d)]}{EW_1}$$

with

$$\begin{aligned} E[\min(X_1, d)] &= \int_0^d x f_X(x)\,\mathrm{d}x + d\,\overline{F_X}(d) \\ &= \frac{\sigma}{\alpha-1}\left(1 - \frac{\sigma+d}{\sigma}\right)^{-(\alpha-1)} + d\left(\frac{\sigma+d}{\sigma}\right)^{-\alpha} \end{aligned}$$

and $EW_1 = \lambda^{-1}$. Now, in return for offering coverage the reinsurance company will charge a premium, $\hat{p}(t)$. The claims process for the reinsurance company can be modelled using the Cramér-Lundberg model and the surplus process, $\hat{U}(t)$, for the reinsurance company is thus given by

$$\hat{U}(t) = \hat{u} + \hat{c}t + \sum_{n=1}^{\hat{N}_3(t)} \max(X_n - d, 0)$$

, where NPC again needs to be satisfied, i.e.

$$\hat{c} = (1 + \hat{\rho}) \frac{E[\max(X_1 - d, 0)]}{E\hat{W}_1}$$

for some safety loading $\hat{\rho}$. Here, $\hat{N}_3(t)$ is the Poisson process for the claims exceeding the deductible d. The task at hand is to determine $E[\max(X_1 - d), 0)]$ and $E\hat{W}_1$. Firstly, $E[\max(X_1 - d), 0)]$ is nothing else than the expected claims cost in the layer (d, ∞), which is the same as the mean excess function encountered, $e_F(d)$, in Section 2.1. Since $X_1 \sim \text{Pareto}(\alpha, \sigma)$, it follows

$$e_F(d) = EX_1 + \frac{u}{\alpha - 1}.$$

To determine $E\hat{W}_1$, let $M(t)$ be the number of claims exceeding the deductible at t, and notice that, conditioned on the event $N_3(t) = n$, $M(t)$ is $\text{Bin}(n, p)$, i.e. binomially distributed with probability of success $p = \overline{F}_X(d)$. Hence,

$$\begin{aligned}
P(M(t) = k) &= \sum_{n=0}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \\
&= e^{-\lambda t} \frac{p^k}{k!} \sum_{n=0}^{\infty} \frac{(1-p)^{n-k}}{(n-k)!} (\lambda t)^{(n-k)} (\lambda t)^k \\
&= e^{-\lambda t} \frac{(\lambda t p)^k}{k!} \sum_{n=0}^{\infty} \frac{(\lambda t (1-p))^{n-k}}{(n-k)!} \\
&= e^{-\lambda t} \frac{(\lambda t p)^k}{k!} e^{\lambda t (1-p)} \\
&= e^{-\lambda t p} \frac{(\lambda t p)^k}{k!}.
\end{aligned}$$

Hence, the number of claims exceeding the deductible, d, is given by a Poisson process with intensity $\lambda t p$. The NPC for the reinsurance company is thus

$$\hat{c} = (1 + \hat{\rho}) \lambda p \left(EX_1 + \frac{d}{\alpha - 1} \right).$$

and the final expression for the surplus process for the insurance company is given by

$$\tilde{U}(t) = u + (\tilde{c} - \hat{c})t + \sum_{n=1}^{N_3(t)} \min(X_n, d).$$

Similarly, the surplus process for the reinsurance company is given by

$$\hat{U}(t) = u + \hat{c}t + \sum_{n=1}^{\hat{N}_3(t)} \max(X_n - d, 0).$$

The NPC for the reinsurance company could also be derived by noting

$$\begin{aligned} P(X > x + d \mid X > d) &= \frac{P(X > x + d)}{P(X > d)} \\ &= \left(\frac{\sigma + d + y}{\sigma}\right)^{-\alpha} \Big/ \left(\frac{\sigma + d}{\sigma}\right)^{-\alpha} \\ &= \left(\frac{x + (\sigma + d)}{\sigma + d}\right)^{-\alpha} \end{aligned}$$

i.e. the distribution of the claims exceeding the deductible d is $\mathrm{Pa}(\alpha, \sigma + d)$ and with expected value $\frac{\sigma+d}{\alpha-1}$. It also follows the conclusions of Theorem 3.5 are valid and the reinsurance company can thus expect its ruin probability to deccay slowly. In practice, the reinsurance company can of course have its own view on both the model choice as well as the actual parameterisation of the model.

With the assumption d is chosen such that $p = 0.1$, the effect on the ruin probability is shown in Table 8 below. Note that choosing $p = 0.1$ implies a self-retention of 21.62, i.e. the solution $\hat{d_1}$ to the equation $\overline{F}_X(d) = 0.1$.

Portfolio/Initial capital	100	120	140	160	180	200
Base	0.16345	0.13055	0.10244	0.08267	0.06756	0.05576
Alternative 3a ($p = 0.1$)	0.04774	0.02096	0.00775	0.00300	0.00101	0.00025

Table 8: Illustration of how the ruin probability depends on the level of initial capital at $T = 100$ for the base portfolio (cf. Table 5) and portfolio with excess-of-loss type reinsurance purchased (Alternative 3a).

As can be seen, purchasing reinsurance reduces the ruin probability below the specified threshold; an initial capital of 160 suffices as the ruin probability is below 0.5% as the reinsurance coverage caps the very large losses, which otherwise would have had a large impact on the ruin probability. This is particularly evident for larger values of initial capital for which the reduction is much larger since the very large losses causing ruin are capped at a comparatively low level.

Thus far, the behaviour of the reinsurance company has not been considered since it has been tacitly assumed that coverage would be provided without restrictions. Nevertheless, the reinsurance company would surely have its own acceptable thresholds for the ruin probability. For argument's sake, assume the reinsurance company has the same threshold as the insurance company's value, below which the ruin probability is considered acceptable, 0.5%, and its possibilities to raise additional capital are limited. What options are then available to the reinsurance company to reduce the ruin probability to an acceptable level? It could e.g.

1. Increase premiums.
2. Reduce offered capacity by increasing the deductible d but still indemnify all losses in the interval (d, ∞).
3. Reduce offered capacity by limiting its commitment to losses in the interval (d_1, d_2), $d_1 < d_2$
4. Retrocede the risk, i.e. buy coverage from other reinsurers.

Of these, Alternative 1, would most likely have a negligible effect similar to the insurance company's attempts to raise premiums, and Alternative 3. gives rise to the same type of discussion but between reinsurance companies instead. Regarding Alternative 2, whilst the expression for $e_F(d)$ readily shows an increased expected claims cost from increasing the deductible, the thinning out of the Poisson process will hopefully mitigate the increase in $e_F(d)$ to result in, at least, an expected decreased total claims cost.

Table 9 below shows the effect on the ruin probability for the reinsurance company from decreasing the exceedance probability from $p_1 = 0.1$ to $p_2 = 0.05$, which corresponds to increasing the attachment point of the layer from $d_1 = 21.62$ to $d_2 = 34.72$. If the reinsurance company wants to decrease the ruin probability below 0.5 %, increasing the attachment point to $d_2 = 34.72$ is not enough even when the initial capital is 200, which, again, shows how severe the losses are causing ruin.

Coverage/Initial capital	100	120	140	160	180	200
Excess-of-loss 1 ($p = 0.1$)	0.03567	0.02962	0.02407	0.02122	0.01712	0.01528
Excess-of-loss 2 ($p = 0.05$)	0.01791	0.01464	0.01211	0.01093	0.00924	0.00818

Table 9: Illustration of how the ruin probability depends on the level of initial capital at $T = 100$ for the reinsurance company for two types of excess-of-loss coverage offered.

Table 10 shows the corresponding effect on the insurance company's ruin probability as a consequence of reduced reinsurance capacity. As expected, the ruin probability increases quite drastically when the self-retention is increased.

Portfolio/Initial capital	100	120	140	160	180	200
Base	0.16345	0.13055	0.10244	0.08267	0.06756	0.05576
Alternative 3a ($p = 0.1$)	0.04774	0.02096	0.00775	0.00300	0.00101	0.00025
Alternative 3b ($p = 0.05$)	0.06943	0.03678	0.01809	0.00880	0.00356	0.00164

Table 10: Illustration of how the ruin probability for the insurance company depends on the level of initial capital at $T = 100$ for the base portfolio and two alternative portfolios with excess-of-loss type reinsurance purchased at level $p = 0.1$ and $p = 0.05$.

In Figure 14 below, the ruin probability for both the insurance company and reinsurance company is shown in the presence of excess-of-loss contracts. The relative flatness of the red line in the right exhibit bears testimony to the severeness of the claims from the Pareto distribution piercing the reinsurance layer.

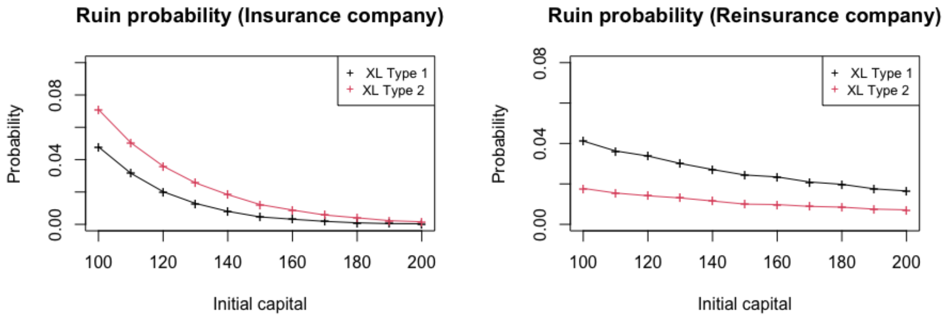


Figure 14: Illustration of the portfolio ruin probability for the insurance company (left exhibit) and the reinsurance company (right exhibit) for two types of excess-of-loss contracts with attachment point $d_1 = 21.62$ (black line) and attachment point $d_2 = 34.72$ (red line).

The above results assume unlimited coverage provided by the reinsurance company, which, although beneficial for the insurance company, is dangerous for the former as the losses from the Pareto distribution can indeed be very severe and therefore require high levels of capital to maintain the ruin probability at an acceptable level. It is consequently doubtful if any reinsurer would enter into such an agreement. A more effective arrangement would instead be to have multiple reinsurers participating and thus assuming a portion of losses, in so-called layers, exceeding the insurance company's deductible. This type of construction limits the liability of the reinsurers to a specified amount thereby reducing the amount of capital necessary to provide coverage. The table below summarises the scheme outlined above with each reinsurer participating in one layer together with its maximum loss. Notwithstanding the above, Reinsurer 5 is, for simplicity, assumed to cover losses in the unbounded layer (d_5, ∞).

Reinsurer	Layer	Attachment points	Maximum loss
Reinsurer 1	Layer 1	$(d_1, d_2]$	$d_2 - d_1$
Reinsurer 2	Layer 2	$(d_2, d_3]$	$d_3 - d_2$
Reinsurer 3	Layer 3	$(d_3, d_4]$	$d_4 - d_3$
Reinsurer 4	Layer 4	$(d_4, d_5]$	$d_5 - d_4$
Reinsurer 5	Layer 5	(d_5, ∞)	Not defined

Table 11: Illustration of an excess-of-loss programme structure with five layers with the fifth being unbounded. Please note $d_1 < d_2 < \cdots < d_5$.

Naturally, this assumes enough reinsurers are around to provide coverage. Now, from the point of view of the insurance company, nothing has changed as it still only covers losses in $[0, d_1$, but the major difference is that each reinsurer only covers losses in its respective layer. Using the in Chapter 3 already described model for modelling the surplus process of an insurance company, it is seen that NPC is unchanged for the insurance company and Reinsurer 5 and a slight modification is needed for Reinsurers 1 to 4 according to

$$\hat{c}_i = (1 + \hat{\rho}) \frac{1}{E\hat{W}_i} \big(E[d_i < X < d_j \mid X > d_i] + (d_j - d_i)\overline{F}(d_j) \big)$$

for $i = 1, \ldots, 4$ and $d_i < d_j$. Lastly, to have actual numerical values to work with, assume each d_i is determined according to specified quantile levels of the Pareto distribution, $F^{-1}(p_i) = d_i$ with $p_1 = 0.9$, $p_2 = 0.95$, $p_3 = 0.975$, $p_4 = 0.99$ and $p_5 = 0.995$.

With all assumptions made, nothing remains other than presenting the results of the exercise as in Table 12 below. Note that chosen levels of initial capital are here 0, 10, 20, 100, 160 and 200.

Initial capital	0	10	20	100	160	200
Insurer	0.90311	0.68959	0.53217	0.02962	0.00148	0.00011
Reinsurer 1	0.26412	0.08894	0.01896	0.00000	0.00000	0.00000
Reinsurer 2	0.09344	0.06812	0.00792	0.00000	0.00000	0.00000
Reinsurer 3	0.02721	0.02331	0.01943	0.00001	0.00000	0.00000
Reinsurer 4	0.00494	0.00430	0.00422	0.00000	0.00000	0.00000
Reinsurer 5	0.00126	0.00113	0.00112	0.00078	0.00058	0.00052

Table 12: Illustration of how the ruin probability depends on the level of initial capital at $T = 100$ for the insurance company (Insurer) and each member of the reinsurance panel (Reinsurer 1 to 5).

As evident, as well as expected, from Table 12, substantially reduces the amount of capital required to keep the ruin probability at an acceptable level for each participating reinsurer; with an initial capital of 100, the ruin probability is below 0.5% for all reinsurers. In comparison, and shown in Table 9, this is not even the case with an initial capital of 200 when one reinsurer is covering losses in the unbounded layer. Noteworthy is also the fact that the ruin probability is very low, indeed below 0.5%, regardless of the level of initial capital for Reinsurer 5.

4.3.3 Conclusion

In conclusion, from the insurance company's perspective, purchasing excess-of-loss reinsurance is an effective way of mitigating the effects on its capital position and thus ruin probability from the severity of the Pareto distribution. From the reinsurance company's perspective, however, the ruin probability exceeds the threshold value of 0.5 % regardless of self-retention level and initial capital when unlimited coverage is offered. To maintain an adequate capital position, it should limit its liability to any bounded layer, as shown in Table 12, the results of which starkly contrasts to the results of Table 9. By the very nature of the situation, the insurance company will thus have to purchase coverage from multiple reinsurers. With this arrangement, the insurance company obtains protection and each reinsurance company limits its liability to claims in its respective layer, with the exception of Reinsurer 5. However, in case of reinsurance capacity constraints, the insurance company would nonetheless have to resort to other measures to manage the risk, e.g. raising more capital or increasing premiums either specifically for the risk in question or for the entire portfolio.

5 Final remarks

The purpose of this thesis has been to try and emphasise the consequences of whether a severity distribution can be considered light-tailed or heavy-tailed and the implications of this classification on the dynamics of the ruin probability of an idealised insurance company's portfolio.
In Chapter 2, some properties were suggested which a light-tailed and heavy-tailed distribution should possess. Amongst these were the suggestion of using the exponential distribution as a reference distribution: if the right-tail decays faster to zero than the right tail of the exponential distribution, then it could be considered light-tailed; if on the other hand it decays slower, it could be considered heavy-tailed. Another suggestion for distinguishing between a light-tailed or heavy-tailed distribution could be to use the mean excess function and require it bounded in

case of a light-tailed distribution and unbounded if heavy-tailed. The notion of a distribution function with a regularly varying right tail was introduced exhibiting the property of being closed under summation implying that heavy-tails can not be "diversified away" in the sense of the Central Limit Theorem. An example were provided to show this. As the piéce de rèsistance, the class of subexponential distributions was introduced, which derives it name from having a right tail decaying faster to zero than any exponential tail. This class of functions is a useful and popular choice in non-life insurance mathematics. Some properties were stated in Theorem 2.2. A corollary to this is the fact that if the distribution is subexponential, then it does not have a moment generating function.

Chapter 3 was a very brief summary of ruin theory with Lundberg's inequality (Theorem 3.2) and Cramér's ruin bound (Theorem 3.3) as classical results in the light-tailed case. In particular, these two theorems state the asymptotic behaviour of the ruin probability and proves the ruin probability to decay exponentially as the amount of initial capital tends to infinity. Under the assumption of an exponential severity distribution it is even possible to derive an explicit expression for the ruin probability. In the heavy-tailed case, Theorem 3.5 provides a fundamentally different asymptotics in the case of subexponential distributions: the probability of ruin $\psi(u)$ is of essentially of the same order as $\overline{F}_{X,I}(u)$, which decays slower than any exponential tail.
The results in Chapter 4 indicate a fundamentally different behaviour of the portfolio in the presence of heavy-tailed severity distribution as opposed to a portfolio containing solely light-tailed risks. In the former instance, the influence is evident through the occurrence of very large claims affecting the surplus of the portfolio to the extent of a non-negligible probability of it becoming negative over the modelled time horizon. Insofar as this probability is unacceptably high, careful capital and risk management is called for to address this. In the absence of raising additional capital, purchasing reinsurance might be an option to lower the ruin probability. However, as shown in 9, the ruin probability for the reinsurance company might as well be considered unacceptably high even when the attachment point of the contract is quite far out in the tail, thus forcing the reinsurance company to reduce capacity even further. As shown in Table 12, an effective way for the reinsurance company to reduce the necessary capital employed is to limit its commitment to a layer $((d_i, d_j)$ and thus its maximum loss. This will again force the insurance company to consider other measures to mitigate the risk. Purchasing reinsurance from multiple reinsurers as specified in Table 11, increasing premiums or raising additional capital might be such measures, whereby some will be preferable to other. In some instances, some might not even be possible.

6 Appendix

6.1 Karamata's Theorem

Karamata's theorem is formulated as

Theorem 6.1 (Karamata's theorem, p. 9 in [19]). *Let L be slowly varying and locally bounded in $[x_0, \infty)$ for some $x_0 \geq 0$. Then,*

- *for $\alpha > -1$,*

$$\int_{x_0}^{x} t^{\alpha} L(t) dt \sim (\alpha + 1)^{-1} x^{\alpha+1} L(x), \quad x \to \infty,$$

- *for* $\alpha < -1$,

$$\int_x^{\infty} t^{\alpha} L(t)dt \sim -(\alpha+1)^{-1} x^{\alpha+1} L(x), \quad x \to \infty.$$

By noting that $f(x) = x^{\alpha} L(x)$, the conclusions of Theorem 6.1 can be expressed as

- for $\alpha > -1$,

$$\lim_{x\to\infty} \frac{\int_{x_0}^{x} f(t)dt}{x f(x)} = \frac{1}{(\alpha+1)}$$

- for $\alpha < -1$,

$$\lim_{x\to\infty} \frac{\int_{x_0}^{\infty} f(t)dt}{x f(x)} = -\frac{1}{(\alpha+1)}$$

6.2 Detailed Numerical Results

Figure 15 shows the distribution of the surplus of Line of Business 1 at $t = 25, 50, 75, 100$.

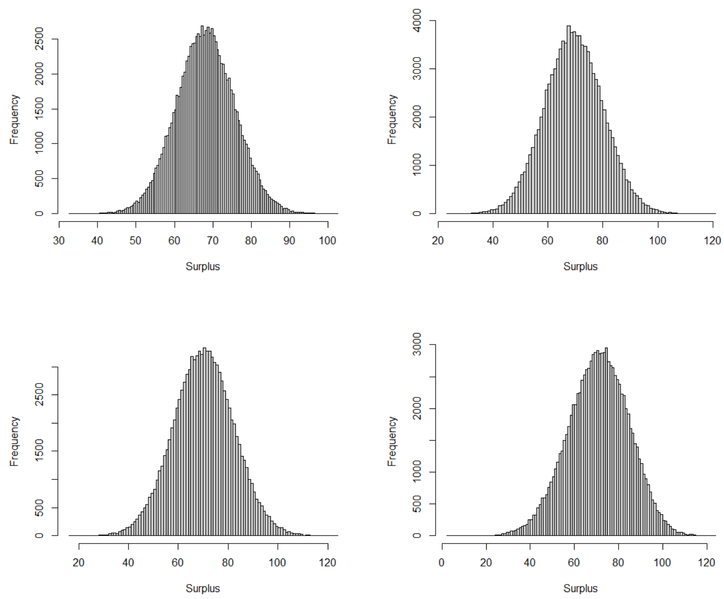


Figure 15: Distribution of the surplus of Line of Business 1 at $t = 25$ (top left), $t = 50$ (top right), $t = 75$ (bottom left) and $t = 100$ (bottom right). Initial capital equal to 66.67.

Table 13 shows some information about the distribution of the surplus for Line of Business 1.

Time	25	50	75	100
0.1th	43.56	35.85	32.39	27.62
0.5th	48.07	41.78	38.69	34.20
1th	50.12	44.63	41.80	37.85
2.5th	53.07	48.58	46.27	43.31
10th	58.30	55.81	54.70	53.12
Mean	68.09	69.40	70.16	70.93
Standard deviation	7.69	10.67	12.14	13.71
CoV	0.11	0.15	0.17	0.19
Min	32.91	23.51	16.00	2.87
Max	102.41	120.12	123.00	123.00

Table 13: Distributional information about Line of Business 1 at $t = 25, 50, 75, 100$. Initial capital equal to 66.67.

By studying both Figure 15 and Table 13 it is evident the influence from Line of Business 1 on the left tail of the distribution, and hence the values of the ruin probability of interest here, is negligible.
Figure 16 shows the surplus of Line of Business 2 t $t = 25, 50, 75, 100$.

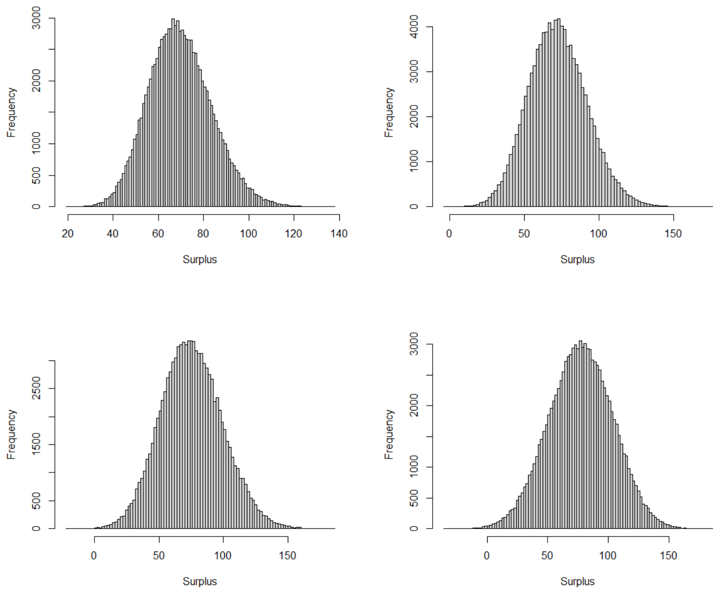


Figure 16: Distribution of the surplus of Line of Business 2 at $t = 25$ (top left), $t = 50$ (top right), $t = 75$ (bottom left) and $t = 100$ (bottom right). Initial capital equal to 66.67.

Table 14 shows some information about the distribution of the surplus for Line of Business 2.

Time	25	50	75	100
0.1th	31.93	18.56	7.39	-3.04
0.5th	37.40	26.33	17.79	9.30
1th	40.29	30.12	22.80	15.86
2.5th	44.40	36.36	30.79	25.30
10th	52.36	48.00	45.52	43.14
Mean	69.66	72.47	75.37	77.26
Standard deviation	13.93	19.51	23.82	26.38
CoV	0.20	0.27	0.32	0.34
Min	19.97	-3.81	-20.91	-34.81
Max	137.17	175.84	185.67	185.67

Table 14: Distributional information about Line of Business 2 at $t = 25, 50, 75, 100$. Initial capital equal to 66.67.

Although somewhat wider, it is evident from Figure 16 and Table 14 that the influence from Line of Business 2 on the left tail of the distribution, and hence the values of the ruin probability of interest here, is negligible.
Figure 17 shows the surplus of Line of Business 3 t $t = 25, 50, 75, 100$.

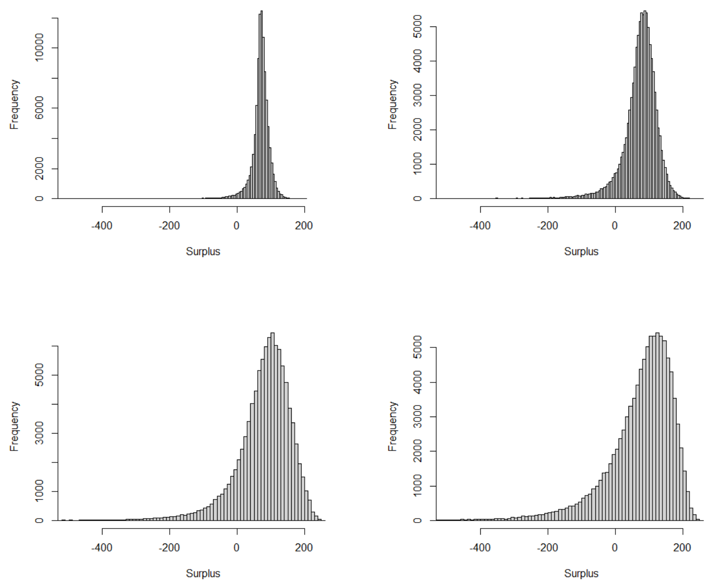


Figure 17: Distribution of the surplus of Line of Business 3 at $t = 25$ (top left), $t = 50$ (top right), $t = 75$ (bottom left) and $t = 100$ (bottom right). Initial capital equal to 66.67.

Table 15 shows some information about the distribution of the surplus for Line of Business 3.

Time	25	50	75	100
0.1th	-340.86	-704.88	-1065.19	-1363.37
0.5th	-114.27	-256.57	-445.77	-568.39
1th	-54.05	-160.39	-288.67	-568.39
2.5th	-3.11	-71.07	-156.98	-222.10
10th	42.36	14.31	-23.37	-56.12
Mean	67.78	69.72	72.77	66.51
Standard deviation	43.15	81.47	120.96	146.73
CoV	0.64	1.17	1.66	2.21
Min	-3980.10	-6348.40	-8343.93	-8730.08
Max	201.32	257.27	258.22	258.22

Table 15: Distributional information about Line of Business 3 at $t = 25, 50, 75, 100$. Initial capital equal to 66.67.

By studying Figure 17 and Table 15, it is evident that the left tail of the Pareto distribution, and thus Line of Business 3, has a significant influence on the left tail of the distribution of the portfolio surplus. This is in stark contrast to the other lines of business.
Figure 18 shows the surplus of the portfolio, at $t = 25, 50, 75, 100$, without any reinsurance.

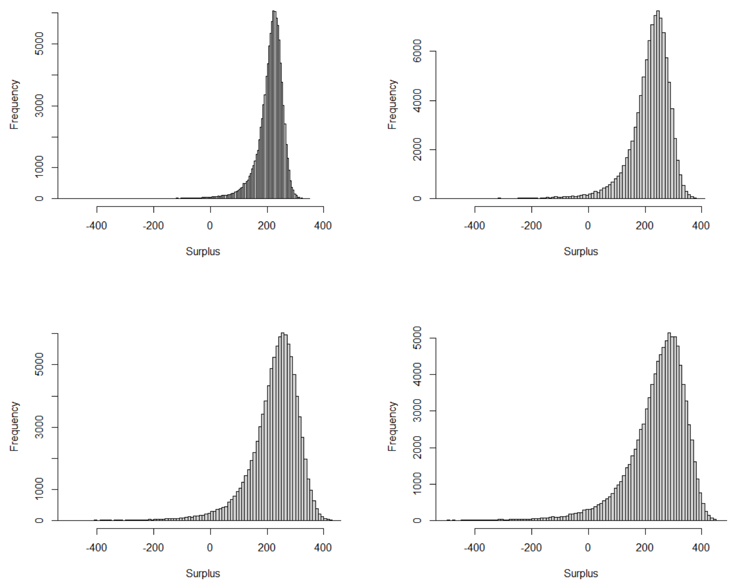


Figure 18: Distribution of the surplus of the portfolio, without any reinsurance, at $t = 25$ (top left), $t = 50$ (top right), $t = 75$ (bottom left) and $t = 100$ (bottom right). Initial capital equal to 200.

Table 16 shows some information about the distribution of the surplus for the portfolio without any reinsurance.

Time	25	50	75	100
0.1th	-474.56	-742.65	-993.89	-1192.69
0.5th	-94.00	-221.09	-317.35	-401.91
1th	-8.62	-96.18	-156.90	-223.05
2.5th	74.62	19.94	-20.03	-56.52
10th	153.61	132.09	116.08	107.67
Mean	207.00	213.39	219.44	234.83
Standard deviation	73.36	105.72	127.84	149.03
CoV	0.35	0.50	0.58	2.21
Min	-5750.16	-8176.96	-8168.95	-8559.23
Max	345.56	407.10	453.25	485.68

Table 16: Distributional information about the portfolio, without reinsurance cover, at $t = 25, 50, 75, 100$. Initial capital equal to 200.

Figure 19 shows the surplus of the portfolio, at $t = 25, 50, 75, 100$, with excess-of-loss type reinsurance with deductible $d_1 = 21.62$.

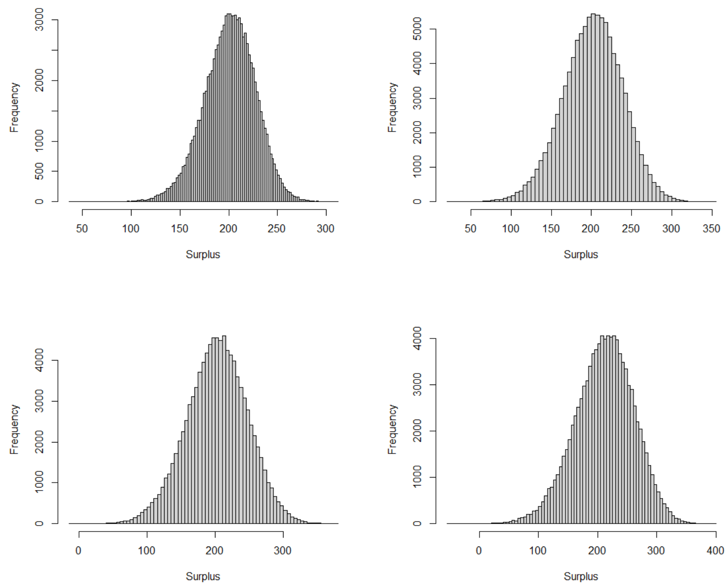


Figure 19: Distribution of the surplus of the portfolio, with reinsurance at deductible $d_1 = 21.62$, at $t = 25$ (top left), $t = 50$ (top right), $t = 75$ (bottom left) and $t = 100$ (bottom right). Initial capital equal to 200.

Table 17 shows some information about the distribution of the surplus for the portfolio with excess-of-loss type reinsurance with deductible $d_1 = 21.62$.

Time	25	50	75	100
0.1th	108.23	78.00	55.45	39.57
0.5th	126.28	100.33	83.02	71.73
1th	134.36	111.03	94.29	86.86
2.5th	146.09	126.35	111.52	109.53
10th	166.73	154.03	144.73	146.88
Mean	201.00	201.58	202.30	211.37
Standard deviation	26.34	36.89	44.545	49.45
CoV	0.13	0.18	0.22	0.24
Min	36.09	22.12	-14.17	-51.53
Max	311.00	350.28	378.39	398.30

Table 17: Distributional information about the portfolio, with excess-of-loss type reinsurance with deductible $d_1 = 21.62$, at $t = 25, 50, 75, 100$. Initial capital equal to 200.

Figure 20 shows the surplus of the portfolio, at $t = 25, 50, 75, 100$, with excess-of-loss type reinsurance with deductible $d_2 = 34.7$.

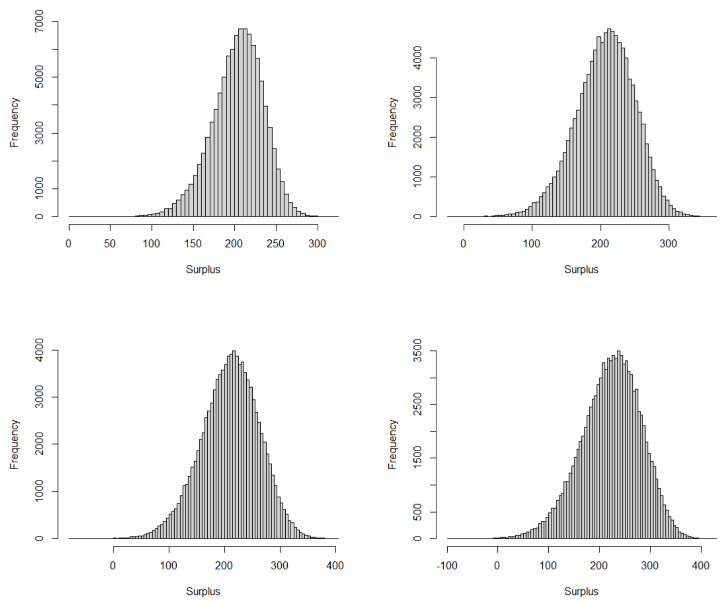


Figure 20: Distribution of the surplus of the portfolio, with reinsurance at deductible $d_2 = 34.7$, at $t = 25$ (top left), $t = 50$ (top right), $t = 75$ (bottom left) and $t = 100$ (bottom right). Initial capital equal to 200.

Table 18 shows some information about the distribution of the surplus for the portfolio with excess-of-loss type reinsurance with deductible $d_2 = 34.7$.

Time	25	50	75	100
0.1th	86.49	52.62	28.77	9.24
0.5th	109.83	81.14	63.63	50.76
1th	120.81	96.38	78.65	69.90
2.5th	136.16	116.09	101.01	97.37
10th	163.31	150.71	142.04	144.85
Mean	203.70	207.07	210.28	222.38
Standard deviation	30.96	43.34	52.35	59.00
CoV	0.15	0.21	0.25	0.27
Min	4.50	-21.77	-76.31	-96.89
Max	322.84	369.71	404.04	428.24

Table 18: Distributional information about the portfolio, with excess-of-loss type reinsurance with deductible $d_2 = 34.7$, at $t = 25, 50, 75, 100$. Initial capital equal to 200.

Figure 21 shows the surplus of the portfolio, at $t = 25, 50, 75, 100$, with excess-of-loss type reinsurance with multiple reinsurers participating in different layers, each layer attaching at $d_1 = 21.7$, $d_2 = 34.7$, $d_3 = 53.2$, $d_4 = 90$ and $d_5 = 131.4$.

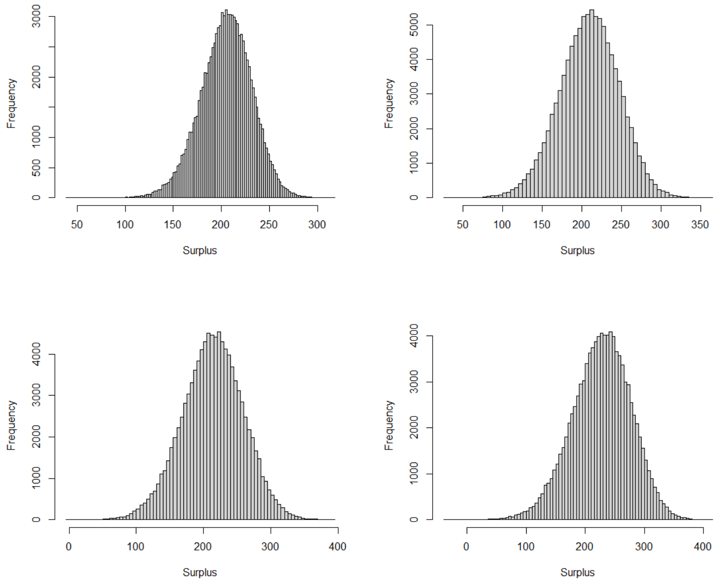


Figure 21: Distribution of the surplus of the portfolio, with excess-of-loss type reinsurance with multiple reinsurers participating in different layers, at $t = 25$ (top left), $t = 50$ (top right), $t = 75$ (bottom left) and $t = 100$ (bottom right). Initial capital equal to 200.

Table 19 shows some information about the distribution of the surplus for the portfolio with excess-of-loss type reinsurance with multiple reinsurers participating in different layers, each layer attaching at $d_1 = 21.7$, $d_2 = 34.7$, $d_3 = 53.2$, $d_4 = 90$ and $d_5 = 131.4$.

Time	25	50	75	100
0.1th	111.74	84.58	65.20	55.26
0.5th	129.68	107.01	92.85	87.26
1th	137.73	117.64	104.39	102.42
2.5th	149.54	133.13	121.68	125.10
10th	170.20	161.17	155.16	144.85
Mean	204.79	209.22	213.40	162.50
Standard deviation	26.67	37.36	45.11	49.50
CoV	0.13	0.18	0.21	0.22
Min	38.50	28.64	-4.84	-36.99
Max	317.26	360.57	391.97	414.15

Table 19: Distributional information about the portfolio, with excess-of-loss type reinsurance with multiple reinsurers participating in different layers, at $t = 25, 50, 75, 100$. Initial capital equal to 200.

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