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Abstract

We study optimal dividend payments and investments of the surplus of with-profit life insurance policies using continuous-time stochastic control. Under some simplifying assumptions, the control problem studied can be treated as a generalisation of the investment-consumption problem first set up and studied by Merton.

We use the dynamic programming method, by which the control problem boils down to solving a second order partial differential equation (PDE) called a *Hamilton-Jacobi- Bellman equation.* We consider cases where the policy holders display constant relative risk aversion, which implies first that the PDE has a semi-explicit solution and second that the optimal investment process is constant. The optimal dividend process is linear in the surplus.

We illustrate the results with simulations for a simple life annuity, where the PDE has an explicit solution.

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Contents

Notation and symbols

Notation	Explanation
$x \wedge y$	$\min(x, y)$
$x \vee y$	$\max(x, y)$
$\mathbb{1}_A$	indicator function of A
$N(0, \sigma^2)$	Normal distribution with mean μ and variance σ^2
$\mathcal{B}(U)$	Borel σ-algebra generated by open subsets of U
$f_t(t,x), f_x(t,x), f_{xx}(t,x)$	$\frac{\partial f}{\partial t}(t,x),\ \frac{\partial f}{\partial x}(t,x),\ \frac{\partial^2 f}{\partial x^2}(t,x)$
ODE	Ordinary differential equation
PDE	Partial differential equation
HJB	Hamilton-Jacobi-Bellman

1 Introduction

An important part of life insurance mathematics deals with proper valuation of the future cash flows between the insured (or other beneficiaries of an insurance policy) and the issuer of the policy as a result of insurance contracts. Valuation is done on the level of individual policies as well as on an aggregate portfolio or company level. Apart from the valuation aspects, life insurance practice also contains decision problems in a stochastic environment, such as deciding on asset allocation strategies, dividend payments to insured and/or equity holders, premium levels etc. A systematic treatment of such decision problems can be achieved using stochastic control.

The motivating control problem for this thesis was controlling dividend payments in the with-profit life annuity that is a part of the Swedish Premium pension system, but we will consider the general setup of controlling the surplus in with-profit life insurance. With-profit life insurance is a traditional form of life insurance where the insurance company in exchange for an insurance premium promises payments to the insured, in the form of lump sums and/or payment streams, depending on states relating to the health and life status of the insured. Premia are calculated in a prudent fashion, which means that the premia are larger than what the insurance company actually needs, under best estimates, to set aside in order to be able to pay out the promised, so-called *guaranteed* payments. This difference gives rise to a surplus that the insurance company should invest and return to the insured as dividends, either in the form of cash or in the form of additional premia which increases the guaranteed payments. Thus, both the investment strategy and the dividend strategy act so as to control the surplus and it is natural to seek ways of optimising these controls. The criterion that will be used for optimisation is maximisation of the expected utility of the insured.

Under some simplyfing assumptions regarding the asset market, where the surplus can be invested, and the product design, it turns out, as first analysed by Steffensen [Ste04], that the problem can be seen as a version of the investment-consumption problem first set-up and solved by Merton [Mer69], [Mer71].

We will solve the stochastic control problems using the the dynamic programming approach, which was originally introduced by Bellman [Bel53]. This approach uses the fact that the *value function*, the optimum of a gain or cost functional over admissible controls, satisfies a certain recursive property, which yields a dynamic programming equation. In the cases considered in this thesis, where the controlled system can be modelled as a stochastic differential equation which satisfies the Markov property, the dynamic programming equation will be a second-order partial differential equation called a *Hamilton-Jacobi-Bellman (HJB)-equation*. Finding a solution to the HJB-equation associated to the control problem is in general difficult. A solution may not even exist in a classical sense, or, if it exists, it may need to be solved numerically. We will however consider cases where classical solutions exist and where explicit or semi-explicit solutions can be found. That the solution to the HJB-equation indeed is the value function of the control problem and that a candidate for an optimal control indeed is an optimal control is shown by verification theorems.

Our presentation of the dividend payment problem will follow that of Steffensen in [Ste04, sections 1-4] and that of Schmidli [Sch08, p 127-132] where we consider the case where the surplus is paid out as cash and not used for increasing the level of guaranteed benefits, as this is the case in the with-profit life annuity in the Premium pension and indeed in most with-profit insurance products in the Swedish market. Using preferences that can be represented by power utility functions with constant relative risk aversion, the optimal portfolio strategy is constant and depends on the mean return and volatility of the risky asset and the risk aversion of the insured. The optimal dividend allocation strategy is linear in the surplus with coefficients determined by the utility functions and the time-dependent part of the value function. The time-dependent part of the value function can be interpreted as a utility-adjusted value of future payments.

In addition to the presentation that largely follows [Ste04, sections 1-4] and Schmidli [Sch08, p 127-132], we consider a new example. This example is a life annuity with two states, for which the HJB-equation can be solved analytically. In particular the impact of mortality on the dividend strategy and the surplus is illustrated and is seen to increase the initial dividend rate since the value of future consumption is discounted by mortality.

The thesis is organized as follows. In order to make the thesis accessible to non-specialists with an undergraduate background in mathematics, statistics or actuarial science, Section 2 introduces necessary definitions, concepts and results from probability theory, stochastic processes and stochastic calculus as well as some results concerning ordinary differential equations. In Section 3 we introduce stochastic control and how to solve stochastic control problems using dynamic programming. This section concludes with a solved example: a version of the so-called Merton problem of optimal consumption and investment from mathematical finance. Section 4 contains the main problem of the thesis. The section starts with introducing relevant concepts and terms from life insurance mathematics, in particular the mathematical formulation of a with-profit policy. This is followed by the presentation and solution of the main problem of optimising the dividend payments in a with-profit life insurance. The section concludes with illustrations in the particular case of a simple life annuity. Finally, Section 5 concludes the thesis.

2 Preliminaries

This section presents some important concepts and results from probability theory and analysis which are used in the remainder of the thesis. We assume that the reader are familiar with basic concepts of probability (such as the concept of a σ-algebra, a probability measure and expectation). In most cases, the results will be given without proof. For proofs and more thorough explanations, the reader can consult one of numerous resources on probability theory, stochastic processes and stochastic calculus. The ones that have been used for this section include [Pha09, ch 1] and [CE15, ch 2 - 3]. Some results are also taken from [AS20].

2.1 Probability and analysis essentials

In this subsection, the main reference is [CE15, ch 2].

Definition 2.1 (Probability space)**.** Let Ω be a set whose elements represent outcomes of some state of the world, which we call *sample space*. Subsets of Ω are called *events*. Let $\mathcal{F}$ be a σ-algebra on Ω and let P be a finite measure on $(\Omega, \mathcal{F})$ such that $P(\Omega) = 1$. The triplet $(\Omega, \mathcal{F}, P)$ is called a *probability space*.

We think of $\mathcal{F}$ as containing events for which we can decide whether they have occurred or not. The random variables and processes which we consider will always be defined on an underlying probability space (and explicit reference to this space will often be suppressed). Properties that hold with probability one (on the given probability space) are said to hold P-almost surely. In the following we will often suppress explicit mentioning of this, so that statements that are claimed as true in fact hold P-almost surely.

Definition 2.2 (Random variable)**.** A map $X : \Omega \to \mathbb{R}$ is called a *random variable* if it is measurable with respect to the Borel sigma algebra on $\mathbb{R}$, i.e. $X^{-1}(U) = \{\omega \in \Omega : X(\omega) \in U\} \in \mathcal{F}$ for any Borel set U.

The σ-algebra generated by X is denoted by $\sigma(X) \subset \mathcal{F}$ and is the smallest σ-algebra such that X is measurable.

It is crucial in the study of probability to be able to model how information already available (from events contained in some σ-algebra which is smaller or coarser than the full σ-algebra $\mathcal{F}$) influences probability and expectation of a random variable. In order to be able to do this one needs the concept of conditional expectation. (cf. [CE15, Definition 2.3.1]).

Definition 2.3 (Conditional expectation)**.** Let X be a random variable on

$(\Omega, \mathcal{F}, P)$ such that $\mathbb{E}(|X|) \leq \infty$. Let $\mathcal{E} \subset \mathcal{F}$ be a σ-algebra. The *conditional expectation* of X given $\mathcal{E}$, denoted by $\mathbb{E}(X|\mathcal{E})$ is given by any $\mathcal{E}$-measurable random variable Y, such that

$$\mathbb{E}(\mathbb{1}_A X) = \int_A X dP = \int_A Y dP = \mathbb{E}(\mathbb{1}_A Y) \text{ for any } A \in \mathcal{E}.$$

If Z also satisfies the conditions above, then $Y = Z$ almost surely.

This corresponds to $\mathbb{E}(X|\mathcal{E})$ providing an average of X over the sets of $\mathcal{E}$ ([CE15, Remark 2.3.3]). The above definition is rather abstract, but it can be shown that a conditional expectation defined in this way satisfies the properties that one should expect (given knowledge from basic probability courses).

Proposition 2.1 (Properties of conditional expectation)**.** *The conditional expectation defined in 2.3 satisfies*

- *Linearity.*
- *Tower property:* $\mathbb{E}(\mathbb{E}(X|\mathcal{E})|\mathcal{D}) = \mathbb{E}(X|\mathcal{D})$ *for a σ-algebra $\mathcal{D}$ such that $\mathcal{D} \subset \mathcal{E}$.*
- *Factoring out known random variable: if Y is $\mathcal{E}$-measurable, X and the product XY are integrable, then* $\mathbb{E}(YX|\mathcal{E}) = Y\mathbb{E}(X|\mathcal{E})$.

From analysis we will also need some of the integral convergence theorems, which provide conditions that allow exchanging the order of limits and integration/expectation.

Proposition 2.2 (Dominated convergence theorem)**.** *Let X_n be a sequence of random variables such that $X_n \to X$ almost surely and let G be non-negative and integrable such that $|X_n| \leq G$. Then*

$$\lim_{n\to\infty} \mathbb{E}(X_n) = \mathbb{E}(X).$$

Proposition 2.3 (Fatou's Lemma)**.** *Let X_n be a sequence of non-negative random variables. Then*

$$\mathbb{E}(\liminf_{n\to\infty} X_n) \leq \liminf_{n\to\infty} \mathbb{E}(X_n).$$

Proposition 2.4 (Monotone convergence theorem)**.** *Let X_n be a sequence of non-negative random variables such that it holds almost surely that $X_n \leq X_{n+1}$ and $X_n \to X$. Then*

$$\lim_{n\to\infty} \mathbb{E}(X_n) = \mathbb{E}(X).$$

Note that propositions 2.2, 2.3 and 2.4 also hold, mutatis mutandis, for conditional expectations.

2.2 Stochastic processes and martingales

As the main object of study in this thesis is controlled stochastic processes (and their applications in an insurance context), we will provide brief definitions of the most important concepts (where we rely mainly on [CE15, $Ch.3, Ch.5$]). As continuous-time stochastic processes is a very technical subject, it is outside the scope of the text to provide all definitions and results in full detail and generality. Some technicalities will only be mentioned in this subsection. We refer to [CE15] and further references given there.

Definition 2.4 (Stochastic process)**.** A stochastic process $X = (X_t)_{t\in\mathbb{T}}$ is a family of random variables indexed by $t \in \mathbb{T}$. Usually $\mathbb{T} = [0, T]$ where $0 < T \leq \infty$ and we think of $\mathbb{T}$ as time. We call X_t (or $X(t)$) the state of X at time t.

The map $X(\omega) : t \in \mathbb{T} \to X_t(\omega)$ is called the sample path of $\omega \in \Omega$.

We also need to account for how information about Ω (which we encode by σ-algebras) increases over time (cf. [Pha09, p 1]), which leads to the concept of a filtration.

Definition 2.5 (Filtration)**.** A filtration $\{\mathcal{F}_t\}_{t\in\mathbb{T}}$ is a family of σ-algebras such that $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for all $s \leq t$. A probability space with a filtration is called a *filtered probability space.*

The filtration generated by a stochastic process X is given by

$$\{\mathcal{F}_t\}^X_{t\in\mathbb{T}} = \sigma(X_s, s \leq t),$$

the σ-algebra generated by all random variables $X_s, s \leq t$.

Whenever we consider a stochastic process X, we will always assume that it is defined on a filtered probability space. In this thesis we will also assume that all filtrations satisfy some technical conditions which are called the *usual conditions*:

- $\mathcal{F}_0$ contains all subsets of P-null sets, i.e. all sets $\{A \subset \Omega : A \subseteq B \in \mathcal{F} \text{ such that } P(B) = 0\}$.
- $\mathcal{F}_t$ is right-continuous, meaning that $\mathcal{F}_t = \cap_{s\geq t}\mathcal{F}_s$.

Definition 2.6 (Version of a stochastic process)**.** Two stochastic processes $(X_t)_{t\in\mathbb{T}}$ and $(Y_t)_{t\in\mathbb{T}}$ are *versions* of each other if

$$P(\{\omega : X_t(\omega) = Y_t(\omega)\}) = 1 \text{ for each } t \in \mathbb{T}.$$

In this thesis, we will assume that there exist versions of the processes studied that exhibits the so-called *càdlàg* property, namely that the process is right-continuous with left limits (and from now on we will not explicitly state that this property is satisfied). The left limit at t of such a process is denoted by $X(t-)$ or X_{t-}.

Definition 2.7 (Adapted process)**.** A stochastic process X is adapted to a filtration $\{\mathcal{F}_t\}_{t\in\mathbb{T}}$ if X_t is $\mathcal{F}_t$-measurable for each t.

Another important concept in the study of stochastic processes is that of a stopping time. Stopping times (relative to a filtration) are, roughly speaking, random times for which it is possible to decide whether they have occured at a fixed time using the information from the filtration.

Definition 2.8 (Stopping time)**.** A random variable $\tau : \Omega \to \mathbb{T} \cup \{\infty\}$ is a stopping time (with respect to a filtration $\{\mathcal{F}_t\}_{t\in\mathbb{T}}$) if

$$\{\tau \leq t\} = \{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t \text{ for each } t \in \mathbb{T}.$$

Remark. An example of why it is useful to assume the usual conditions is given in [Pha09, Prop 1.1.4]: it guarantees that if X is *càdlàg*, then the hitting time of any open set $U \subset \mathbb{R}$ is a stopping time. If X is continuous, then the exit time of any open $U \subset \mathbb{R}$ is a stopping time.

Definition 2.9 (Stopped stochastic process)**.** Let X be a stochastic process on a filtered probability space and τ a stopping time. Then the stopped stochastic process X^τ is given by

$$X^\tau(t) = \begin{cases} X(t) & \text{if } t \leq \tau \\ X(\tau) & \text{if } t > \tau. \end{cases}$$

We have now come to an important class of stochastic processes, which will occur frequently in the following, namely the class of *martingales*.

Definition 2.10 (Martingale (sub, super))**.** A stochastic process $(X_t)_{t\in\mathbb{T}}$ is a *supermartingale* with respect to a filtration $\{\mathcal{F}_t\}_{t\in\mathbb{T}}$ if it is adapted and the following properties hold:

- $\mathbb{E}(|X_t|) < \infty$ for all $t \in \mathbb{T}$.
- $\mathbb{E}(X_t|\mathcal{F}_s) \leq X_s$ for each $s \leq t$.

If, in the last property, we replace "$\leq$" by "$\geq$", we say that $(X_t)_{t\in\mathbb{T}}$ is a *sub-martingale*. A process which is both a supermartingale and a submartingale is called a *martingale*.

It turns out that, for the martingales considered here, this property also extends to stopping times ([Pha09, Theorem 1.1.3]):

Theorem 2.5 (Optional sampling)**.** *Let $(X_t)_{t\in\mathbb{T}}$ be a càdlàg martingale (with respect to some filtration) and let S, T be bounded stopping times such that $S \leq T$ and $S, T \in \mathbb{T}$. Then*

$$\mathbb{E}(X_T|\mathcal{F}_S) = X_S \text{ almost surely}$$

which yields the following corrollary ([Pha09, Corollary 1.1.1]).

Corollary 2.5.1. *Let $(X_t)_{t\in\mathbb{T}}$ be càdlàg and adapted to $\{\mathcal{F}_t\}_{t\in\mathbb{T}}$.*

1. *X is a martingale iff for any bounded stopping time T we have that $\mathbb{E}(|X_T|) < \infty$ and $\mathbb{E}(X_T) = X_0$.*
2. *If X is a martingale and T a stopping time, then the stopped process X^T is a martingale.*

Another useful concept is that of a local martingale, which is a process that is a martingale if we stop it at any finite stopping time. In the following, it will sometimes be the case that showing that a local martingale is in fact a martingale is a crucial part of establishing the desired result.

Definition 2.11 (Local martingale)**.** Let $(X_t)_{t\in\mathbb{T}}$ be *càdlàg* and adapted to $\{\mathcal{F}_t\}_{t\in\mathbb{T}}$. Then X is a local martingale if there exists a sequence $(T_n)_{n\in\mathbb{N}}$ of stopping times such that $\lim_{n\to\infty} T_n = \infty$ almost surely and that the stopped processes X^{T_n} is a martingale for each n. This sequence is called a localizing sequence for X.

Using Fatou's lemma (Proposition 2.3) we get the following result.

Proposition 2.6. *A nonnegative local martingale is a supermartingale.*

Proof. Let T_n be a localizing sequence for X. We have that $\mathbb{E}(X_0) = \mathbb{E}(X^{T_n}) < \infty$. Moreover, we have, for $s \leq t$, that

$$X_s = \lim_{n\to\infty} X_s^{T_n} = \lim_{n\to\infty} \mathbb{E}(X_t^{T_n}|\mathcal{F}_s) \geq \mathbb{E}(\lim_{n\to\infty} X_t^{T_n}|\mathcal{F}_s) = \mathbb{E}(X_t|\mathcal{F}_s)$$

where we used the local martingale property in the second step and Fatou in the second to last step. □

Another class of stochastic processes of fundamental importance is the class of Markov processes, where the current state of the process encodes all information about the future of the process. Another way of putting it, informally, is that future states of a Markov process only depend on its history through the present state. We give the following definition (from [CE15, Def. 17.2.1]).

Definition 2.12 (Markov process)**.** A stochastic process X is Markov with respect to a filtration $\{\mathcal{F}_t\}_{t\in\mathbb{T}}$ if, for every $s \leq t$ we have

$$\mathbb{E}[\phi(X_t)|\mathcal{F}_s] = \mathbb{E}[\phi(X_t)|X_s] \text{ for any bounded, measurable function } \phi : \mathbb{R} \to \mathbb{R}.$$

or, equivalently, that $P(X_t \in A|\mathcal{F}_s) = P(X_t \in A|X_s)$ for any Borel set $A \in \mathbb{R}$. If the property also holds when s, t are stopping times, the process is said to be a *strong* Markov process or have the *strong* Markov property.

In life insurance mathematics, the basic building blocks of models are Markov processes where the state space is finite, i.e. $X_t : \Omega \to \mathcal{J} = \{0, 1, ..., J\}$ (see for example [AS20, Appendix A.3]). Each state corresponds to a policy state (e.g "alive", "dead", "disabled").

Definition 2.13 (Continuous time Markov Chain)**.** A continuous time Markov chain is a Markov process X which take values in $\mathcal{J} = \{0, 1, ..., J\}$. It can be characterised by its transition probabilities $p_{ij}(t, s) = P(X(t) = j|X(s) = i)$ or equivalently by its transition intensities $\lambda_{ij}(t)$ where

$$\begin{cases} p_{ij}(t+h,t) & = \lambda_{ij}(t)h + o(h) \\ p_{ii}(t+h,t) & = 1 - \sum_{i\neq j} \lambda_{ij}(t)h + o(h). \end{cases}$$

Another type of stochastic process which we will encounter are counting processes. We will consider counting processes counting the number of jumps between states in a continuous time Markov Chain. We follow [CE15, p 128] and [JYC09, p 458].

Definition 2.14 (Counting process)**.** A stochastic process N, defined on a filtered probability space, is called a counting process if it is non-decreasing, càdlàg, adapted and takes values in the positive integers $\mathbb{Z}^+$.

The associated jump process $(\Delta N_t)_{t\in\mathbb{T}}$ is defined by $\Delta N_t = N_t - N_{t-}$ and takes values in $\{0, 1\}$.

We will next give some important examples of stochastic processes that will form building blocks for the processes used in the later sections. One of the most fundamental processes is the Wiener process, which can be seen as the continuous time analogue of a random walk and is used ubiquitously in applications.

Definition 2.15 (Wiener process (1-dimensional version))**.** A *Wiener process* W is an adapted stochastic process with $W_0 = 0$ satisfying

1. For any $0 \leq s < t$, the increments $W_t - W_s$ are independent of $\mathcal{F}_s$.
2. The increments $W_t - W_s$ are normally distributed $N(0, t-s)$.

We also have the following property which follows from the distributional properties of the increments.

Proposition 2.7. *Any Wiener process has a version which is almost surely continuous.*

It is clear that the Wiener process is a martingale. In the following, we want to study processes that have dynamics which contains Wiener processes, and it is thus important to be able to set up and study differential equations containing Wiener processes. In particular, we need to be able to define integrals with respect to (certain) stochastic processes. This needs the concept of the variation of a process.

Definition 2.16 (Variation of a process)**.** The variation of a path $X(\omega)$ of a process X on the interval $[0,T]$ is given by

$$V_T(X(\omega)) = \sup \left(\sum_i^n |X_{t_{i+1}}(\omega) - X_{t_i}(\omega)| \right)$$

where the supremum is taken over all partitions $0 < t_0 < ... < t_n = T$. If this is finite for almost all paths and any $t \in \mathbb{T}$ the process has finite variation.

The following related concept is important in order to be able to define a stochastic integral.

Definition 2.17 (Cross-variation and Quadratic variation)**.** If X, Y are continuous local martingales, there exists a process called the cross-variation process (or bracket) which is given by

$$\langle X, Y \rangle_t = \lim_{n\to\infty} \sum_i^n (X_{t_{i+1}^n} - X_{t_i^n})(Y_{t_{i+1}^n} - Y_{t_i^n}) \text{ with convergence in probability}$$

where the mesh size $|t_{i+1}^n - t_i^n| \to 0$ as $n \to \infty$. This process is continuous and of bounded variation.

We call $\langle X \rangle_t = \langle X, X \rangle_t$ the quadratic variation of X.

Proposition 2.8 (Variation and Quadratic variation of a Wiener process)**.** *The Wiener process has infinite variation for almost all paths. Its quadratic variation process is given by $\langle W \rangle_t = t$.*

One important consequence of the above is that it is not possible to define an integral of an adapted process as a Lebesgue-Stieltjes integral with respect to the Wiener process pathwise, since this requires the process to be of bounded variation. It is however possible to define a stochastic integral of an adapted process pathwise if one uses a partition which is carefully chosen to depend on the local roughness of the process, see [Kar95]. That the standard definition of the stochastic integral with respect to the Wiener process as a Lebesgue-Stieltjes integral fails is the starting point for stochastic calculus, which is the topic of the next subsection.

2.3 Stochastic calculus

In order to be able to do analysis on stochastic processes and in particular to formulate and solve stochastic differential equations, it is necessary to define stochastic integration, i.e. integrals where the integrator is some stochastic process. We will define an integral with respect to the Wiener process but it is possible to define stochastic integrals with respect to more general processes [1].

In this subsection, we partly follow [Pha09, ch 1], but also [Øks13, ch 3].

We will define an integral with respect to the Wiener process for processes α with time index set $[0, T]$ such that

- α is progressively measurable, i.e. for any $t \in [0, T]$ the map $(t, \omega) \to \alpha(t, \omega)$ is measurable with respect to $\mathcal{B}([0, t]) \times \mathcal{F}_t$.
- α is adapted to the Wiener filtration.
- $\mathbb{E}\left(\int_0^T |\alpha_t|^2 dt\right) < \infty$.

Such processes are said to belong to $L^2(W)$. In order to define an integral, the so-called Itô integral, for processes in $L^2(W)$, we start by defining it for a subspace of processes where it can be given a natural definition.

Definition 2.18 (Elementary process and its Itô integral)**.** An elementary process ϕ is a process of the form

$$\phi_t = \sum_{k=1}^{n} b_k \mathbb{1}_{(t_k, t_{k+1}]}(t)$$

where b_k is $\mathcal{F}_{t_k}$-measurable and bounded and $t_1, ..., t_n$ is a sequence of stopping times in $[0, T]$. For such elementary processes, the Itô integral is defined as

$$\int_0^t \phi_s dW_s = \sum_{k=1}^{n} b_k (W_{t_{k+1} \wedge t} - W_{t_k \wedge t}).$$

[1] See [CE15, ch 8-12].

This integral will then be extended from the smaller subspace of elementary processes to the full space $L^2(W)$ in view of the following results[2].

Proposition 2.9. *The elementary processes are dense in $L_2(W)$ in the sense that for each $\alpha \in L_2(W)$ there exists a sequence $(\phi^n)_{n\in\mathbb{N}}$ of elementary processes such that*

$$\lim_{n\to\infty} \mathbb{E}\left(\int_0^T (\phi_s^n - \alpha_s)^2 ds\right) = 0.$$

Lemma 2.10 (Itô isometry)**.** *For elementary processes $\phi \in L^2(W)$ it holds that*

$$\mathbb{E}\left(\int_0^T \phi_s^2 ds\right) = \mathbb{E}\left[\left(\int_0^T \phi_s dW_s\right)^2\right]$$

and in particular $\int_0^T \phi_s dW_s$ is a square integrable martingale.

Since the elementary processes are dense in $L_2(W)$, we can define the Itô integral on $L_2(W)$ as follows.

Definition 2.19 (Itô integral with respect to Wiener process)**.** Let $\alpha \in L^2(W)$. Then the Itô integral with respect to the Wiener process is given by

$$\int_0^T \alpha_t dW_t = \lim_{n\to\infty} \int_0^T \phi_s^n dW_s$$

where $(\phi^n)_{n\in\mathbb{N}}$ is a sequence of elementary processes such that

$$\lim_{n\to\infty} \mathbb{E}\left(\int_0^T (\phi_s^n - \alpha_s)^2 ds\right) = 0.$$

and the convergence [3] is in the sense that

$$E\left[\sup_{u\in[0,T]}\left(\int_0^u \phi_s^n dW_s - \int_0^u \alpha_s dW_s\right)^2\right] \to 0$$

Note that the Itô integral is not defined pathwise since convergence of the stochastic integral is in mean square. That the Itô integral is a square integrable martingale now yields (in view of corollary 2.5.1) that it has expectation 0.

We will also consider the less involved stochastic integral with respect to a counting process (cf. [JYC09, p. 458]).

[2] For more details see [Øks13, ch 3.1].
[3] See [CE15, Example 12.1.11]].

Definition 2.20 (Stochastic integral with respect to counting process)**.** Let N be a counting process such that it has a finite number of jumps in $[0, T]$. Let C be a bounded and measurable stochastic process. Then the stochastic integral of C with respect to N is defined as

$$\int_s^t C_u dN_u = \sum_{s<u\leq t} C_u \Delta N_u \text{ where } 0 \leq s < t \leq T.$$

We define the following class of stochastic process following [Pha09, Definition 1.2.11].

Definition 2.21 (Itô process)**.** Let W be a Wiener process on a filtered probability space. A (1-dimensional) Itô process is a process of the form

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s$$

where X_0 is $\mathcal{F}_0$-measurable and b_s and σ_s are adapted and

$$\int_0^t |b_s| ds + \int_0^t |\sigma_s|^2 ds < \infty.$$

For such processes we will use the differential notation $dX_t = b_t dt + \sigma_t dW_t$.

We note that X_t is the sum of a process of bounded variation and a (local) martingale, a class of processes which is called *semimartingales*.

Using the definition to compute stochastic integrals will be tedious and we will therefore make repeated use of the following fundamental result which is similar to the chain rule in standard calculus. We will actually formulate the result, Itô's lemma, following [Pha09, sec 1.2.3, p 17], for a slightly more general class of processes than that for which we have defined the stochastic integral (as it is outside the scope of this thesis to provide a thorough introduction to stochastic integration[4]).

Theorem 2.11 (Itô's lemma for a (class of) semimartingales)**.** *Let X be a process of the form $X = (X^1, ..., X^k)$ where $X^i = M^i + A^i$ where M^i is a continuous martingale and A^i is an adapted process of bounded variation. Let $f \in \mathscr{C}^{1,2}(\mathbb{T} \times \mathbb{R}^k)$. Note that A^i might have jumps and denote by $A^{i,c}$ the continuous part of A^i. Then the process $(f(t, X_t))_{t\in\mathbb{T}}$ is also a semimartingale*

[4]For this we again refer to [CE15, ch 8-12].

(i.e. a sum of a martingale and a process of bounded variation) and is given by

$$f(t, X_t) = f(0, X_0) + \int_0^t f_t(s, X_s)ds + \sum_{i=1}^k \int_0^t f_{x^i}(s, X_s)dM_s^i$$
$$+ \frac{1}{2}\sum_{i,j} \int_0^t f_{x^i x^j}(s, X_s)d\langle M^i, M^j\rangle_s +$$
$$\sum_{i=1}^k \int_0^t f_{x^i}(s, X_s)dA_s^{i,c} + \sum_{0<s\leq t} [f(s, X_s) - f(s, X_{s-})]$$

where for Wiener processes W^i, W^j we have that $\langle W^i, W^j\rangle_t = t$ if $i = j$ and 0 otherwise and for continuous processes of bounded variation the brackets are 0. In particular, for X an Itô process we have that $(f(t, X_t))_{t\in\mathbb{T}}$ is also an Itô process and

$$f(t, X_t) = f(0, X_0) + \int_0^t f_t(s, X_s)ds + \sum_{i=1}^k \int_0^t f_{x^i}(s, X_s)dX_s^i$$
$$+ \frac{1}{2}\sum_{i,j} \int_0^t f_{x^i x^j}(s, X_s)d\langle X^i, X^j\rangle_s.$$

The processes that are to be controlled are defined as stochastic differential equations (SDE) and we will therefore end this subsection by introducing that concept. Here we follow [Pha09, ch 1.3], but see also [Øks13, ch 5]. We will consider what is called *strong* solutions, which means that the solution is adapted to the filtration generated by the driving process of the SDE. This is in contrast to a *weak* solution, for which it is only required that the solution is adapted to some filtration for which the driving process also is adapted. This filtration may be strictly larger than the one generated by the driving process.

Definition 2.22 (Stochastic Differential Equation (driven by Wiener process))**.** Assume that we have a filtered probability space such that W is a d-dimensional Wiener process with respect to the filtration. We consider functions $b = b(t, x, \omega)$ and $\sigma = \sigma(t, x, \omega)$ defined on $\mathbb{T} \times \mathbb{R}^n \times \Omega$, where $b = (b_1, ..., b_d)$ and $\sigma = \sigma_{ij}, 1 \leq i \leq n, 1 \leq j \leq d$. For all $x \in \mathbb{R}^n$ we assume that $b(., x, .)$ and $\sigma(., x, .)$ are progressively measurable processes which we shorten to $b(., x)$ and $\sigma(., x)$. [5]

A stochastic differential equation is an expression of the form

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t.$$

A (strong) solution of the SDE starting at time t is an Itô process such that

$$\int_t^s |b(v, X_v)|dv + \int_t^s |\sigma(v, X_v)|^2 dv < \infty \text{ almost surely}$$

[5] See [Pha09, ch 1.3.1, p 22].

and

$$X_s = X_t + \int_t^s b(v, X_v)dv + \int_t^s \sigma(v, X_v)dW_v$$

for any $t \leq s \in \mathbb{T}$.

The solution exists and is unique provided σ and b satisfy some growth conditions, which are given below.

Proposition 2.12 (Existence and uniqueness of solutions)**.** *Consider an SDE as in 2.22. The SDE has a unique solution provided that the following Lipschitz and growth conditions are satisfied for any $t \in \mathbb{T}$:*

$$|b(t, x, \omega) - b(t, y, \omega)| + |\sigma(t, x, \omega) - \sigma(t, y, \omega)| \leq K|x - y|$$
$$|b(t, x, \omega)| + |\sigma(t, x, \omega)| \leq C(1 + |x|).$$

for some constants K and C. The uniqueness is pathwise and the solution X satisfies

$$E\left(\int_0^T |X_t|^2 dt\right) < \infty.$$

We denote by $X^{s,z}$ the strong solution to the SDE starting from z at time s.

Moreover X satisfies the strong Markov property (definition 2.12).

2.4 Ordinary differential equations

When solving the (partial) differential equations that are associated to the stochastic control problem in this thesis, the setup is such that we will use ansatz solutions which makes the PDEs separable in time and space. Typically, we will then need to solve ordinary differential equations in time, which is why we will recall some results concerning solutions to linear ODEs with non-constant coefficients. The following result is taken from [AS20, Appendix A.2].

Consider the following linear ODE:

$$f_t(t) = g(t)f(t) + h(t) \tag{2.1}$$

where $t \in [a, b]$ with the boundary condition $f(t_0) = f_0$.

Proposition 2.13. *If g, h are continuous on $[a, b]$, then* (2.1) *has a solution, which is given by*

$$f(t) = f_0 e^{\int_{t_0}^t g(u)du} + \int_{t_0}^t e^{\int_s^t g(u)du} h(s)ds$$

In the case that $t_0 = b$ (final value problem), the solution can be written

$$f(t) = f_0 e^{-\int_t^b g(u)du} - \int_t^b e^{-\int_t^s g(u)du} h(s) ds$$

We will also discuss existence of solutions to systems of ODEs where we do not solve the system explicitly. These results can be found in [AO08, ch 15].

Consider the following system of ODEs:

$$\begin{aligned} u_t^1 &= g_1(t, u^1, u^2, ..., u^n) \\ u_t^2 &= g_2(t, u^1, u^2, ..., u^n) \\ \cdots \\ u_t^n &= g_n(t, u^1, u^2, ..., u^n) \end{aligned} \tag{2.2}$$

where $u^i(t)$ is a continuously differentiable function on $[0, T]$.
With $u = (u^1(t), \ldots, u^n(t))$, we can write the system on vector form as

$$u_t = g(t, u) \text{ where } u_t = (u_t^1, \ldots, u_t^n) \text{ and } g(t, u) = (g_1(t, u), \ldots, g_n(t, u)).$$

Here, u_t and u are maps from $[0, T]$ taking values in E (a compact) subset[6] of $\mathbb{R}^n$ and $g(t, u)$ is a map from $[0, T] \times E$ to E.

We are interested in sufficient conditions that guarantee the existence of solutions.

Remark. Usually the following existence theorem are stated for initial value problems, i.e. with boundary conditions $u^0 = (u^1(0), \ldots, u^n(0))$. In our case, we will have boundary conditions $u^T = (u^1(T), \ldots, u^n(T))$ for the endpoint T. The theorem relies on successive iterations [AO08, ch 8-9], which could also be used in the case of endpoint conditions. We will not prove this and instead state without proof that the existence theorem also holds with boundary conditions $u^i(T)$.

The following version then follows from [AO08, Theorem 15.4].

Theorem 2.14. *[Peano existence theorem for system of ODEs] Assume that the following conditions are satisfied in the set $|t - T| \leq T$:*

1. $||u|| < \infty$.
2. *$g(t, u)$ is continuous and bounded.*
3. *$u^T(t)$ is continuous.*

Then the system (2.2) has at least one solution on $|t - T| \leq T$.

[6] Since $u^i(t)$ is a continuously differentiable function defined on a compact set.

3 Stochastic Optimal Control using Hamilton-Jacobi-Bellman Equations

In this section we will discuss at some length the types of stochastic optimal control problems that are the focus of this thesis. We will formulate a class of control problems and outline a method, the so-called Hamilton-Jacobi-Bellman or dynamic programming approach, in order to step-by-step set up and solve these problems. We will also solve a version of one of the most classical stochastic control problems in finance, Merton's problem of optimal consumption and investment. This problem also serves as a foundation for the main problem discussed in this thesis, that of optimal allocation of dividends in a with-profit life insurance, which is the topic of the next section. The scope of the thesis does not permit the inclusion of the most general formulations of stochastic control problems and we will make a number of simplifying assumptions in order to assure that the stochastic differential equations have strong solutions and that the desired solutions of the Hamilton-Jacobi-Bellman PDE are smooth enough. In order to simplify the presentation we only consider one-dimensional SDEs.

We will rely on the so-called *dynamic programming principle* introduced by Bellman [Bel53] in order to solve the stochastic control problems. Control problems where this principle can be used satisfy a certain recursive property which allows us to split the problem into smaller parts where the optimization can be done iteratively over each subinterval. By shrinking the length of the subinterval to 0 we will be able to formally derive a partial differential equation called the *Hamilton-Jacobi-Bellman* (HJB) equation associated to the control problem. That a particular solution of the HJB equation indeed solves the control problem is then established by proving a so-called verification theorem. However, in many but the simplest cases, explicit smooth solutions may be hard to come by.

3.1 Dynamic programming for control

In this subsection, we will define a prototypical stochastic control problem. We will also derive the dynamic programming principle and indicate how the dynamic programming principle together with Itô's lemma can be used to formally derive the associated HJB equation. Here, our main source of inspiration and reference is [Pha09, p. 35-45], but we have also drawn on [vH07, p 141-148]. The time horizon is finite and we write $\mathbb{T} = [0, T], T < \infty$.

Definition 3.1 (Controlled diffusion process)**.** Consider a filtered probability space $(\Omega, \mathcal{F}, P)$ where the filtration is generated by a (1-dimensional) Wiener process W. A controlled diffusion process is an SDE with dynamics described by

$$dX_t^\alpha = b(t, X_t^\alpha, \alpha_t)dt + \sigma(t, X_t^\alpha, \alpha_t)dW_t$$

where

$$b : \mathbb{T} \times \mathbb{R} \times A \to \mathbb{R}$$
$$\sigma : \mathbb{T} \times \mathbb{R} \times A \to \mathbb{R}$$

Here, α_t is progressively measurable and taking values in $A \subset \mathbb{R}$. The functions b and σ are measurable functions satisfying a Lipschitz condition for each $t \in \mathbb{T}$ and each $a \in A$:

$$|b(t, x, a) - b(t, y, a)| + |\sigma(t, x, a) - \sigma(t, y, a)| \leq K|x - y|$$

Next, we define an admissible control strategy (cf. [vH07, def. 6.1.1, p 141]):

Definition 3.2. A control strategy α_t is said to be an admissible control if

1. $(\alpha_t)_{t\in\mathbb{T}}$ is $\mathcal{F}_t$-adapted (which follows from being progressively measurable).
2. the SDE stated in Definition 3.1 has a unique solution.

It is a *Markov* strategy if it has the form $\alpha_t = a(t, X_t^\alpha)$ for some measurable function $a : \mathbb{T} \times \mathbb{R} \to A$. This means that the control process α does not depend on the history of X_t^α [7]. In this thesis, we will assume that the control strategies are indeed Markov strategies.

The prototypical control problem which will be considered is the following (where we follow [Pha09, p 38-39]).

Definition 3.3 (Control problem for a diffusion process)**.** Let X_t^α be a controlled diffusion process as in Definition 3.1 and α an admissible control. The functions $f : \mathbb{T} \times \mathbb{R} \times A \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are measurable functions satisfying the following growth conditions:

$$\begin{aligned} |f(t, x, a)| &\leq C(1 + |x|^2) + D(1 + b(t, 0, a)^2 + \sigma(t, 0, a)^2) \\ |g(x)| &\leq E(1 + |x|^2) \\ g(x) &\geq F \end{aligned}$$

where $C, D, E > 0$ and $F \in \mathbb{R}$ are constants. The *finite-horizon gain function* $(T < \infty)$ is given by

$$J(t, x, \alpha) = \mathbb{E}_{t,x}\left[\int_t^T f(s, X_s^\alpha, \alpha_s)ds + g(X_T^\alpha)\right] \tag{3.1}$$

[7] In many cases this is not a very restricting assumption, see [Øks13, Theorem 11.2.3].

where the notation $\mathbb{E}_{t,x}$ means conditioning on $X_t^\alpha = x$ (which will often be suppressed). The associated *value function* is given by

$$V(t,x) = \sup_\alpha J(t,x,\alpha).$$

The control process α^* is an optimal control for the control problem if $V(t,x) = J(t,x,\alpha^*)$.

Remark. The growth conditions above ensures that the gain is finite for any admissible control α [8].

We will also tacitly assume (see [Pha09, p 39]) that the value function $V(t,x)$ is measurable in all arguments which is not obvious and require results which are outside our scope.

The dynamic programming principle for the control problem makes use of the Markov property of the controlled process, showing that the value function satisfies a certain recursive property. This principle allows for splitting the control problem into subproblems, where the control problem is solved recursively on subintervals of $[0,T]$ By splitting the problem into intervals $[t_k, t_{k+1}]$ and letting $|t_{k+1} - t_k| \to 0$ we will be able to (formally) derive a PDE which the value function should satisfy.

Proposition 3.1 (Dynamic programming principle - finite horizon)**.** *Consider the control problem defined in Definition 3.3. Let $\mathcal{T}_{t,T}$ be the set of stopping times for the process in the interval $[t,T]$. Then the value function satisfies the following recursive property*

$$V(t,x) = \sup_\alpha \mathbb{E}\left[\int_t^\tau f(s, X_s^\alpha, \alpha_s)ds + V(\tau, X_\tau^\alpha)\right]$$

for any stopping time $\tau \in \mathcal{T}_{t,T}$.

The proof combines ideas from the sources [Pha09, p. 43-45], [vH07, p 143-45] and [Sch08, p 30-31].

Proof sketch. Fix τ and consider an arbitrary admissible control strategy α. We

[8] See [Pha09, Remark 3.2.1].

get that

$$\begin{aligned}
J(t,x,\alpha) &= \mathbb{E}\left[\int_t^T f(s,X_s^\alpha,\alpha_s)ds + g(X_T^\alpha)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\int_t^T f(s,X_s^\alpha,\alpha_s)ds + g(X_T^\alpha)\middle| X_\tau^\alpha\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\int_t^\tau f(s,X_s^\alpha,\alpha_s)ds + \int_\tau^T f(s,X_s^\alpha,\alpha_s)ds + g(X_T^\alpha)\middle| X_\tau^\alpha\right]\right] \\
&= \mathbb{E}\left[\int_t^\tau f(s,X_s^\alpha,\alpha_s)ds + J(\tau,X_\tau^\alpha,\alpha)\right] \\
&\leq \mathbb{E}\left[\int_t^\tau f(s,X_s^\alpha,\alpha_s)ds + V(\tau,X_\tau^\alpha)\right]
\end{aligned}$$

where we have used the tower property of conditional expectation and the Markov property of the controlled diffusion process X_s^α (which yields that $J(\tau,X_\tau^\alpha,\alpha)$ does not depend on the path of X_s^α for $s<\tau$).

Taking the supremum of the left hand side yields

$$V(t,x) \leq \sup_\alpha \mathbb{E}\left[\int_t^\tau f(s,X_s^\alpha,\alpha_s)ds + V(\tau,X_\tau^\alpha)\right]$$

Consider the following strategy

$$\widehat{\alpha} = \begin{cases} \alpha & \text{if } s\in[0,\tau] \\ \alpha^\varepsilon & \text{if } s\in[\tau,T] \end{cases}$$

where the strategy α^ε is such that $V(\tau,X_\tau^\alpha)-\varepsilon \leq J(\tau,X_\tau^\alpha,\alpha^\varepsilon)$ for some arbitrary $\varepsilon>0$. That the strategy $\widehat{\alpha}$ is progressively measurable (and hence an admissible control) is not a priori obvious, but it can be shown (see [Pha09, p 42]) that this is the case. We can then use a similar argument as above to obtain

$$\begin{aligned}
V(t,x) \geq J(t,x,\widehat{\alpha}) &= \mathbb{E}\left[\int_t^\tau f(s,X_s^\alpha,\alpha_s)ds + J(\tau,X_\tau^\alpha,\alpha^\varepsilon)\right] \\
&\geq \mathbb{E}\left[\int_t^\tau f(s,X_s^\alpha,\alpha_s)ds + V(\tau,X_\tau^\alpha)\right] - \varepsilon
\end{aligned}$$

Since $\varepsilon>0$ was arbitrary, we can take supremum of the right hand side to obtain

$$V(t,x) \geq \sup_\alpha \mathbb{E}\left[\int_t^\tau f(s,X_s^\alpha,\alpha_s)ds + V(\tau,X_\tau^\alpha)\right]$$

and with both inequalities established, the proposition follows. □

3.2 Formal derivation of the Hamilton-Jacobi-Bellman equation and a verification theorem

In this subsection we mainly rely on [Pha09, p 43]. We will make use of a particular linear differential operator associated to the control problem given in Definition 3.3. Here, and in the following, we will use the notation $\mathscr{C}^{1,2}(\mathbb{R}^+ \times \mathbb{R})$ to mean the space of functions which are continuously differentiable in its first argument and twice continuously differentiable in its second argument.

Definition 3.4 (Differential operator of a controlled diffusion)**.** Let X_s^α be a diffusion process as in Definition 3.1. The differential operator $\mathcal{L}^a$ associated to the controlled diffusion process acts on functions in $\mathscr{C}^{1,2}(\mathbb{R}^+ \times \mathbb{R})$ by

$$\mathcal{L}^a f(t,x) = b(t,x,a)\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma^2(t,x,a)\frac{\partial^2 f}{\partial x^2}(t,x).$$

In order to (formally) derive the associated Hamilton-Jacobi-Bellman equation of the control problem, we use the dynamic programming principle (Proposition 3.1) on a small interval $[t, t+h]$ and a constant control a:

$$V(t,x) \geq \mathbb{E}\left[\int_t^{t+h} f(s, X_s^a, a)ds + V(t+h, X_{t+h}^a)\right]. \tag{3.2}$$

But, assuming that V is smooth enough (and with X_s^α as in Definition 3.1), we can apply Itô's lemma to obtain an expression for $V(t+h, X_{t+h}^a)$:

$$\begin{aligned} V(t+h, X_{t+h}^a) &= V(t,x) + \int_t^{t+h} V_t(s, X_s^a)ds + \int_t^{t+h} V_x(s, X_s^a)dX_s + \\ &\frac{1}{2}\int_t^{t+h} V_{xx}(s, X_s^a)d\langle X_s, X_s\rangle \\ &= V(t,x) + \int_t^{t+h} V_t(s, X_s^a) + \mathcal{L}^a V(s, X_s^a)ds + \\ &\int_t^{t+h} \sigma(s, X_s^a, a)V_x(s, X_s^a)dW_s. \end{aligned}$$

The last term is a stochastic integral with respect to a Brownian motion and, assuming that σV_x satisfies some growth conditions, a martingale with expectation 0. In this formal derivation we will assume this to be the case whereas a part of establishing the so-called verification theorems is to indeed show that this is the case. Inserting into Equation 3.2 and dividing by h yields

$$\begin{aligned} 0 &\geq \mathbb{E}\left[\int_t^{t+h} f(s, X_s^a, a) + V_t(s, X_s^a) + \mathcal{L}^a V(s, X_s^a)ds\right] \\ \Rightarrow 0 &\geq \frac{1}{h}\mathbb{E}\left[\int_t^{t+h} f(s, X_s^a, a) + V_t(s, X_s^a) + \mathcal{L}^a V(s, X_s^a)ds\right]. \end{aligned}$$

Since the integrands are continuous in x, the mean-value property of integral calculus can be applied, and we can (at least formally) pass to the limit ($h \to 0$) to obtain

$$f(t, x, a) + V_t(t, x) + \mathcal{L}^a V(t, x) \leq 0$$

Since this holds for arbitrary $a \in A$, we have that

$$V_t(t, x) + \sup_{a \in A} [f(t, x, a) + \mathcal{L}^a V(t, x)] \leq 0$$

For the optimal control α^* (assuming it exists), we have equality in Equation 3.2 and thus, using similar steps as in the derivation above, we obtain

$$f(t, x, \alpha_t^*) + V_t(t, x) + \mathcal{L}^{\alpha^*} V(t, x) = 0.$$

This indicates that the value function should satisfy

$$V_t(t, x) + \sup_{a \in A} [\mathcal{L}^a V(t, x) + f(t, x, a)] = 0. \tag{3.3}$$

with a terminal condition (from Equation 3.1) given by $V(T, x) = g(x)$ and the supremum, if attained, is the optimal control.

The next, and crucial, step in the HJB approach, is to show that a given smooth solution to the HJB equation is in fact the value function of the control problem. This step also yields an optimal control. Such a verification theorem can be formulated for the prototypical stochastic control problem stated in Definition 3.3.

Theorem 3.2 (Verification theorem - finite horizon)**.** *Let $w \in \mathscr{C}^{1,2}([0, T) \times \mathbb{R})$ and continuous at $T \times \mathbb{R}$. Let moreover w satisfy a quadratic growth condition:*

$$|w(t, x)| \leq C(1 + x^2) \text{ for all } t, x \in [0, T] \times \mathbb{R}.$$

Suppose that w is a solution to the HJB equation 3.3 with terminal condition $w(T, x) = g(x)$ and that there exists an admissible control $\alpha^*(t, x)$ such that

$$\begin{aligned}\frac{\partial w}{\partial t}(t, x) + \sup_{a \in A} [\mathcal{L}^a w(t, x) + f(t, x, a)] &= \frac{\partial w}{\partial t}(t, x) + \mathcal{L}^{\alpha^*(t,x)} w(t, x) + f(t, x, \alpha^*(t, x)) \\ &= 0.\end{aligned}$$

Then $V(t, x) = w(t, x)$ on $[0, T] \times \mathbb{R}$ and $\alpha^*(t, x)$ is an optimal Markovian control for the control problem given in Definition 3.3.

Proof. See [Pha09, p 47-48]. □

In this approach to stochastic control the usage of the DPP to derive the HJB equation is usually done in a formal, non-rigourous way. Given an HJB equation obtained by a formal derivation, a verification theorem is established (which can be done in order to prove that a (sufficiently smooth) solution to the HJB equation indeed is the value function of the associated control problem. Once a verification theorem has been established, it remains to find a solution to the HJB equation. In some cases a solution can be found using a clever guess (ansatz), but in many cases solutions have to be found numerically (if one has theorems at hand which establishes existence of solutions).

Remark. In this section we have outlined the HJB/verification approach for solving stochastic control problems in the case of a diffusion process and for a finite time horizon. The approach can be generalized in several ways. One way is to allow for an indefinite time-horizon (where the process is stopped at a random stopping time) or an infinite time horizon. Another generalization is to extend the approach beyond pure diffusion processes to processes involving jumps, in which case the HJB equation will have an integral part and the verification theorems will be slightly more involved.

If the value function is not smooth enough, it is possible to relax the smoothness requirements by introducing the concept of *viscosity solutions* to PDEs. This approach is described in [Pha09, ch 4], but will not be used in this thesis.

Summing up this subsection, we outline the procedure for solving stochastic control problems using the Hamilton-Jacobi-Bellman approach with verification theorems for sufficiently smooth value functions.

- Set up the stochastic control problem.
- Derive (formally) its associated Hamilton-Jacobi-Bellman equation.
- Prove a verification theorem - that a given solution to the Hamilton-Jacobi-Bellman equation is the value function of the control problem, thereby obtaining an optimal control.
- Obtain (or show existence of) a (sufficiently smooth) solution to the Hamilton-Jacobi-Bellman equation (which then coincides with the value function of the control problem).

This is the approach that will be used for attacking the control problems in this thesis.

3.3 Example: Merton's Problem of Optimal Consumption

One of the pioneering applications of stochastic control in finance is due to Merton. In a series of seminal papers, [Mer69] and [Mer71], Merton set up and solved a so-called investment-consumption problem, where the control problem involves maximising the expected utility of consumption of an agent. The agent can control the rate of consumption and the allocation of wealth into investments in different assets, where risky assets are modelled by a geometric Brownian motion. This problem has been studied in detail and generalized in several directions since. We will solve a basic version of this problem, in order to illustrate the procedure outlined in the previous subsection. It will also be the starting point and reference for the main problem discussed in section 4.

The presentation of Merton's problem of optimal investment and consumption in the current subsection draws on [Sch08, sec 3.1, p 114-120] and [AS20, ch XII, p 400-408]. In this formulation, we will consider consumption over a finite time horizon $[0, T]$ with no terminal utility. Investments can be allocated to a risk-free asset and a risky asset whose dynamics are given by a geometric Brownian motion.

Definition 3.5 (Controlled wealth process - Merton problem)**.** Consider a filtered probability space with the filtration generated by the Brownian motion W. Denote the controlled wealth process by $X^{\pi,c}$. The wealth can be invested in a market with the following price dynamics

$$
\begin{aligned}
dB_t &= rB_t dt,\ B(0) = 1 \\
dS_t &= mS_t dt + \sigma S_t dW_t,\ S(0) = s_0.
\end{aligned}
$$

The agent invests through the portfolio process $\pi = (\pi_t)_{t\in\mathbb{T}}$, which is the proportion of wealth X invested in the risky asset. The agent consumes according to the consumption rate process $c = (c_t)_{t\in\mathbb{T}}$. The control processes $(\{\pi_t, c_t\})_{t\in\mathbb{T}}$ are chosen to be admissible in the sense of Definition 3.2. Moreover, $c_t \geq 0$.

With the state of the portfolio process given by π_t, the agent will at time t hold a portfolio of $\frac{\pi_t X_t}{S_t}$ of the risky asset and $\frac{(1-\pi_t)X_t}{B_t}$ of the risk-free asset (as the price processes are given by S and B respectively). The dynamics of the value of the portfolio is then given by $\frac{\pi_t X_t}{S_t} dS_t + \frac{(1-\pi_t)X_t}{B_t} dB_t$.

Hence, the dynamics of the full controlled wealth process is given by

$$
\begin{aligned}
dX_t^{\pi,c} &= \pi_t X_t^{\pi,c} \frac{dS_t}{S_t} + (1-\pi_t) X_t^{\pi,c} \frac{dB_t}{B_t} - c_t dt \\
&= [(1-\pi_t)r + m\pi_t]\, X_t^{\pi,c} dt + \sigma \pi_t X_t^{\pi,c} dW_t - c_t dt. \qquad (3.4)
\end{aligned}
$$

The problem consists of deciding on a control process $(\{\pi_t, c_t\})_{t\in\mathbb{T}}$ which maximises the expected utility of consumption over $[0,T]$, where the preferences of the agent are assumed to be represented by a class of utility functions which we define next.

Definition 3.6 (Utility function - Merton problem)**.** We say that $\phi(c,t) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, which is differentiable in c and t is a utility function if it fulfills the following conditions:

- ϕ is increasing and strictly concave in c.
- ϕ is non-increasing in t.
- $\phi(0,t) = 0$.

The first condition implies that the agent is risk-averse, so that it gains less marginal utility from consumption for larger values of c. The second condition, also common in economics, rules out that consumption in the future is more valuable than consumption today. The last condition is a normalization factor.

Definition 3.7 (Optimal consumption - finite horizon)**.** Let $\phi(c,t) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be a utility function. Let $X^{\pi,c}$ be a controlled wealth process as in Definition 3.5. The control problem is given by

$$V(t,x) = \sup_{c,\pi} \mathbb{E}\left[\int_t^{T\wedge\tau} \phi(c(s),s)ds\right]$$

where $c(t)$ is the consumption rate of the controlled wealth process and τ is a stopping time determining the time of bankruptcy:

$$\tau = \inf\{s > 0 : X^{\pi,c} \leq 0\}.$$

Remark. The bankruptcy condition prevents gaining unbounded utility from unbounded borrowing. Note that the absence of any final utility also implies the boundary condition $V(T,x) = 0$. As in [Sch08, p 115], we will in the following simplify the notation somewhat by omitting the stopping time and instead take as given that $c(t) = 0$ whenever $t \geq \tau$.

The next step in the procedure outlined in the previous subsection is to formally derive the Hamilton-Jacobi-Bellman equation associated to the control problem, by using the dynamic programming principle. Note that we in this formal derivation assume that $V \in \mathscr{C}^{1,2}([0,T]\times\mathbb{R})$. The differential operator (cf. Definition 3.4) for the controlled wealth process given in Equation 3.4 with constant controls is given by

$$\mathcal{L}^{c,\pi} f(t,x) = ([(1-\pi)r + m\pi]\, x - c)\,\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\pi^2x^2\frac{\partial^2 f}{\partial x^2}$$

Proceeding as in the previous subsection, applying Itô's lemma on $[t, t+h]$ where $t \in (0, T)$, we get that

$$V(t+h, X_{t+h}^{c,\pi}) = V(t,x) + \int_t^{t+h} V_t(s, X_s^{c,\pi}) + \mathcal{L}^{c,\pi} V(s, X_s^{c,\pi}) ds + \int_t^{t+h} \sigma \pi_s X_s V_x(s, X_s^{c,\pi}) dW_s.$$

If now $\pi_s X_s$ is bounded, then the integrand in the stochastic integral above satisfies growth conditions that assures it belongs to $L_2(W)$. The stochastic integral is then a square integrable martingale which, in view of Corollary 2.5.1 has expectation 0. Using equation (3.2) (a consequence of the dynamic programming principle, Proposition 3.1), we find that

$$\mathbb{E}\left[\int_t^{t+h} \phi(c(s), s) + V_s(s, X_s^{c,\pi}) + \mathcal{L}^{c,\pi} V(s, X_s^{c,\pi}) ds\right] \leq 0.$$

By dividing by h and (formally) passing to the limit by letting $h \to 0$ we have found the Hamilton-Jacobi-Bellman equation associated to the control problem:

$$V_t(t,x) + \sup_{c,\pi} \left[\mathcal{L}^{c,\pi} V(t,x) + \phi(c,t)\right] = 0. \tag{3.5}$$

Before proving the verification theorem for the problem, we need a few lemmata. The second lemma essentially states that we can find optimal controls by optimising π and c pointwise on $[0,T] \times \mathbb{R}^+$.

Lemma 3.3. *V(t,x) is strictly increasing, concave in x and V(t,0) = 0*

Proof. Omitted, see [Sch08, Lemma 3.1, p 115]. □

Lemma 3.4. *Assume $V(t,x)$ is a solution to the Hamilton-Jacobi-Bellman equation* (3.5) *such that $V(t,x) \in \mathscr{C}^{1,2}([0,T) \times \mathbb{R})$ and continuous on $T \times \mathbb{R}$, which is increasing and stricly concave in x. The supremum for π is given by*

$$\pi^*(t,x) = -\frac{(m-r)V_x(t,x)}{\sigma^2 x V_{xx}(t,x)}$$

and there exists $c^*(t,x)$ where the supremum is attained.

Proof. Since V is increasing and strictly concave in x, we have that $V_{xx} < 0$ for almost all x, t. Let $H(t, x, V_x, V_{xx}, c, \pi) = \mathcal{L}^{c,\pi} V(t,x) + \phi(c(t,x), t)$ from (3.5). By differentiating H with respect to π we obtain a first-order condition for a local extremum:

$$\frac{\partial H}{\partial \pi} = (m-r) x V_x + \pi \sigma^2 x^2 V_{xx} = 0$$

which yields the local extremum

$$\pi^* = -\frac{(m-r)V_x(t,x)}{\sigma^2 x V_{xx}(t,x)}$$

Since the second derivative of H is negative, the local extremum is indeed a local maximum which is global since H is quadratic in π.

Since ϕ is strictly concave in c, we have that $H(t,x,V_x,V_{xx},c,\pi)$ is strictly concave in c. Hence, it has a global maximum for which

$$\frac{\partial H}{\partial c} = \phi_c(c(t,x),t) - V_x(t,x) = 0$$

so that $c^*(t,x)$ solves the equation $\phi_c(c^*(t,x),t) = V_x(t,x)$. □

The next step is then to prove a verification theorem - that a given smooth solution to the Hamilton-Jacobi-Bellman equation (3.5) coincides with the value function and that the optimal control is given by the functions given in Theorem 3.4. The formulation (and elements of the proof sketch) of the verification theorem follows [Sch08, p 117-118]. Some steps of the proof sketch also use ideas from [Pha09, p 47-48]. As in the formal derivations, it involves applications of Itô's lemma. We also indicate why the stochastic integrals involved are in fact martingales.

Theorem 3.5 (Verification theorem - Merton problem finite horizon)**.** *Assume that there exists a solution $f(t,x)$ to the Hamilton-Jacobi-Bellman equation* (3.5) *such that $f(t,x) \in \mathscr{C}^{1,2}([0,T]\times\mathbb{R})$ and increasing in x with boundary conditions $f(T,x)=0$. Then $V(t,x) \le f(t,x)$. If, in addition, $\pi^*(t,x)$ is given by*

$$\pi^*(t,x) = -\frac{(m-r)V_x(t,x)}{\sigma^2 x V_{xx}(t,x)}$$

and is bounded and $f(t,0)=0$, then $V(t,x)=f(t,x)$ and an optimal strategy is given by $\{\pi^*(t,X_t^*), c^*(t,X_t^*)\}$ where $c^*(t,x)$ solves the equation $\phi_c(c^*(t,x),t) = V_x(t,x)$.

Proof sketch. Taking $\pi = c = 0$, the fact that f is a solution to (3.5) implies $f_t(t,x) \le 0$. Together with the boundary condition $f(T,x)=0$ this yields that $f(t,x) \ge 0$.

We consider an arbitrary strategy $\{\pi, c\}$ which is admissible.

By Itô's lemma we have for any stopping time τ that

$$f(s\wedge\tau, X_{s\wedge\tau}^{c,\pi}) = f(t,x) + \int_t^{s\wedge\tau} f_t(v, X_v^{c,\pi}) + \mathcal{L}^{c,\pi} f(v, X_v^{c,\pi})dv + \int_t^{s\wedge\tau} \sigma\pi_v X_v^{c,\pi} f_x(v, X_v^{c,\pi})dW_v$$

In particular, we consider stopping times

$$\tau_n = \inf\{s > t : \int_t^s |\sigma\pi_v X_v^{c,\pi} f_x(v, X_v^{c,\pi})|^2 dv \geq n\}$$

which is a localisation sequence for $\int_t^s \sigma\pi_v X_v^{c,\pi} f_x(v, X_v^{c,\pi}) dW_v$. Since f is a solution to the HJB-equation (3.5), we get

$$f(s\wedge\tau_n, X_{s\wedge\tau_n}^{c,\pi}) + \int_t^{s\wedge\tau_n} \phi(v, c(v, X_v))dv \leq f(t,x) + \int_t^{s\wedge\tau_n} \sigma\pi_v X_v^{c,\pi} f_x(v, X_v^{c,\pi})dW_v.$$

We note that, for any n, the stopped process

$$(\int_t^{s\wedge\tau_n} \sigma\pi_v f_x(v, X_v^{c,\pi})dW_v)_{s\in[t,T]}^{\tau_n}$$

is a martingale (with expectation 0). Thus we have

$$\mathbb{E}\left(f(s\wedge\tau_n, X_{s\wedge\tau_n}^{c,\pi}) + \int_t^{s\wedge\tau_n} \phi(v, c(v, X_v))dv\right) \leq f(t,x).$$

Since $\int_t^{s\wedge\tau_n} \phi(v, c(v, X_v)dv = \int_t^s \phi(v, c(v, X_v))\mathbb{1}_{v\leq\tau_n} dv$ and $\{\phi(v, c(v, X_v))\mathbb{1}_{v\leq\tau_n}\}_n$ is an increasing sequence of non-negative functions, the monotone convergence theorem yields that

$$\lim_{n\to\infty} \int_t^{s\wedge\tau_n} \phi(v, c(v, X_v)dv = \int_t^s \phi(v, c(v, X_v)dv$$

so that, when taking $n \to \infty$, we can apply MCT to obtain

$$\mathbb{E}(f(s, X_s^{c,\pi}) + \mathbb{E}(\int_t^s \phi(v, c(v, X_v)dv) \leq f(t,x).$$

This holds for any $s \in [t,T]$, so we can take the limit as $s \to T$ to obtain (using the boundary condition $f(T,x) = 0$)

$$\mathbb{E}(\int_t^T \phi(v, c(v, X_v)dv) \leq f(t,x) \Rightarrow V(t,x) \leq f(t,x)$$

by considering the supremum of all admissible strategies $\{\pi, c\}$.

Assume now that $\{\pi^*, c^*\}$ is an optimal strategy and denote the (optimally) controlled process by X^*. By assumption, $\pi^*(t, X_t^*)$ is bounded. As in the first part, we apply Itô's lemma to obtain

$$f(s\wedge\tau_n, X_{s\wedge\tau_n}^*) = f(t,x) + \int_t^{s\wedge\tau_n} f_t(v, X_v^*) + \mathcal{L}^{c*,\pi*} f(v, X_v^*)dv + \int_t^{s\wedge\tau_n} \sigma\pi_v^* X_v^* f_x(v, X_v^*)dW_v$$

where we consider the stopping time

$$\tau_n = \inf\{s > t : \int_t^s |\sigma\pi_v X_v^{c,\pi} f_x(v, X_v^{c,\pi})|^2 dv \geq n\}.$$

Since $\{c_t^*, \pi_t^*\}$ attains the supremum in equation (3.5), we get that

$$f(s\wedge\tau_n, X_{s\wedge\tau_n}^*) = f(t,x) - \int_t^{s\wedge\tau_n} \phi(v, c^*(v, X_v^*))dv + \int_t^{s\wedge\tau_n} \sigma\pi_v^* X_v^* f_x(v, X_v^*)dW_v.$$

As in the first part of the proof sketch, the stochastic integral stopped at τ_n is a martingale, hence

$$\mathbb{E}(f(s\wedge\tau_n, X_{s\wedge\tau_n}^*)) = f(t,x) - \mathbb{E}(\int_t^{s\wedge\tau_n} \phi(v, c^*(v, X_v^*))dv).$$

Again, we use the monotone convergence theorem to obtain

$$\lim_{n\to\infty} \mathbb{E}(\int_t^{s\wedge\tau_n} \phi(v, c^*(v, X_v^*))dv) = \mathbb{E}(\int_t^s \phi(v, c^*(v, X_v^*))dv)$$

and

$$\lim_{n\to\infty} \mathbb{E}(f(s\wedge\tau_n, X_{s\wedge\tau_n}^*) = \mathbb{E}(f(s, X_s^*).$$

Now, by sending s to T we get that $\mathbb{E}\left(f(T, X_T^*) = 0\right.$ by the boundary condition. Hence

$$f(t,x) = \mathbb{E}(\int_t^T \phi(v, c^*(v, X_v^*))dv) = V(t,x).$$

□

The final step is to provide a suitable solution of the Hamilton-Jacobi-Bellman equation (3.5) satisfying the boundary conditions as in the verification theorem above, which then will yield the value function and the optimal investment and consumption processes (control processes).

For some classes of utility functions, it turns out that it is possible to find explicit solutions to the HJB equation by the use of an ansatz. We will consider such a class next. The utility functions that we consider are separable in time and consumption, a property that will be inherited by the solutions.

Proposition 3.6. *Assume* $\phi(c,t) = \exp(-\rho t)\frac{1}{1-\gamma}c^{1-\gamma}$ *where* $\gamma \in (0,1)$*. Then a solution to the Hamilton-Jacobi-Bellman equation* (3.5) *satisfying the conditions in Theorem 3.5 is given by*

$$V(t,x) = \frac{1}{1-\gamma} f(t)^\gamma x^{1-\gamma}$$

where

$$f(t) = \begin{cases} c_0 e^{-r^* t} + \frac{e^{-\frac{\rho}{\gamma}t}}{\frac{\rho}{\gamma} - r^*} & \text{if } r^* \neq \frac{\rho}{\gamma} \\ (t-T)e^{\frac{-\rho}{\gamma}t} & \text{if } r^* = \frac{\rho}{\gamma} \end{cases}$$

with $r^* = \frac{1-\gamma}{\gamma}(r + \frac{(m-r)^2}{2\gamma\sigma^2})$ *and* $c_0 = \frac{\exp(-T(\frac{\gamma}{\rho} - r^*))}{r^* - \frac{\rho}{\gamma}}$.

The optimal controls are given by

$$\pi^*(t,x) = \frac{m-r}{\gamma\sigma^2}, \ c^*(t,x) = \exp(-\frac{\rho}{\gamma}t)\frac{x}{f(t)}.$$

Proof. We will demonstrate the proposition by using an ansatz of the form given in the statement of the proposition. We consider a continuously differentiable function $f : [0,T] \to \mathbb{R}^+$ with $f(T) = 0$. Computing the relevant derivatives yield

$$\begin{aligned} V_t(t,x) &= \frac{\gamma}{1-\gamma}\left(\frac{x}{f(t)}\right)^{1-\gamma} \cdot f'(t), \\ V_x(t,x) &= \left(\frac{x}{f(t)}\right)^{-\gamma}, \\ V_{xx}(t,x) &= -\gamma\left(\frac{x}{f(t)}\right)^{-\gamma-1} \cdot \frac{1}{f(t)}. \end{aligned}$$

Considering the left hand side of the HJB equation (3.5), we notice that it is concave in c. Thus, there is a unique c^* where the supremum is attained which satisfies $\phi_c(c^*, t) = V_x(t,x)$. With $\phi_c(c^*, t) = (c^*)^{-\gamma}\exp(-\rho t)$ this yields

$$(c^*)^{-\gamma}\exp(-\rho t) = \left(\frac{x}{f(t)}\right)^{-\gamma} \Rightarrow c^*(t,x) = \exp(-\frac{\rho}{\gamma}t)\frac{x}{f(t)}.$$

From Lemma 3.4 we get

$$\begin{aligned} \pi^* &= -\frac{(m-r)V_x(t,x)}{\sigma^2 x V_{xx}(t,x)} = -\frac{(m-r)}{\sigma^2}\left(\frac{x}{f(t)}\right)^{-\gamma}\frac{1}{-\gamma}\left(\frac{x}{f(t)}\right)^{\gamma} \\ \Rightarrow \pi^*(t,x) &= \frac{m-r}{\gamma\sigma^2}. \end{aligned}$$

Here, we in particular notice that π^* does not depend on t or x. We can now plug in the expressions for the candidate optimal controls $\pi^*(t,x)$ and $c^*(t,x)$ and the derivatives of the ansatz solution $V(t,x)$ in the HJB equation (3.5). After some tedious calculations we notice that we can factor out $\left(\frac{x}{f(t)}\right)^{1-\gamma}$:

$$\left(\frac{x}{f(t)}\right)^{1-\gamma}\left(\frac{\gamma}{1-\gamma}f'(t) + (r + \frac{1}{2}\frac{(m-r)^2}{\gamma\sigma^2})f(t) + \frac{\gamma}{1-\gamma}e^{-\frac{\rho}{\gamma}t}\right) = 0.$$

Assuming that we can divide by $\left(\frac{x}{f(t)}\right)^{1-\gamma}$, we obtain the following ordinary differential equation in t:

$$\begin{aligned} f'(t) + \frac{1-\gamma}{\gamma}(r + \frac{1}{2}\frac{(m-r)^2}{\gamma\sigma^2})f(t) + e^{-\frac{\rho}{\gamma}t} &= 0 \\ \Rightarrow f'(t) + r^* f(t) + e^{-\frac{\rho}{\gamma}t} &= 0 \end{aligned}$$

with boundary condition $f(T) = 0$ where we have set $r^* = \frac{1-\gamma}{\gamma}(r + \frac{1}{2}\frac{(m-r)^2}{\gamma\sigma^2})$. This ODE has to be satisfied in order for the ansatz $V(t,x)$ to solve the HJB equation. It is a linear ODE with constant coefficients which has the solution

$$f(t) = c_0 e^{-r^* t} + \frac{e^{-\frac{\rho}{\gamma}t}}{\frac{\rho}{\gamma} - r^*}$$

with $c_0 = \frac{\exp(-T(\frac{\gamma}{\rho} - r^*))}{r^* - \frac{\rho}{\gamma}}$. This holds whenever $r^* \neq \frac{\rho}{\gamma}$. If $r^* = \frac{\rho}{\gamma}$ we instead have the solution

$$f(t) = (t - T)\exp^{\frac{-\rho}{\gamma}t}.$$

□

Remark. This type of utility function (power utility function) exhibits a property called constant relative risk aversion (CRRA) [AS20, ch I.2], which holds for any (twice differentiable) utility function u that satisfy

$$x\frac{-u_{xx}(x)}{u_x(x)} = C \text{ (constant)}.$$

We notice that the rate of consumption relative to current wealth $\frac{c^*(t,x)}{x}$ tends to infinity when $t \to T$. This seems reasonable and is due to the fact that in this particular setup of the optimal consumption problem, we have not included any utility from consumption at $t = T$. What is also striking in this problem is that π^*, the investment strategy, does not depend on x or t, but it does depend on properties of the risky asset (its return and volatility) and the risk aversion γ of the agent.

4 Optimal Allocation of Dividends in With-profit Insurance

In this section, we will consider the main problem of optimally allocating dividends to the insured in a with-profit life insurance. It will turn out that our particular formulation makes it a generalized version of the consumption-investment problem (Merton's problem) from the previous section. We will start by introducing necessary concepts from life insurance mathematics and define the relevant processes that relate to a with-profit life insurance policy. In this, we follow the presentation, and in most cases also the notation used, in [AS20, ch. V and ch. VI.4].

4.1 Life insurance - the with-profits contract

We consider a life insurance policy as a stream of payments (possibly including lump sum transfers) which are exchanged between the insured (or other beneficiaries of the policy) and the issuer (insurance company) depending on a finite number of states $\mathcal{J} = \{0, \dots, J\}$. The states are typically depending on the life and/or health status of one or more beneficiaries. The contract is issued at time 0 and runs until time T[9]. The state evolution is taken to be an inhomogeneous continuous-time Markov chain $Z = (Z_t)_{t\in[0,T]}$ defined on a filtered probability space $(\Omega, \mathcal{F}, P)$ where the filtration $(\mathcal{F}_t^Z)_{t\in[0,T]}$ is generated by the process. As described in Definition 2.13, its distribution is determined by its transition intensities $\mu_{ij}(t)$, which are given by

$$P\left(Z(t+h) = j | Z(t) = i\right) = \mu_{ij}(t)h + o(h).$$

The associated counting process

$$N^j(t) = \#\{s \in (0,t] : Z(s^-) \neq j, Z(s) = j\} \tag{4.1}$$

counts the number of jumps to state j until time t.

Definition 4.1 (Benefit process)**.** By the *Benefit process* we mean the $(\mathcal{F}_t^Z)_{t\in[0,T]}$-adapted process B which gives the total net benefits (benefits minus premiums) to be paid out during the course of the contract. The state $B(t)$ gives total net benefits paid until time t. Its dynamics is given by

$$dB(t) = b^{Z(t)}(t)dt + \Delta B^{Z(t)}(t) + \sum_{k\neq Z(t-)} b^{Z(t-)k}(t)dN^k(t)$$

meaning that it includes (continuous) payment streams ($b^j(t)$) as well as lump sum payments due either at deterministic times ($\Delta B^j(t)$) or at jumps between states ($b^{jk}(t)$) .

Example 4.1 (Survival model). A canonical and simple model is the survival model with two states $\mathcal{J} = \{0, 1\}$ corresponding to $\{alive, dead\}$.

Typical insurance contracts in this model include

- *pure endowment* that pays 1 unit upon survival until time m, which have all coefficients equal to zero except $\Delta B^0(m) = 1$.
- *term insurance* that pays 1 unit upon the time of death if death occurs before T, which have all coefficients equal to zero except $b^{01}(t) = \mathbb{1}_{t<T}$.

[9] We consider $T < \infty$ but it is possible in practice to consider lifelong contracts in this model by choosing T large enough.

- *deferred annuity* that pays 1 unit per time from the time of withdrawal m until the time of death or contract termination, which have all coefficients equal to zero except $b^0(t) = \mathbb{1}_{m<t<T}$.

Given a certain benefit process and an associated Markov process, it is also fundamental to consider how to valuate the payment stream given by the benefit process. This concept is called the *reserve.* For our purposes, valuation will be done using a deterministic interest rate $r(t)$.

Definition 4.2 (Reserve)**.** Let r be a deterministic interest rate process and Z be an inhomogeneous Markov process with state space $\mathcal{J} = \{0, \ldots, J\}$ and transition intensities $\mu_{ij}(t)$ determining the state of the policy.

The *reserve* associated to the insurance policy is given by

$$\begin{aligned} R(t) &= \mathbb{E}\left[\int_t^T exp(-\int_t^s r(\tau)d\tau)dB(s) \,\Big|\, \mathcal{F}_t^Z\right] \\ &= \mathbb{E}\left[\int_t^T exp(-\int_t^s r(\tau)d\tau)dB(s)|Z(t)\right], \end{aligned} \tag{4.2}$$

the conditional expected present value of the payment stream using r as discount rate. We also write $R^{Z(t)}(t)$ for the reserve function in the state $Z(t)$.

A well-known relation in life insurance mathematics is that the dynamics of the reserve follow the so-called *Thiele differential equation.*

Proposition 4.1 (Thiele differential equation)**.** *Let $R(t)$ be as in equation (4.2). Then $R^j(t)$ has the following dynamics at points t such that $\Delta B^j(t) = 0$:*

$$\frac{dR^j(t)}{dt} = r(t)R^j(t) - b^j(t) - \sum_{i:i\neq j} \mu_{ji}(t)\left(b^{ji}(t) + R^i(t) - R^j(t)\right) \tag{4.3}$$

$$R^j(T) = 0$$

At points where $\Delta B^j(t) \neq 0$ we have $R^j(t-) = R^j + \Delta B^j(t)$. The term $\Psi^{ji}(t) = b^{ji}(t) + R^i(t) - R^j(t)$ is called sum at risk.

Proof. See [AS20, p 128-129] for the standard proof using Kolmogorov's backward equations and [AS20, p 130-132] for a proof using martingale methods. □

As mentioned in [AS20, p 130], this gives an interpretation to the reserves as evolving as a (policy) account from which payments $b^j(t)$ are paid out, interest $r(t)R^j(t)$ is accrued and the risk premia $\sum_{i:i\neq j}\mu_{ji}(t)\Psi^{ji}(t)$ is debited the account (or credited in case the sum at risk is negative). The risk premium

$\mu_{jk}(t)\Psi^{jk}(t)$ should cover the cost of the transition payments $b_{jk}(t)$ as well as the net cost of setting up the reserve in state k.

The reserve at the start of a contract with a given benefit process can be used to determine its premium. In the case of the with-profit insurance contract, the insurance company specifies a set $(\overline{r}(t), \overline{\mu_{ij}(t)})$ of interest rates and transition intensities, called *technical basis*, under which benefits and premiums are calculated such that they satisfy

$$R^*(0-) = \mathbb{E}\left[\int_{0-}^{T} e^{-\int_0^t \overline{r}} \overline{dB(t)} \;\Big|\; Z_0\right] = 0, \tag{4.4}$$

the so-called *equivalence principle*. Here,

$$\overline{dB(t)} = b^{Z(t)}(t)dt + \Delta B^{Z(t)}(t) + \sum_{k \neq Z(t-)} b^{Z(t-)k}(t) dN^k(t)$$

is the benefit process of the contract where the underlying Markov chain Z is defined by the transition intensities $\overline{\mu_{ij}(t)}$ of the technical basis.

The payments specified by the benefit process determined by the technical basis are usually denoted *first-order payments* [AS20, p 166]. The technical basis is set in a *prudent* manner, which means that $(\overline{r}(t), \overline{\mu_{ij}(t)})$ should be chosen so that the reserve evaluated using best estimates on interest rates and and transition intensities is smaller than when using the technical basis.

The difference that hereby arises determines a free reserve, which, after taking into account possible capital requirements on the side of the insurance company, should be returned to the policy holders as dividends. This free reserve, or contract surplus, will be invested in some kind of asset market. Thus, during the duration of the contract, the insurance company has to decide on a *dividend strategy* and an *investment strategy* (given an asset market). The dividend strategy can be specified in terms of a dividend process.

Definition 4.3 (Dividend process)**.** The *dividend process* D gives the total dividends that are to be paid out (in addition to the payments specified by the benefit process) during the course of the contract. The state $D(t)$ gives total dividends paid out until time t. Its dynamics is given by

$$dD(t) = \delta^{Z(t)}(t)dt + \Delta D^{Z(t)}(t) + \sum_{k \neq Z(t-)} \delta^{Z(t-)k}(t) dN^k(t) \tag{4.5}$$

meaning that it includes (continuous) payment streams $(\delta^j(t))$ as well as lump sum payments due either at deterministic times $(\Delta D^j(t))$ or at jumps between states $(\delta^{jk}(t))$. Here N^j is a counting process as in equation (4.1). The dividend process is adapted to the filtration $\mathcal{F}_t = (\mathcal{F}^{Z_t} \bigvee \mathcal{F}^{S_t})$ where, for each t, $\mathcal{F}^{S_t}$ is the σ-algebra generated by the asset market process. Note that in this case the intensities specifying the Markov process Z_t is the objective measure (and not the technical basis).

Thus, the total payments which the policy holders receive will be given by $B + D$. Characterising for the with-profit contract is that dividends can never be negative, i.e. D is positive and increasing, and we then speak of the first-order payment process B as *guaranteed*, since the total payments will always be greater than or equal to B.

4.2 Optimal dividends in a with-profit life insurance as an extension of Merton's Problem

The main problem of this thesis, which will be considered in this subsection, is to determine dividend and investment strategies for a with-profit insurance policy using the stochastic optimal control methods described in previous parts of the thesis.

We will make a number of simplifying assumptions (see also [Ste04, p 9-10] and [Sch08, p 127]):

- The guaranteed payments B will be perfectly hedged (outside of the model of the free reserve) so that no gains nor losses stemming from covering these payments affect the free reserve.
- Premium is paid as a one-time premium (lump sum) at $t = 0-$. This implies in particular for the dividend process that $\delta^{Z(t)}(t) \geq 0$ and $\delta^{Z(t-)k}(t) \geq 0$.
- $\Delta D^{Z(t)}(t) = 0$ for $t \in (0, T)$, no lump sum payments apart from the premium, termination or at transitions between states.

This problem was initially put forward in [Ste04]. The presentation here draws on that paper and that in [Sch08, p 127 - 132]. The state of the policy is given by an inhomogeneous Markov process Z with state space $\mathcal{J} = \{0, \ldots, J\}$ and has associated counting processes $N^k(t)$ as in Subsection 4.1. The free reserves can be invested in an asset market which is the same as for the Merton problem in Subsection 3.3 (one risk-less asset and one risky asset whose price process is given by a geometric Brownian motion).

Given a with-profit life insurance policy with a free reserve, we consider a dividend process $D(t)$ which is specified by the collection of processes $(\delta^i(t))_{t\in\mathbb{T}}$,

$(\delta^{ij}(t))_{t\in\mathbb{T}}$ where $i, j \in \mathcal{J}$ and random variables $\Delta D^i(T)$ (as in Equation 4.5), and an investment process $(\pi_t)_{t\in\mathbb{T}}$, which as in Definition 3.5 denotes the portfolio weight of the risky asset. These processes will be the control processes and the free reserve as a controlled process will be denoted, following [Sch08, p 127] the surplus process.

Definition 4.4 (Controlled Surplus process)**.** Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\{\mathcal{F}_t\}_{t\geq 0}$ a filtration satisfying the usual conditions, generated by the processes $\{(Z, W)\}$ defined on the probability space. Here, Z is an inhomogeneous Markov process representing the state of the policy and W is a Brownian motion. By X we denote the surplus process, whose value, for each $t \in [0, T]$, can be invested in a market with the following price dynamics

$$
\begin{aligned}
dB_t &= rB_t dt,\ B(0) = 1 \\
dS_t &= mS_t dt + \sigma S_t dW_t,\ S(0) = s_0
\end{aligned}
$$

where r, m, σ are positive constants satisfying $m - r > 0$ and from which dividends can be paid out according to the processes $(\delta^i(t))_{t\in\mathbb{T}}, (\delta^{ij}(t))_{t\in\mathbb{T}}$ for $i, j \in \mathcal{J}$ [10]. Let $(\pi_t)_{t\in\mathbb{T}}$ be the process denoting the proportion invested in the risky asset.

The dynamics of the controlled surplus process will then be given by (cf. Equation 3.4):

$$
\begin{aligned}
dX_t^{\pi,\delta} = {}& [(1-\pi_t)r + m\pi_t]\, X_t^{\pi,\delta} dt + \sigma\pi_t X_t^{\pi,\delta} dW_t \\
& -\delta(t)^{Z(t)} dt - \delta^{Z(t-)Z(t)}(t) dN^{Z(t)}(t).
\end{aligned} \tag{4.6}
$$

Here, $N^j(t)$ is a counting process as in equation (4.1).

We will consider control strategies $(\delta^i(t))_{t\in\mathbb{T}}, (\delta^{ij}(t))_{t\in\mathbb{T}}, (\pi_t)_{t\in T}$ that are admissible in the following sense (cf. [Sch08, p 127]):

- They are adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ of Definition 4.4.
- The stochastic differential equation (4.6) is well defined and admits a unique solution.
- $\delta^i(t), \delta^{ij}(t) \geq 0$ for all i, j, t and $X_{T-} = \Delta D^{Z(T)}(T) \geq 0$.

The last condition ensures that dividends are non-negative and that all remaining surplus are paid out at contract termination.

Comparing the current setup to that of the Merton problem discussed in the previous section, we note that in place of the consumption process $(c_t)_{t\in\mathbb{T}}$ in

[10] We will often supress explicit mention of the state space and assume that i, j runs over $\mathcal{J}$ when referring to dividend strategies.

Definition 3.5, we have dividend processes $(\delta^i(t)), (\delta^{ij}(t))$ and that we have introduced jumps corresponding to changes in the state of the policy. The control problem can be formulated by introducing utility functions which are similar to those defined in Definition 3.6, but are also state dependent (cf. [Sch08, p 128]).

Definition 4.5. The functions $\phi^c(i,d,t) : \mathcal{J} \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ and $\phi^l(i,j,d,t)$: $\mathcal{J} \times \mathcal{J} \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ which are differentiable in d and t represent the utility that the policy holder experiences from the dividend payments. The function $\phi^T(i,D) : \mathcal{J} \times \mathbb{R}^+ \to \mathbb{R}^+$ represent the utility gained from the terminal payment at T. As utility functions they satisfy

- ϕ is increasing and strictly concave in d.
- $\lim_{d\to\infty} \frac{\partial \phi}{\partial d} = 0$.
- $\phi^c(i,0,t) = \phi^l(i,j,0,t) = \phi^T(i,0) = 0$.

Note that ϕ^c represents the utility gained from a dividend rate whereas ϕ^l and ϕ^T represent utilities gained from lump sum payments. In this approach we assume that the utilities of rates and lump sums can be added so as to represent the total utility of the policy holder. See [Sch08, Remark 3.9, p 128].

We are now finally able to formulate the stochastic control problem that maximizes the utility of the dividend payments (cf. [Sch08, p 128]).

Definition 4.6 (Optimal dividend allocation)**.** Let $\phi^c(i,d,t), \phi^l(i,j,d,t)$ and $\phi^T(i,D)$ be functions as in Definition 4.5 representing the utility from dividend payments. Let X be a controlled surplus process as in Definition 4.4. The gain function of an investment-dividend strategy $(\delta^i_t), (\delta^{ij}_t), (\pi_t)$ is given by

$$\begin{aligned} J(t,i,x,\delta,\pi) = \mathbb{E}\Bigg[\int_t^T & \phi^c(Z(s),\delta^{Z(s)},s)ds \\ &+ \int_t^T \phi^l(Z(s-),Z(s),\delta^{Z(s-)Z(s)},s)dN^{Z(t)}(t) \\ &+ \phi^T(Z(T),\Delta D^{Z(T)}(T)) \,\Bigg|\, Z(t)=i, X_t = x\Bigg] \end{aligned}$$

and the value function

$$V(t,i,x) = \sup_{\delta,\pi} J(t,i,x,\delta,\pi)$$

where supremum is taken over admissible strategies.

In order to perform the formal derivation of the Hamilton-Jacobi-Bellman of the control problem, we proceed in similar fashion as in Subsection 3.3, and assume that V is smooth enough and that stochastic integrals exist and are zero mean martingales. For simplicity of notation we will drop the superscripts for the controlled wealth process X and suppress the dependence of δ and π on X, Z. We will also sometimes use the compressed notation $\mathbb{E}_{t,i,x}$ for denoting conditional expectation given the state $Z(t) = i, X(t) = x$. Applying Itô's lemma (Theorem 2.11):

$$
\begin{aligned}
V(t+h, Z(t+h), X(t+h)) &= V(t,i,x) + \int_t^{t+h} V_s(s, Z(s), X(s))ds \\
&+ \int_t^{t+h} V_x(s, Z(s), X(s))dX(s) + \frac{1}{2}\int_t^{t+h} V_{xx}(s, Z(s), X(s))d\langle X(s), X(s)\rangle \\
&+ \sum_{t<s\leq t+h} [V(s, Z(s), X(s)) - V(s-, Z(s-), X(s-))] \\
&= V(t,i,x) + \int_t^{t+h} V_s(s, Z(s), X(s))ds + \\
&\int_t^{t+h} \Big([(1-\pi_s)r + m\pi_s] X(s) - \delta^{Z(s)} \Big) V_x(s, Z(s), X(s))ds \\
&+ \int_t^{t+h} \frac{1}{2}\sigma^2\pi(s)^2 X(s)^2 V_{xx}(s, Z(s), X(s))ds + \int_t^{t+h} \sigma\pi_s V_x(s, Z(s), X(s))dW(s) \\
&+ \sum_{t<s\leq t+h; Z(s-)\neq Z(s)} [V(s, Z(s), X(s)) - V(s-, Z(s-), X(s-))] .
\end{aligned}
$$

Applying the dynamic programming principle, Proposition 3.1, with constant controls, we get that

$$
\begin{aligned}
V(t,i,x) \geq \mathbb{E}\Big[&\int_t^{t+h} \phi^c(Z(s), \delta^{Z(s)}, s)ds \\
&+ \int_t^{t+h} \phi^l(Z(s-), Z(s), \delta^{Z(s-)Z(s)}, s)dN^{Z(t)}(t) \\
&+ V(t+h, Z(t+h), X(t+h)) \;\Big|\; Z(t) = i, X(t) = x \Big].
\end{aligned}
$$

Combining the previous two equations yields

$$
\begin{aligned}
0 \geq \mathbb{E}_{t,i,x}\Bigg[&\int_t^{t+h} \phi^c(Z(s), \delta^{Z(s)}, s)ds \\
&+ \int_t^{t+h} \phi^l(Z(s-), Z(s), \delta^{Z(s-)Z(s)}, s)dN^{Z(t)}(t) \\
&+ \int_t^{t+h} \Big(\left[(1-\pi_s)r + m\pi_s\right] X(s) - \delta^{Z(s)} \Big) V_x(s, Z(s), X(s))ds \\
&+ \int_t^{t+h} V_s(s, Z(s), X(s)) + \frac{1}{2}\sigma^2\pi_s^2 X(s)^2 V_{xx}(s, Z(s), X(s))ds \\
&+ \sum_{t<s\leq t+h; Z(s-)\neq Z(s)} \left[V(s, Z(s), X(s)) - V(s-, Z(s-), X(s-)\right] \\
&+ \int_t^{t+h} \sigma\pi_s V_x(s, Z(s), X(s))dW(s)\Bigg].
\end{aligned}
$$

In this formal derivation, we assume that the stochastic integral of the last line is a zero mean martingale. Comparing to the previous HJB-equations, the above expression also contains a jump part related to the Markov process determining the state of the policy. Using the properties of the transition intensities defining the Markov process and the assumption that $\delta^{ii} = 0$, the expectation of the jump part becomes

$$
\begin{aligned}
\mathbb{E}_{t,i,x}\Bigg(&\int_t^{t+h} \phi^l(Z(s-), Z(s), \delta^{Z(s-)Z(s)}, s)dN^{Z(t)}(t) \\
&+ \sum_{t<s\leq t+h; Z(s-)\neq Z(s)} \left[V(s, Z(s), X(s)) - V(s-, Z(s-), X(s-))\right]\Bigg) = \\
&\sum_{j=0}^{J} \phi^l(i, j, \delta^{ij}, t) \cdot \mu_{ij}(t)h + o(h) + \\
&\sum_{j=0}^{J} \left[V(t, j, x-\delta^{ij}) - V(t, i, x)\right] \cdot \mu_{ij}(t)h + o(h).
\end{aligned}
$$

In order to arrive at the HJB-equation, we divide by h and consider the limit as $h \to 0$, which means that the above expression becomes

$$
\sum_{j=0}^{J} \mu_{ij}(t) \left[V(t, j, x-\delta^{ij}) - V(t, i, x) + \phi^l(i, j, \delta^{ij}, t)\right]
$$

which, together with the diffusion part, yields the following HJB-equation (cf.

[Sch08, eq 3.8, p 128]):

$$\sup_{\pi,\delta}\Bigg\{\phi^c(i,\delta^c,t) + \left(\left[(1-\pi)r + m\pi\right]x - \delta^c\right)V_x(t,i,x) + \frac{1}{2}\sigma^2\pi^2x^2V_{xx}(t,i,x) + V_t(t,i,x) + \sum_{j=0}^{J}\mu_{ij}(t)\left[V(t,j,x-\delta^l) - V(t,i,x) + \phi^l(i,j,\delta^l,t)\right]\Bigg\} = 0, \quad (4.7)$$

where we have used the notation δ^c and δ^l to distinguish between dividend rates and dividend lump sums paid at transitions between states (corresponding to the control processes $(\delta^i(t))$ and $(\delta^{ij}(t))$).

As in the previous section, the value function should satisfy some regularity conditions, in particular that it is twice continuously differentiable in x and continuously differentiable in t. The following property also holds for the value function (cf. [Sch08, p 129]).

Lemma 4.2. *V(t,i,x) is increasing and concave in x and satisfies the boundary conditions $V(t,i,0) = 0$.*

Proof. The proof is omitted but is similar to that of Lemma 3.3. □

Moreover, at the termination of the contract, we require the value function to coincide with the terminal utility, which leads to the boundary condition $V(T,i,x) = \phi^T(i,x)$.

Lemma 4.3. *Assume $V(t,i,x)$ is a solution to the Hamilton-Jacobi-Bellman equation (Equation 4.7). The supremum for π is given by*

$$\pi^*(t,i,x) = -\frac{(m-r)V_x(t,i,x)}{\sigma^2 x V_{xx}(t,i,x)}.$$

The optimal strategy for dividends is given by the solutions to the equations

$$\phi^c_\delta(i,\delta^{c*}(t,i,x),t) = V_x(t,i,x)$$
$$\phi^l_\delta(i,j,\delta^{l*}(t,i,j,x),t) = V_x(t,j,x-\delta^{l*}(t,i,j,x))$$

provided that the solutions exist. If no solution $\delta^{c*}(t,i,x)$ exists, the optimal strategy is given by $\delta^{c*}(t,i,x) = 0$. If no solution $\delta^{l*}(t,i,j,x)$ exists, then $\delta^{l*}(t,i,j,x) = 0$.

Proof. The proof is similar to that of Lemma 3.4 - finding first order conditions for the right hand side of (4.7) which have to be zero in the case of a local extremum. The supremum is unique since (4.7) is strictly concave in π. □

The statement and proof of the verification theorem will resemble that of Theorem 3.5, but the presence of jumps complicate the derivations somewhat. The strategy is again to use Itô's lemma and then establish that a particular part of the process is a martingale which have expectation zero.

Theorem 4.4 (Verification theorem - Optimal Dividends in Life Insurance)**.** *Assume that there exists a solution $f(t,i,x)$ to the Hamilton-Jacobi-Bellman equation (4.7) such that $f(t,i,x)$ is increasing in x, is continuously differentiable in t and twice continuously differentiable in x. Assume moreover that it satisfies the boundary condition $f(T,i,x) = \phi^T(i,x)$. Then $V(t,i,x) \leq f(t,i,x)$. If, in addition, $\pi^*(t,x)$ is given by*

$$\pi^*(t,i,x) = -\frac{(m-r)V_x(t,i,x)}{\sigma^2 x V_{xx}(t,i,x)}$$

and is bounded and $f(t,i,0) = 0$, then $V(t,i,x) = f(t,i,x)$ and an optimal strategy is given by $(\pi^*(t,i,X_t^*)), (\delta^{c*}(t,i,X_t^*)), (\delta^{l*}(t,i,j,X_t^*))$ and $\Delta D^i(T,x) = x$. Here, $(\delta^{c*}(t,i,X_t^*))$ and $(\delta^{l*}(t,i,j,X_t^*))$ are as in Lemma 4.3.

The outline of the proof sketch follows [Sch08, Theorem 3.10, p 129-130] and the proof of Theorem 3.5. As in subsection 3.3 we introduce the following linear differential operator associated to the controlled wealth process:

$$\mathcal{L}^{\delta,\pi} f(t,i,x) = \left([(1-\pi_t)r + m\pi_t]\, x - \delta^c\right) \frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 \pi_t^2 x^2 \frac{\partial^2 f}{\partial x^2}.$$

and we let

$$\mathcal{A}^{\delta,\pi} f(t,i,x) = \frac{\partial f}{\partial t} + \mathcal{L}^{\delta,\pi} f(t,i,x).$$

Proof sketch. We start by noting that the conditions that f be increasing in x and that $f(t,i,0) = 0$ imply that f is non-negative and bounded from below.

For the first part, we consider an arbitrary strategy $\{\pi, \delta^c, \delta^l\}$ which is admissible. We fix $t \in [0,T)$ for which we have $f(t, Z(t), X(t)) = f(t,i,x)$ and consider the evolution up to $s \in (t,T]$. We let

$$H_s = \{v \in (t,s] : Z(v) \neq Z(v-)\} = \{\eta_1, ..., \eta_k\}$$

the set of jump times of Z in $(t,s]$ and assume that the set is finite. Between the jumps, we apply Itô's lemma as follows (where $j \in \{k, ...2\}$):

$$\begin{aligned} h_k &= f(s, Z(s), X(s)) - f(\eta_k, Z(\eta_k), X(\eta_k)) - \int_{\eta_k}^s \mathcal{A}^{\delta,\pi} f(v, Z_v, X_v)dv \\ &= \int_{\eta_k}^s \sigma \pi_v f_x(v, Z_v, X_v) dW_v \end{aligned}$$

$$
\begin{aligned}
h_{j-1} &= f(\eta_j, Z(\eta_j), X(\eta_j) - f(\eta_{j-1}, Z(\eta_{j-1}), X(\eta_{j-1})) - \int_{\eta_{j-1}}^{\eta_j} \mathcal{A}^{\delta,\pi} f(v, Z_v, X_v) dv \\
&= \int_{\eta_{j-1}}^{\eta_j} \sigma \pi_v f_x(v, Z_v, X_v) dW_v
\end{aligned}
$$

$$
\begin{aligned}
h_0 &= f(\eta_1, Z(\eta_1), X(\eta_1)) - f(t, Z(t), X(t)) - \int_t^{\eta_1} \mathcal{A}^{\delta,\pi} f(v, Z_v, X_v) dv \\
&= \int_t^{\eta_1} \sigma \pi_v f_x(v, Z_v, X_v) dW_v
\end{aligned}
$$

where we note that each term is a stochastic integral, which will be the subject of the following localization. Consider (as in the proof of Theorem 3.5) stopping times

$$
\tau_n = \inf\{u > t : \int_t^u |\sigma \pi_v f_x(v, Z(v), X(v))|^2 dv \geq n\}, \tag{4.8}
$$

where we note that $\tau_n \to \infty$ as $n \to \infty$. Consider the process which is the sum of the above terms

$$
\mathfrak{h}(s) = \sum_{i=0}^{k} h_i
$$

which is a local martingale, since

$$
(\mathfrak{h})^{\tau_n} = \sum_{i=0}^{k \wedge |H_{s \wedge \tau_n}|} h_i
$$

is a martingale.

For the jump part, we have, according to [Sch08, p 129], that the following compensated jump process is a martingale (with respect to the filtration of Definition 4.4):

$$
\begin{aligned}
\tilde{M}(s) = \sum_{\eta \in H_s} & f(\eta, Z(\eta), X(\eta)) + \phi^l(Z(\eta-), Z(\eta), \delta^l(\eta, Z(\eta-), Z(\eta), X(\eta)), \eta) \\
& - f(\eta, Z(\eta-), X(\eta-)) \\
& - \int_t^s \sum_{j \in \mathcal{J}} \mu_{Z(v)j}(v) \left[f(v, j, X_v - \delta^l(v, Z(v), j, X(v))) \right. \\
& \left. + \phi^l(Z(v), j, \delta^l(v, Z(v), j, X(v)), v) - f(v, i, X(v)) \right] dv
\end{aligned}
$$

Now, the sum

$$
\tilde{M}(s) + \sum_{i=0}^{k} h_i
$$

is a telescoping sum where the terms $f(\eta, Z(\eta), X(\eta))$ and $f(\eta, Z(\eta-), X(\eta-))$ cancel out so that we get

$$\begin{aligned}\tilde{M}(s) + \sum_{i=0}^{k} h_i = f(s, Z(s), X(s))& \\ &+ \sum_{\eta \in H_s} \phi^l(Z(\eta-), Z(\eta), \delta^l(\eta, Z(\eta-), Z(\eta), X(\eta)), \eta) \\ &- \int_t^s \big(\mathcal{A}^{\delta,\pi} f(v, Z(v), X(v)) \\ &+ \sum_{j \in \mathcal{J}} \mu_{Z(v)j}(v)[f(v, j, X_v - \delta^l(v, Z(v), j, X(v))) \\ &+\phi^l(Z(v), j, \delta^l(v, Z(v), j, X(v)), v) - f(v, i, X(v))]\big)\, dv.\end{aligned}$$

Moreover, the telescoping sum is a local martingale, whose stopped version, according to the localisation in (4.8), can be written as

$$\tilde{M}(s \wedge \tau_n) + \sum_{i=0}^{k \wedge |H_{s\wedge\tau_n}|} h_i.$$

Since f is a solution to the HJB-equation (4.7) we have, for any $u \in [0, T]$, $i, j \in \{0, ..., J\}$ and $y \in \mathbb{R}^+$,

$$\sup_{\pi,\delta} \Bigg\{ \phi^c(i, \delta^c, u) + \mathcal{A}^{\delta,\pi} f(u, i, y) + \sum_{j=0}^{J} \mu_{ij}(u) \left[f(u, j, y - \delta^l) - f(u, i, y) + \phi^l(i, j, \delta^l, u) \right] \Bigg\} = 0$$

which leads to the following inequality for the stopped processes

$$\begin{aligned}f(s\wedge\tau_n, Z(s\wedge\tau_n), X(s\wedge\tau_n)) + \sum_{\eta \in H_{s\wedge\tau_n}} \phi^l(Z(\eta-), Z(\eta), \delta^l(\eta, Z(\eta-), Z(\eta), X(\eta)), \eta) \\ + \int_t^{s\wedge\tau_n} \phi^c(Z(v), \delta^c(v, Z(v), X(v)), v) dv \geq \tilde{M}(s \wedge \tau_n) + \sum_{i=0}^{k \wedge |H_{s\wedge\tau_n}|} h_i.\end{aligned}$$

Hence, the left hand side of the equation above defines a supermartingale. Since the process is a supermartingale we have that

$$\begin{aligned}f(t, i, x) \geq \mathbb{E}\,(f(s \wedge \tau_n, Z(s \wedge \tau_n), X(s \wedge \tau_n))& \\ &+ \sum_{\eta \in H_{s\wedge\tau_n}} \phi^l(Z(\eta-), Z(\eta), \delta^l(\eta, Z(\eta-), Z(\eta), X(\eta)), \eta) \\ &+ \int_t^{s\wedge\tau_n} \phi^c(Z(v), \delta^c(v, Z(v), X(v)), v) dv.\end{aligned}$$

where we in particular note that each term is non-negative. We can then apply Fatou's lemma (Proposition 2.3) when letting $n \to \infty$ to obtain

$$
\begin{aligned}
&f(t,i,x) \geq \\
&\quad \mathbb{E}\Bigg(f(s,Z(s),X(s)) + \sum_{\eta \in H_s} \phi^l(Z(\eta-),Z(\eta),\delta^l(\eta,Z(\eta-),Z(\eta),X(\eta)),\eta) \\
&\qquad\qquad + \int_t^s \phi^c(Z(v),\delta^c(v,Z(v),X(v)),v)dv \Bigg).
\end{aligned}
$$

and, applying the boundary condition $f(T,i,x) = \phi^T(i,x)$, we get that

$$
\begin{aligned}
&f(t,i,x) \geq \\
&\quad \mathbb{E}\Bigg(\phi^T(J_T,X_T) + \sum_{\eta \in H_T} \phi^l(Z(\eta-),Z(\eta),\delta^l(\eta,Z(\eta-),Z(\eta),X(\eta)),\eta) \\
&\qquad\qquad + \int_t^T \phi^c(Z(v),\delta^c(v,Z(v),X(v)),v)dv \Bigg) \\
&\qquad\qquad\qquad = J(t,i,x,\pi,\delta).
\end{aligned}
$$

Taking supremum of the right hand side yields that $f(t,i,x) \geq V(t,i,x)$.

The second part amounts to showing that the process

$$
\begin{aligned}
&f(s,Z(s),X^*(s)) + \sum_{\eta \in H_s} \phi^l(Z(\eta-),Z(\eta),\delta^{l*}(\eta,Z(\eta-),Z(\eta),X(\eta)),\eta) \\
&\qquad\qquad + \int_t^s \phi^c(Z(v),\delta^{c*}(v,Z(v),X(v)),v)dv
\end{aligned}
$$

is a martingale whenever we have the optimal strategies

$$
(\pi^*(t,i,X_t^*)), (\delta^{c*}(t,i,X_t^*)), (\delta^{l*}(t,i,j,X_t^*))
$$

which attains the supremum of equation (4.7). For this part we refer to [Sch08, p 130].

□

In order to make further progress towards existence of solutions, it is necessary to introduce some structure to the utility functions. As in the previous section, a natural starting point is to consider utility functions with constant relative risk aversion $\gamma \in (0,1)$. Following [Ste04, p 12] and [Sch08, p 130-131] [11] we

[11] We have changed the notation slightly so as to make γ the coefficient of relative risk aversion, instead of $1-\gamma$. This is in line with [AS20, ch XII].

specify the preferences over time using a set of weight functions $a(t,i)$, $a(t,i,j)$ and $\Delta A(i)$ so that the utility functions become

$$\begin{aligned}\phi^c(t,i,x) &= \frac{1}{1-\gamma}x^{1-\gamma}a(t,i)^\gamma,\\ \phi^l(t,i,j,x) &= \frac{1}{1-\gamma}x^{1-\gamma}a(t,i,j)^\gamma,\\ \phi^T(i,x) &= \frac{1}{1-\gamma}x^{1-\gamma}A(i)^\gamma.\end{aligned} \tag{4.9}$$

The weight functions do not represent a dividend stream per se, but can be interpreted as stating preferences on the side of the policy holder on how the dividends should be distributed over time and over policy states. As mentioned in [Ste04, p 12], one way for the insurance provider to determine the weight functions $a(t,i)$, $a(t,i,j)$ and $\Delta A(i)$ would be to set

$$\begin{aligned}a(t,i) &= b^i(t)\\ a(t,i,j) &= b^{ij}(t)\\ \Delta A(i) &= \Delta B^i(t)\end{aligned}$$

where $b^i(t)$, $b^{ij}(t)$ and $\Delta B^i(t)$ are taken from the benefit process (Definition 4.1) that determine the guaranteed payments. But one could also imagine situations where policy holders would have preferences over time and policy states that diverge from that. For example, in the case of a life annuity the rate of guaranteed payments might be constant but policy holders might prefer the dividend rate to increase or decrease over time.

As in the previous section, we will consider an ansatz solution to the HJB-equation [12] separating x and t:

$$V(t,i,x) = \frac{1}{1-\gamma}g(t,i)^\gamma x^{1-\gamma}. \tag{4.10}$$

Here, $g(t,i)$ is continuously differentiable in t.

With this separation, the optimal strategies will have a simple form. In particular, the investment strategy is identical to that of the Merton problem in the previous section.

Proposition 4.5. *Consider the ansatz solution (Equation 4.10) to the HJB-equation of the control problem (Equation 4.7) where the utility functions are*

[12] This can also be considered as a system of $J+1$ equations if one chooses the representation $(V(t,x))_i, i=0,...,J$.

specified in equation (4.9)*. Then, candidates for the optimal strategies are*

$$\begin{aligned}
\pi^*(i,t,x) &= \frac{m-r}{\gamma\sigma^2} \\
\delta^{c*}(t,i,x) &= \frac{a(t,i)}{g(t,i)}x \\
\delta^{l*}(t,i,j,x) &= \frac{a(t,i,j)}{g(t,j)+a(t,i,j)}x
\end{aligned}$$

Proof. For the ansatz (4.10) and utility functions (4.9) we have

$$\begin{aligned}
V_x(t,i,x) &= g(t,i)^\gamma x^{-\gamma} \\
V_{xx}(t,i,x) &= -\gamma g(t,i)^\gamma x^{-\gamma-1} \\
\phi^c_\delta(t,i,\delta) &= a(t,i)^\gamma \delta^{-\gamma} \\
\phi^l_\delta(t,i,j,\delta) &= a(t,i,j)^\gamma \delta^{-\gamma}
\end{aligned}$$

which, when inserted in the expressions for candidate solutions from lemma 4.3, after some intermediary steps, yield the result. □

We also note the structure of the optimal dividends strategies. The dividend strategies are linear functions of the surplus, with the functions a and g appearing in the numerator and denominator respectively. In order to obtain more information on the structure of g, it is necessary to attack the HJB-equation directly (which is similar in spirit to what is done on [Sch08, p 131]).

Theorem 4.6. *Let utility functions ϕ^c, ϕ^l and ϕ^T be specified as in equation* (4.9)*. Then the function*

$$V(t,i,x) = \frac{1}{1-\gamma} g(t,i)^\gamma x^{1-\gamma}$$

is a solution to the HJB-equation (Equation 4.10) satisfying the conditions of Theorem 4.4 whenever there exists a solution to the following (system of) ordinary differential equations:

$$\begin{aligned}
g_t(t,i) &= r^* g(t,i) - a(t,i) - \sum_{j=0}^{J} \mu_{ij}(t) R^{g,ij}(t,i,j) \ \textit{where} \\
r^* &= -\frac{1-\gamma}{\gamma}\left(r + \frac{(m-r)^2}{2\sigma^2\gamma}\right) \ \textit{and} \\
R^{g,ij}(t,i,j) &= \frac{1}{\gamma}\left(g(t,i)^{1-\gamma}(g(t,j)+a(t,i,j))^\gamma - g(t,i)\right)
\end{aligned} \tag{4.11}$$

with the boundary condition $g(T,i) = A(i)$.

Existence is guaranteed whenever

$$h(t,g) = (h(t,g(t,0)),...,h(t,g(t,J))) \text{ where}$$
$$h(t,g(t,i)) = r^* g(t,i) - a(t,i) - \sum_{j=0}^{J} \mu_{ij}(t) R^{g,ij}(t,i,j)$$

is continuous and bounded.

Proof. The conditions for existence of solutions to the system of ordinary differential equations follows from Theorem 2.14. From now on, we solve the equation assuming that these conditions are satisfied. In particular, it is sufficient that $\mu_{ij}(t)$, $a(t,i)$ and $a(t,i,j)$ are continuous on $[0,T]$.

Using the ansatz solution (Equation 4.10) to the HJB-equation (Equation 4.7) we obtain the following expressions for the derivatives of the candidate value function:

$$\begin{aligned} V_t &= \frac{\gamma}{\gamma-1} g(t,i)^{\gamma-1} x^{1-\gamma} g_t(t,i), \\ V_x &= g(t,i)^{\gamma} x^{-\gamma}, \\ V_{xx} &= -\gamma g(t,i)^{\gamma} x^{-\gamma-1}. \end{aligned}$$

We then plug in these expressions, together with the candidates for optimal strategies given in Proposition 4.5, in the HJB-equation (4.7) with utility functions as in (4.9).

Some tedious calculations yield that it is possible to factor out $\frac{1}{1-\gamma}\left(\frac{x}{g(t,i)}\right)^{1-\gamma}$ to obtain

$$\frac{1}{1-\gamma}\left(\frac{x}{g(t,i)}\right)^{1-\gamma}\Bigg(\gamma a(t,i) + (1-\gamma)(r + \frac{(m-r)^2}{2\sigma^2\gamma})g(t,i) + \gamma g_t(t,i) + \sum_{j=0}^{J} \mu_{ij}(t)\left(g(t,i)^{1-\gamma}(g(t,j)+a(t,i,j))^{\gamma} - g(t,i)\right)\Bigg) = 0$$

which after rearrangement becomes

$$\begin{aligned} g_t(t,i) = &-\frac{1-\gamma}{\gamma}\left(r + \frac{(m-r)^2}{2\sigma^2\gamma}\right) g(t,i) - a(t,i) \\ &- \frac{1}{\gamma}\sum_{j=0}^{J} \mu_{ij}(t)\left(g(t,i)^{1-\gamma}(g(t,j)+a(t,i,j))^{\gamma} - g(t,i)\right). \end{aligned}$$

The boundary condition $V(T,i,x) = \phi^T(i,x)$ means that

$$\frac{1}{1-\gamma} g(T,i)^{\gamma} x^{1-\gamma} = \frac{1}{1-\gamma} A(i)^{\gamma} x^{1-\gamma} \Rightarrow g(T,i) = A(i).$$

Thus, when the system of ODEs (4.11) has a solution, the ansatz (4.10) is a solution to the HJB-equation (4.7) attaining the supremum.

□

When the conditions of Theorem 4.6 are fulfilled, the verification theorem (Theorem 4.4) then implies that the function given in (4.10) is the value function of the control problem and the candidate control processes given in Proposition 4.5 are optimal controls.

Note that Equation (4.11) can be represented as a system of $J + 1$ ordinary differential equations in t and that this system is determined by the parameters γ, σ, m, r and the transition intensities of the Markov process determining the state of the policy. It is thus not dependent on the surplus.

The structure of the system of equations also resembles the Thiele differential equations (proposition 4.1), which was an expression for the reserve.

In fact, it is possible to show (see [Ste04, p 16]) that $g(t, i)$ can be written on conditional expectation form and can be given a similar interpretation as a kind of reserve. However, it is not the reserve of the actual dividend stream, but rather of the artificial stream determined by the weight functions $a(t, i), a(t, i, j)$ and $A(i)$ which specified the policy holders preferences over time and policy states. Thus, one could consider $g(t, i)$ as a *utility reserve*, which, in addition to the preference weight functions, depends on the asset returns and volatility and transition intensities between states.

With this interpretation, the optimal dividend payment stream is to pay a share of current surplus corresponding to the current value of the weight function $a(t, i)$ divided by the utility reserve. The optimal lump sum payments can be given a similar interpretation, with the optimal payment being to pay a share of current surplus corresponding to the current value of the weight function $a(t, i, j)$ divided by the utility reserve including the lump sum payment itself.

4.2.1 Example: Life annuity

We will illustrate the results from subsection 4.2 by considering an example not found in the sources [Ste04] and [Sch08].

We consider a simple survival model with two states $\mathcal{J} = \{0, 1\}$ corresponding to $\{alive, dead\}$. There is no value or utility in the state "dead", i.e. $V(t, 1, x) = 0$. This means that the notation can be simplified by dropping the explicit state references, so that $V(t, 0, x) = V(t, x)$ and $g(t, 0) = g(t)$. It also means that we will have a single HJB-equation. We consider the utility functions

$$\begin{aligned}\phi^c(t,x) &= \frac{1}{1-\gamma}a(t)^\gamma x^{1-\gamma}\\ \phi^l(t,0,1,x) &= 0\\ \phi^T(x) &= \frac{1}{1-\gamma}A_T^\gamma x^{1-\gamma}.\end{aligned}$$

The utility function $\phi^l(t,0,1,x)$ being identically 0 (which also implies that $\delta^l = 0$) corresponds to a with-profit life annuity without any survivor benefit paid out at death (a very typical contract). This yields the following HJB-equation:

$$\sup_{\pi,\delta}\Big\{\phi^c(\delta^c,t) + \left([(1-\pi)r + m\pi]\,x - \delta^c\right)V_x(t,x) \\ + \frac{1}{2}\sigma^2\pi^2x^2V_{xx}(t,x) + V_t(t,x) - \mu_{01}(t)V(t,x)\Big\} = 0.$$

with the boundary condition $V(T,x) = \frac{1}{1-\gamma}x^{1-\gamma}A_T^\gamma$. This is similar to the specification in the problem considered in subsection 3.3, except for the presence of the last term and the boundary condition. With the utility function specified above we use the results from Proposition 4.5 and the ansatz

$$V(t,x) = \frac{1}{1-\gamma}g(t)^\gamma x^{1-\gamma}$$

as in (4.10), to obtain the candidate optimal strategies

$$\begin{aligned}\pi^* &= \frac{m-r}{\gamma\sigma^2}\\ \delta^{c*}(t,x) &= \frac{a(t)}{g(t)}x.\end{aligned}$$

From Theorem 4.6 we get that the ansatz is a solution to the HJB-equation whenever $g(t)$ solves the differential equation

$$g_t(t) = \left(r^* + \frac{\mu_{01}(t)}{\gamma}\right)g(t) - a(t)$$

with the boundary condition $g(T) = A_T$. The verification theorem (Theorem 4.4) then gives that the solution to the HJB-equation is the value function of the control problem and that the candidate strategies are indeed the optimal controls.

In this case, we can use Proposition 2.13 to write down an explicit, unique solution to the ODE, given by

$$g(t) = A_T \exp\left(-\int_t^T (r^* + \frac{\mu_{01}(u)}{\gamma})du\right) + \int_t^T \exp\left(-\int_t^s (r^* + \frac{\mu_{01}(u)}{\gamma})du\right) a(s)ds.$$

We then have to specify parameters and weight functions. We let $a(t) = \exp(\frac{-\rho}{\gamma}t)$ and $A_T = 0.2$. Note that A_T is the coefficient specifying preferences for a final payment at the end of the contract. Thus, if A_T is relatively larger, the payment profile will be "back-loaded". The force of mortality

$$\mu_{01}(t) = \mu_{65+t} = a + b\exp\left(c(65+t)\right)$$

is a Gompertz-Makeham function, where the set of parameters $\{a, b, c\}$ is taken from the pay-out basis[13] used by the Swedish Pensions Agency for insured born 1950-1959, see [Pen22, bilaga A]. It is displayed in figure 4.1.

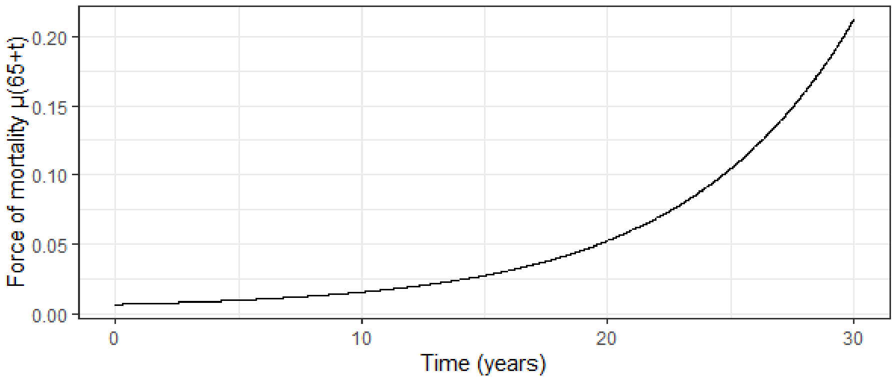

Figure 4.1: Force of mortality $\mu_{01}(t) = \mu_{65+t}$

The rest of the parameters are given in Table 4.1 below.

The simulation of the controlled surplus process are done using the Euler-Maruyama method [KP92, ch 9.1] with step size $\Delta t = 1/1000$. This gives the following expressions for the simulated processes:

$$X^*_{t+\Delta t} = X^*_t + \left((1-\pi^*_t)r + \pi^*_t m\right) X^*_t \Delta t - \delta^*_t \Delta t + \sigma \pi^*_t X^*_t \Delta W_t$$
$$\delta^*_t = \frac{a(t)}{g(t)} X^*_t$$

where $\Delta W_t \sim N(0, \Delta t)$ and $t = i \cdot \Delta t, i \in \{0, ..., \frac{T}{\Delta t}\}$.

[13] Swedish: *prognosgrunder*.

Parameter	Value
age (when $t = 0$)	65
T	30
r	0.02
m	0.08
σ	0.33
γ	0.6
ρ	0.03
X_0	100000

Table 4.1: Parameter values used for simulated trajectory

We have chosen to illustrate the example using one simulated trajectory, with and without including the force of mortality μ_{65+t} (which enters the expressions through $g(t)$) [14]. Doing so illustrates the effect that mortality has when allocating the dividend optimally. Since g is the expected value of discounted preferred future payments - the *utility reserve* - the inclusion of mortality risk reduces the value of g, thus increasing the dividend rate relative to current surplus.

As the contract starts at age 65 and runs for 30 years (until age 95 when $\mu_{65+t} > 0.2$), we would expect to be able to observe this effect. For illustrative purposes we assume that T_{65}, the remaining lifetime at age 65 satisfies $T_{65} > 30$ in this simulation.

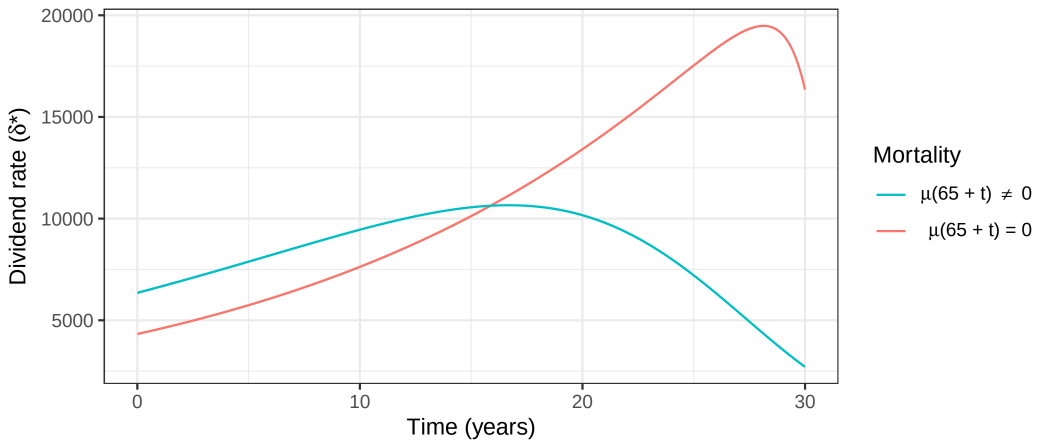

Figure 4.2: Simulation of dividend process

[14]The same normal random sample is used in both cases.

In Figure 4.2, which is a simulated trajectory of the dividend rate process, we see that in the case where mortality is accounted for, the dividend rate is initially higher. This reflects that the mortality risk is taken into account when discounting the future preferred payment stream. The remaining surplus is paid out as a lump sum at $T = 30$.

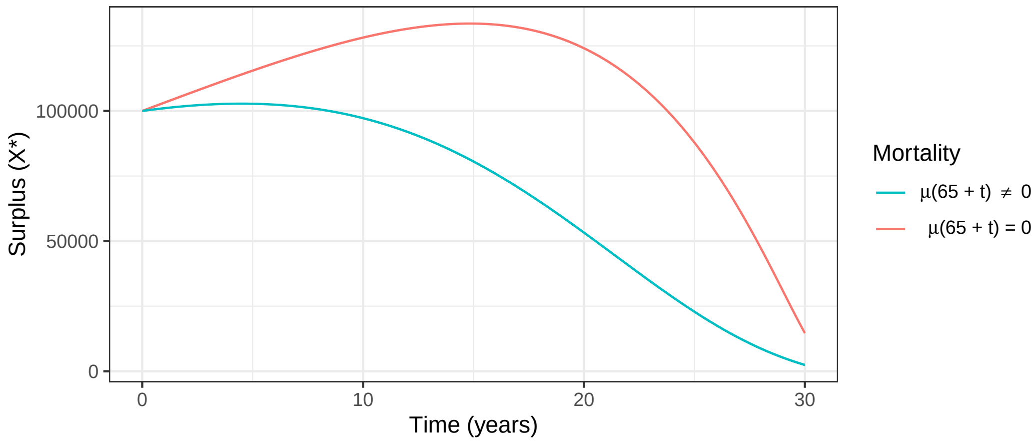

Figure 4.3: Simulation of surplus process

Higher initial dividends, on the other hand, means slower growth of the surplus process. In Figure 4.3, which illustrate the trajectory of the surplus process from the same simulation, it is evident that the surplus process grows faster in the case without mortality risk. We note that there is a larger remaining surplus, to be paid out at $T = 30$, in the case without mortality risk ($X_T^* = 14642$) than in the case with mortality risk ($X_T^* = 2427$).

The value function from the same simulation is shown in Figure 4.4. As we would expect, the value function is strictly larger in the case where the policy holder runs no risk of dying and entering a state of zero value. At the endpoint, $V(T, X^*) = \frac{1}{1-\gamma} A_T^{\gamma} (X_T^*)^{1-\gamma}$, the utility of the remaining surplus that is paid out at termination of the contract.

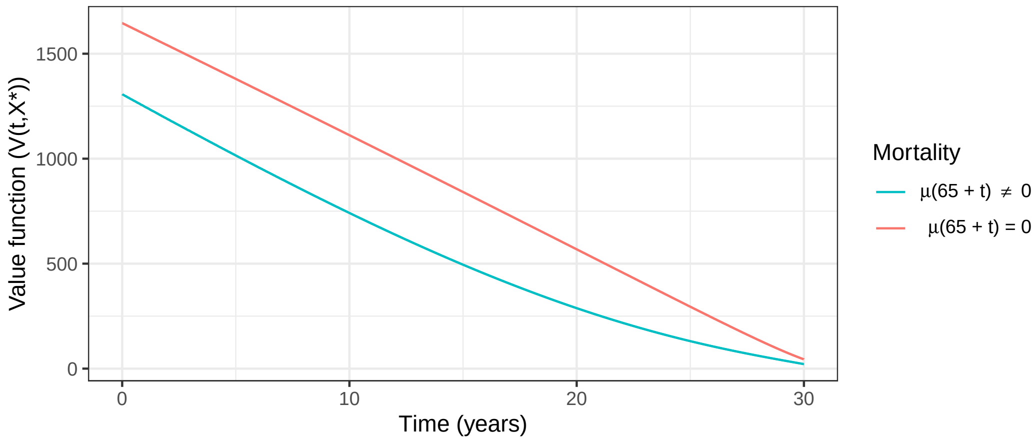

Figure 4.4: Simulation of value function

We also plot the value function as a function of x for some fixed values of t in Figure 4.5. From this illustration, it is clear that the value function indeed is concave in x and that the reducing effect of mortality on the value function diminishes with t, as the remaining time at risk gets smaller. At the endpoint, as noted above, we have for both cases that $V(T, x) = \frac{1}{1-\gamma} A_T^{\gamma} x^{1-\gamma}$.

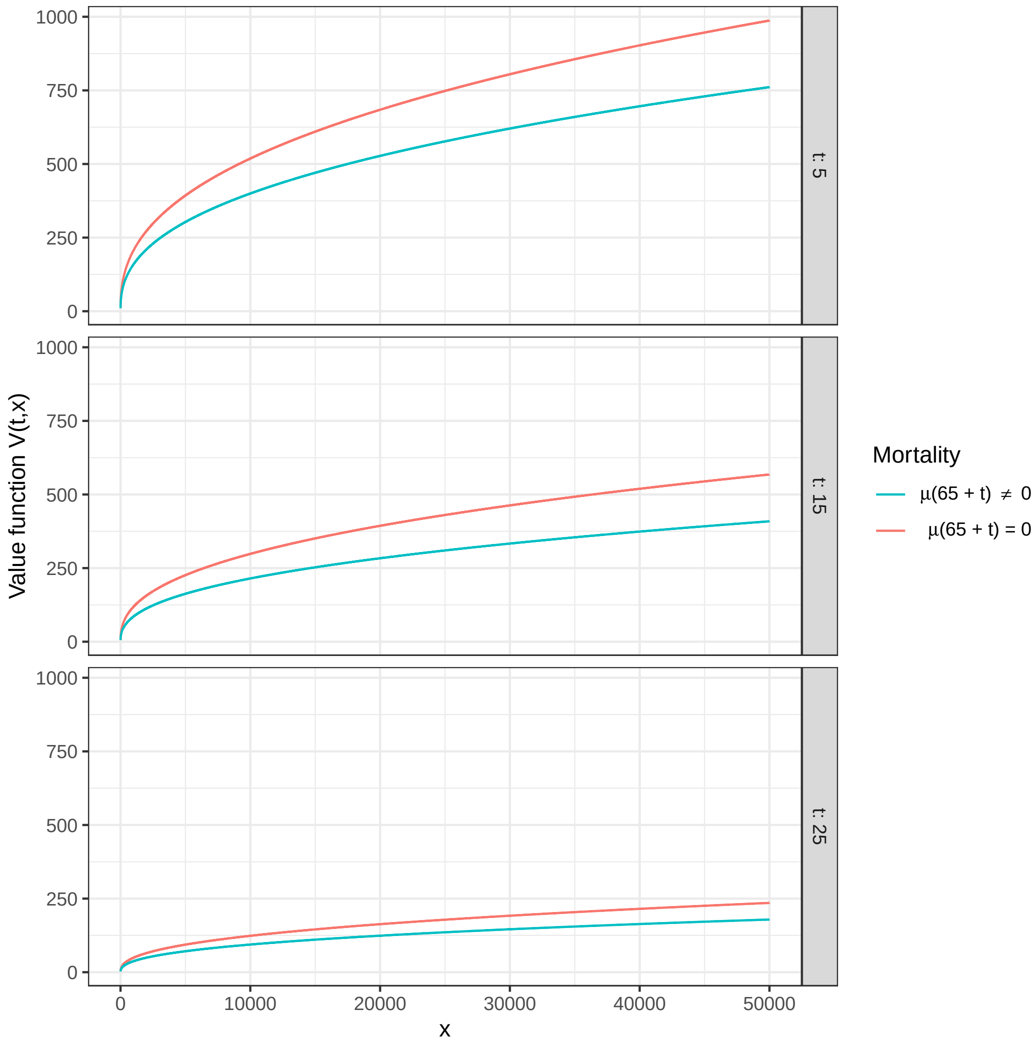

Figure 4.5: Value function $V(t_0, x)$ for $t_0 \in \{5, 15, 25\}$

5 Discussion

This thesis deals with applying stochastic control methods to problems in life insurance mathematics. We used the dynamic programming principle method to derive a PDE called the Hamilton-Jacobi-Bellman equation associated to the control problem. The connection between solutions to the HJB-equation and the value function of the stochastic control problem is established through verification theorems - proving that a solution to the HJB-equation coincides with the value function. A candidate for the optimal control is obtained by pointwise optimisation of the HJB-equation over the control space. If the HJB-equation with this candidate optimal control has a (smooth) solution it is indeed the value function of the control problem and, perhaps more importantly in applications, the optimal control candidate is indeed the optimal control.

The particular control problem that we had in mind was to control the surplus process of a with-profit life insurance contract, where the surplus is to be invested in an asset market and paid out to the policy holders as dividends so as to maximise their utility. Under some simplifying assumptions regarding the product design and the asset market, it turned out that the dividend allocation problem could be formulated as an investment-consumption problem generalizing the well-known Merton problem from mathematical finance.

Finding explicit solutions to the HJB-equation depended crucially on the form of the utility function. With the power utility function with constant relative risk aversion, a semi-explicit solution to the HJB-equation could be found which separated variables. The time-dependent part turned out to be solutions to a system of ordinary differential equations whose structure resembled that of the Thiele differential equations for the reserve of a life insurance contract. The optimal portfolio process was in this case constant, whereas the dividend processes were linear functions of the surplus with the time-dependent part involving the preference weights and solutions to a system of ODEs.

In order to illustrate and interpret the results, we used a simple version of the model with only two states ("alive" and "dead") and without lump sum payments due at transitions between states. This simple version corresponds to a life annuity where the premium is paid in the form of a lump sum at retirement. In this simple version we illustrated the effect of mortality on the dividend and surplus processes.

This particular simplification is however not that serious of a limitation considering the motivation behind writing this thesis. One of the purposes was to illuminate possible strategies for the with-profit life annuity which is offered as part of the Premium pension in the Swedish public pension system. This with-profit life annuity has a very simple design with no lapse or surrender options and can, for all practical purposes, be considered as having a single premium at $t = 0-$. One interpretation of the result in this case is that if one would consider subportfolios (or individual accounts) with similar risk aversion and mortality, the optimal controls yield portfolio weights and weights for paying out the surplus to the policy holders.

There are a number of more serious limitations, however, which require attention and limit the direct applicability of the results.

The model used for the asset market is the simple Black-Scholes asset market model, with a risk-free asset with deterministic interest rate and a risky asset modelled as a geometric Brownian motion with constant coefficients. This is not very realistic, as mean returns and volatility typically are non-constant, non-deterministic and might as processes not even be fully observable. It has also been argued that (see e.g. [CT04, ch 1]) that assuming continuity of sample paths and Gaussian returns of the risky asset are somewhat unrealistic. In the context of the standard Merton problem (without insurance policy states), generalizations in these directions have been made. Allowing for non-deterministic coefficients is dealt with in detail in [KS98, ch 3] and the case of partial observability of returns and/or volatility is considered in several sources, e.g. [HS17]. Versions of the consumption-investment problem with jumps in the risky asset price process can be found in [ØS19, Example 5.2] and [APW14]. It would be interesting to investigate to what extent these extensions of the standard Merton problem would carry over to the version considered in this thesis.

Another limitation is the representation of preferences of the policy holders by utility functions with constant relative risk aversion. This structure allowed for the HJB-equation to have a semi-explicit solution and implied constant optimal portfolio weights. However, this does not seem to be in line with empirical evidence (see e.g. [BC97]). It also does not agree with the practice of target date funds (lifecycle funds), which are common default options in pension plans, where there is a shift away from risk with increasing age (i.e. the risk aversion of the policy holder is assumed to increase with age). The case of time-dependent relative risk aversion is considered in [Ste11] and [Aas17]. The first paper uses dynamic programming, wheras the latter (which also considers the effect of mortality) uses other methods. It would also have been interesting to explore this setting using dynamic programming methods.

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