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Abstract

Ahlberg and de la Riva recently proved noise sensitivity of the indicator of a travel time being above its median as the first evidence of noise sensitive behavior in first-passage percolation. We extend the BKS theorem from indicator functions from the hypercube to real-valued functions from the hypercube, making use of the hypercontractive inequality in a Markovian framework instead of a Fourier analysis framework. This allows us to deduce noise sensitivity of the first-passage percolation left-right travel time in the square with restricted vertical fluctuations in the case of a binary weight distribution, following the work of Ahlberg and de la Riva.

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Contents

1 Introduction

1.1 The model of first-passage percolation

First-passage percolation is a setting where edges of a graph (V, E) are assigned independent identically distributed weights $(\omega_e)_{e\in E} \in [0,+\infty)^E$ and the object of interest is the random metric on the vertex set, defined by the smallest total weight of a path between two vertices. If Γ is a neighbor-to-neighbor path, the total weight of Γ is the random quantity $\sum_{e\in\Gamma}\omega_e$, and the object of interest is the distance on the vertex set defined for all pairs of vertices (x, y) by

$$T(x,y) = \inf\{T(\Gamma) : \Gamma \text{ is a path from } x \text{ to } y\}\,.$$

The quantity $T(x, y)$ is commonly called geodesic length or travel time from x to y. On the square lattice $\mathbb{Z}^2$, this infimum is almost surely attained and paths reaching this minimum are called geodesics. Understanding the long-range properties of the random metric T, especially on lattices, is a long-term goal in the study of first-passage percolation. Fundamental results in this direction are shape theorems, establishing existence of an almost sure limit shape of the ball of radius n around the origin. Results about the fluctuations of that limit shape remain general and this motivates the study of noise within the context of first-passage percolation. Note that travel times from a set of point to another set of point can also be considered, for example the travel time from left to right in the square $[0, n]^2$, for some positive integer n. This is relevant because the symmetry properties of the lattice might make the study easier.

First-passage percolation has first been introduced by Hammersley and Welsh [HW65] in the 1960s to describe the behavior of a fluid running through a porous medium. This is an intermediate setting between discrete percolation models such as Bernoulli percolation, which is realized as a special case of first-passage percolation where the weights are distributed in $\{0, +\infty\}$, and ferromagnetic models in random environments, of which first-passage percolation can be interpreted as a 0-temperature version. A positive-temperature version would be to pick a path Γ between two points with probability proportional to $e^{-\beta T(\pi)}$, where β denotes the inverse temperature parameter. This is referred to as polymer model by Chatterjee [Cha08; Cha14], although polymer models can design a more general class of models consisting in a space of lattice subsets endowed with an energy function and a so-called Gibbs probability measure.

1.2 Noise sensitivity, superconcentration and chaos

The year 1999 marked the beginning of the study of reaction to noise in percolation models, due to the publication by Benjamini, Kalai and Schramm [BKS99] of a paper introducing the notion of noise sensitivity of Boolean functions, together with a proof of noise sensitivity of indicator functions of crossings in Bernoulli percolation. Noise sensitivity is a property of asymptotic decorrelation under the effect of a small noise. Historically, it applies to Boolean functions, or algorithms, or 0-1 valued functions from the hypercube $\{0, 1\}^m$ for some typically large integer m, due to the study of such functions in the Computer Science literature. It makes sense to study such functions in discrete probability theory because they correspond exactly to indicator functions of events when the probability space is a discrete hypercube. More precisely, given a parameter $p \in (0, 1)$ and an integer $m \geq 1$, we endow the hypercube $\{0, 1\}^m$ with the product probability measure

$$\mathbb{P}_p := \prod_{i=1}^{m} \big(p\delta_1 + (1-p)\delta_0\big)\,.$$

The word "noise" generally refers to a small, random, perturbation of an object, here these probability measures. Letting ω^0 be a random variable on $\{0,1\}^m$ of law $\mathbb{P}_p$ and a noise parameter $t > 0$, which we later understand as time, we let ω^t be a random variable obtained from ω^0 by resampling each coordinate independently with probability $1 - e^{-t}$. According to this definition, $\big((\omega_e, \omega_e^t)\big)_{e\in E}$ are independent random variables and for all $e \in E$ and

$$\begin{pmatrix} \mathbb{P}(\omega_e^0, \omega_e^t = 1, 1) & \mathbb{P}(\omega_e^0, \omega_e^t = 1, 0) \\ \mathbb{P}(\omega_e^0, \omega_e^t = 0, 1) & \mathbb{P}(\omega_e^0, \omega_e^t = 0, 0) \end{pmatrix} = \begin{pmatrix} p\big(e^{-t} + p(1 - e^{-t})\big) & p(1-p)(1-e^{-t}) \\ p(1-p)(1-e^{-t}) & (1-p)\big(e^{-t} + (1-p)e^{-t}\big) \end{pmatrix}.$$

Note that this matrix goes to $\begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix}$ when t goes to 0 and to $\begin{pmatrix} p^2 & p(1-p) \\ p(1-p) & (1-p)^2 \end{pmatrix}$ when t goes to infinity, which is a way to see that ω^t interpolates from ω^0 to an independent variable with the same law. Benjamini, Kalai and Schramm [BKS99] defined noise sensitivity of a sequence of Boolean function (f_n) as the following property:

$$\forall t > 0,\ \mathrm{Cov}_p\big(f_n(\omega^0), f_n(\omega^t)\big) \underset{n\to+\infty}{\longrightarrow} 0\,. \tag{1}$$

Note that, when t is close to infinity, the coordinates are resampled with high probability and ω^t is nearly independent of ω^0, so $f(\omega^t)$ has very good reasons to be nearly independent of $f(\omega^0)$. This justifies that noise sensitivity has to be thought of as a property taking place for small values of t. Later, Chatterjee [Cha08; Cha09; Cha14] introduced the related notions of superconcentration and chaos and proved that superconcentration and chaos hold in certain Gaussian disordered systems. The notion of superconcentration is suited to a setting where a random variable is defined as the maximum (or minimum) of a large number of correlated random variables having similar distribution. Such a random extremum is called superconcentrated when the variance of the extremum is way smaller than the variance of a single one of the underlying random variables. As travel times in first-passage percolation are minimums of random lengths on a set of paths, it is reasonable to study superconcentration properties of travel times. It has been shown by Benjamini, Kalai and Schramm [BKS03] that the variance of the travel time between points at distance n is less than $\frac{n}{\log n}$, which is a superconcentration result. On the other hand, chaos is, broadly, the property that the random variables realizing the extremum before and after noising are weakly correlated. In first-passage percolation, the random variables are lengths indexed on paths. Denoting by $\pi^s(x, y)$ the intersection of all geodesic paths from x to y in ω^s, for all $s \geq 0$, chaos in first-passage percolation would then be expressed as

$$\mathbb{E}\big[\pi^0(0, ne_1) \cap \pi^t(0, ne_1)\big] = o(n)\ \big(n \to +\infty\big)\,,$$

which states that, in expectation, the geodesic at time 0 and the geodesic at time t share a significantly low number of edges. Common edges are indeed the source of correlation between two path lengths as they are defined as sums of independent random variables. In the initial definition given by Chatterjee in the context of the maximum of a Gaussian vector $(X_i^0)_{i\in E}$ of covariance matrix R (where E and R depend on an integer n going to infinity) and a noise version of it (X_i^t) obtained from X^0 by running an Ornstein-Uhlenbeck dynamic for a time t, chaos is defined as

$$\mathbb{E}\big[R\big(\mathrm{argmax}\{X_i^0, i \in E\}, \mathrm{argmax}\{X_i^t, i \in E\}\big)\big] = o\big(\max\{R(i,j),\ (i,j) \in E^2\}\big)\,.$$

Chatterjee [Cha08; Cha14] proved that superconcentration is equivalent to chaos in that Gaussian context using the properties of the Ornstein-Uhlenbeck semigroup and a differential equality for the covariance of a function of a Gaussian vector at two different times. Proposition 3.1 of the present document introduces analogous properties for a Markov semigroup corresponding to a resampling dynamic on the hypercube. An approach inspired by that of Chatterjee has been used in first-passage percolation and the other spatial growth model of last-passage percolation by Ahlberg, Deijfen and Sfragara [ADS23; ADS24] in order to exhibit chaos properties in these models. In parallel, noise sensitivity seems more challenging to define and to study than superconcentration and chaos in first-passage percolation. A first noise sensitivity result in first-passage percolation has been proved by Ahlberg and de la Riva [AR23]. Due to the difficulty of studying noise sensitivity in first-passage percolation, it applies to a travel time from left to right in the square, with vertical path fluctuation bounded by a small polynomial. Assume the edge weights ω_e are distributed uniformly in $\{a, b\}$ for some real numbers $b > a > 0$. Letting $\tau(n, k)$ be the minimum of $T(\Gamma)$ over paths Γ with ends in $\{0\} \times [0, n]$ and $\{n\} \times [0, n]$, contained in the square $[0, n]^2$, and such that the difference between the maximal height reached by Γ and its minimal height is less than k, their result is the following.

Theorem 1.1. *Let $(k_n)_{n\geq 1}$ be a sequence of integers such that $k_n = o(n^\alpha)$ for some $\alpha < \frac{1}{22}$. Let $(m_n)_{n\geq 1}$ be a sequence of real numbers such that for all $n \geq 1$, m_n is a median of the distribution of $\tau(n, k_n)$. Then*

$$\mathbb{P}\big(\tau(n, k_n) \geq m_n\big) \to \frac{1}{2}\,,$$

and the sequence of Boolean functions $\mathbb{1}_{\{\tau(n,k_n)\geq m_n\}}$ is noise sensitive in the sense defined by (1).

Note that the initial result of Ahlberg and de la Riva is stronger, allowing for m_n to be a sequence of β-quantiles of the distribution of $\tau(n, k_n)$ for any fixed $\beta \in (0, 1)$. In this paper, we propose a similar result exhibiting a noise sensitivity property of $\tau(n, k_n)$ in the form of the upcoming Theorem 1.2. It states that for all $t > 0$, the correlation between $\tau(n, k_n)(\omega)$ and $\tau(n, k_n)(\omega^t)$ goes to 0 as n goes to infinity.

1.3 Generalizations using a Markovian dynamic

The reason why we interpret the noise parameter as time is that noise can be seen as running a Markovian dynamic for a small time and noise sensitivity and chaos has been studied with the use of dynamical formulas and differential calculus instead of Fourier analysis. The dynamic is the following: the initial configuration ω^0 in the hypercube is drawn according to $\mathbb{P}_p$ and the coordinates are resampled independently according to Poisson clocks of parameter 1, creating a right-continuous Markov process $(\omega^t)_{t\geq 0}$ with values in the hypercube and invariant measure $\mathbb{P}_p$. The dynamic is reversible as well, and these properties imply

$$\mathrm{Cov}_p\big(f(\omega^0), f(\omega^t)\big) = \mathrm{Var}_p\big(\mathbb{E}_p[f(\omega^{\frac{t}{2}})|\omega^0]\big)\,. \tag{2}$$

Studying the time derivative of $t \mapsto \mathbb{E}_p[f(\omega^{\frac{t}{2}})|\omega^0]$ has proved to lead to noise sensitivity results such as such as Talagrand's inequality and the Kahn-Kalai-Linial theorem. These considerations first appeared in a paper of Cordero-Erausquin and Ledoux [CL12], Talagrand [Tal93] also worked in such a setting and Ledoux [Led01] relates hypercontractive inequalities with a dynamic defined this way to concentration results, which would correspond to noise sensitivity results in the case of a Markovian dynamic representing a noise. Chatterjee [Cha14] also makes use of such a setting with an Ornstein-Uhlenbeck Markov semigroup instead of a random walk on the hypercube in his study of superconcentration and chaos for polymer models and other models of definition relying on Gaussian fields. These concepts are also manipulated in lecture by van Handel [Han16; RH20], explaining how the Markovian framework can be used to deal with functions having values in spaces larger than just $\{0, 1\}$, for example the set of real numbers or even a multidimensional space. In 2023, Tassion and Vanneuville [TV23] provide a proof of quantitative noise sensitivity in Bernoulli percolation which is inspired from these methods in view of extending noise sensitivity to percolation models where Fourier analysis fail.

Remember that we would like to prove noise sensitivity results in first-passage percolation, and especially we are looking for a noise sensitive property applicable to travel times. It seems more natural to ask directly for correlation of the travel time instead of an indicator function, which motivates finding a definition of noise sensitivity which does not only apply to 0-1 valued functions. A first remark that could be made is that using only the covariance leads to a bit of a restrictive notion of noise sensitivity. For real-valued functions, we cannot expect the expectation of the function to be neither close to 0 or infinity unless we renormalize it by a standard deviation, which amounts to studying the correlation instead of the covariance. This motivates the following alternative definition of noise sensitivity for a sequence of real-valued functions (f_n) defined on hypercubes of increasing sizes:

$$\forall t > 0, \; \frac{\mathrm{Cov}_p\big(f(\omega^0), f(\omega^t)\big)}{\mathrm{Var}_p(f)} \underset{n\to+\infty}{\longrightarrow} 0 \,. \tag{3}$$

The main tool known to provide a sufficient condition for a sequence of Boolean functions to be noise sensitive is the BKS theorem and is expressed in terms of influences. In the initial Boolean function setting, the influence of a coordinate i on the Boolean function f is the probability that changing the value of ω_i^0 alone changes the output of the function, so that $\mathrm{Inf}_i^p(f)$ is defined as $\mathbb{P}_p\Big(f\big(\sigma_i(\omega^0)\big) \neq f(\omega^0)\Big)$, where the flipping operator σ_i is defined by, for all $e \in E$ and $\omega \in \Omega$,

$$\sigma_i(\omega)_j := \begin{cases} 1-\omega_j & \text{if } i = j \\ \omega_j & \text{if } j \neq i \end{cases} .$$

Behavior of influences is closely linked to noise sensitivity behavior. The BKS theorem [BKS99] states that if a sequence of Boolean functions f_n satisfies

$$\sum_i \mathrm{Inf}_i^p(f_n)^2 \to 0 \,,$$

then the sequence (f_n) is noise sensitive in the sense prescribed by (1). However, defining influences this way only makes sense when dealing with 0-1 valued functions. For real-valued functions, the amount by which the output of f changes when resampling ω^0 matters in view of obtaining noise sensitivity results. This justifies the definition of influence as an L^1 quantity involving a difference obtained in changing only one coordinate. The operators of **difference under resampling of a coordinate** $(D_i)_{1\le i\le m}$ are defined as

$$D_i f(\omega) := \mathbb{E}_{\xi\sim p}\Big[f\big(\sigma_i^\xi(\omega)\big)\Big] - f(\omega) \,,$$

where the expectation is taken with respect to a Bernoulli variable ξ and $\sigma_i^\xi(\omega)$ stands for the vector ω with i^{th} coordinate replaced with ξ. We then propose the following definition of **influence** for real-valued functions :

$$\mathrm{Inf}_i^p(f) := \mathbb{E}_p\Big[\big|D_i f\big|\Big] \,. \tag{4}$$

Note that in the case of Boolean functions, this coincides up to constants with the usual definition of influence.

Remark 1.1. The influence $\mathbb{E}[|D_i f|]$ should in some way be linked to the probability that i belongs to the geodesic path. This is emphasized in the articles by Ahlberg, Deijfen and Sfragara [ADS23; ADS24] about chaos in first-passage percolation, where it is proven that

$$\mathrm{Inf}_i(f) \asymp \mathbb{P}(i \in \pi) \,,$$

for an even more general definition of influence called co-influence, which we do not detail here. However, in Section 4, we prove a result upper-bounding the influence of the travel time with restricted fluctuations by such a probability of belonging to the path of minimal weight.

1.4 Main results

We now come back to first-passage percolation with these new definitions of noise sensitivity. Recall the definition of $\tau(n, k_n)$, letting $\mathcal{P}_k(n)$ be the set of paths

$$\mathcal{P}_k(n) = \{\Gamma : \Gamma \text{ is a path from left to right in } [0,n]^2 \text{ and with vertical fluctuations less than } k\}$$

Ahlberg and de la Riva made progress in the study of noise sensitivity in first-passage percolation by studying a well-defined Boolean function: the indicator function that a travel time is above the median of its distribution, where the weight distribution is binary (this question was first asked to Ahlberg by Benjamini), that is

$$f_n = \mathbb{1}_{\{\tau(n,k_n) \geq m_n\}}$$

Having defined noise sensitivity in the case of real-valued functions, we may ask whether travel times are themselves noise sensitive. We conjecture that the sequence $T(0, nv)$ is noise sensitive for any non-zero $v \in \mathbb{R}^2$. This result is however out of reach with our techniques, mainly by lack of a good enough lower bound on the variance and influences of $T(0, nv)$. In light of Theorem 1.1, Ahlberg and de la Riva [AR23] asked whether $\tau(n, k_n)$ is noise sensitive in the sense of (3), under the same assumption on k_n as in Theorem 1.1. The answer is yes, this is our main result.

Theorem 1.2. *Let $(k_n)_{n\geq 0}$ be a sequence of positive integers. Assume $k_n = O(n^\alpha)$ for some $\alpha < \frac{1}{22}$. Then, the sequence $(\tau(n, k_n))_{n\geq 0}$ is noise sensitive in the sense prescribed by (3).*

Whether noise sensitivity of travel time or noise sensitivity of being above the median imply the other is an interesting question. It makes sense to think that noise sensitivity of the travel time itself is a bit stronger than noise sensitivity of it being above the median, but it seems that we would actually need a stronger result than noise sensitivity of the travel time to obtain an implication. We infer that what is needed is a property of asymptotic independence of the deviation to the mean before and after resampling rather than just a decorrelation property. The initial proof of noise sensitivity of being above the median by Ahlberg and de la Riva relies on the BKS theorem and bounding the influences in the initial Boolean sense. In order to prove Theorem 1.2, we generalize the BKS theorem to real-valued functions, using the notion of influence defined by (4). It takes the form of an inequality, which is established for $p = \frac{1}{2}$ by van Handel [RH20] in lectures he gave in 2020. He attributes the proof to Falik-Samoroditsky [FS07] and Rossignol [Ros06] independently. Their initial proof uses a martingale argument and is robust to the output space being even larger than $\mathbb{R}$, which we do not make use of. We give a slightly different proof, using hypercontractivity and Hölder's inequality in the same way but replacing the martingale argument by manipulations using properties of the Markov semigroup of the random walk on the hypercube in a different way, as explained to the author by Vanneuville.

Theorem 1.3. *For any $p \in (0,1)$ and $t > 0$ there exists $C_p, \theta_p(t) > 0$ such that for all $f : \{0,1\}^m \to \mathbb{R}$,*

$$\frac{\mathrm{Cov}_p\big(f(\omega^0), f(\omega^t)\big)}{\mathrm{Var}_p(f)} \leq \left(C_p \frac{\sum_{i=1}^n \mathrm{Inf}_i^p(f)^2}{\mathrm{Var}_p(f)}\right)^{\theta_p(t)}$$

As announced, this can be interpreted as a quantitative version of the BKS theorem. It is straightforward to deduce from it that, for any sequence of real-valued functions (f_n), if

$$\frac{\sum_{i=1}^n \mathrm{Inf}_i^p(f)^2}{\mathrm{Var}_p(f)} \underset{n\to+\infty}{\to} 0\,,$$

then the sequence (f_n) is noise sensitive in the sense of (3). This quantitative version of the BKS theorem is similar to the one given by Keller and Kindler [KK13] in 2013, which applies to Boolean functions and is obtained from Fourier analysis.

1.5 Comments and further works

Noise sensitivity of Bernoulli percolation is now understood way better than from the initial work of Benjamini, Kalai and Schramm, with quantitative results on noise sensitivity, given by Schramm and Steif [SS10]in 2010, and a precise description of the Fourier spectrum by Garban, Pete and Schramm [GPS10]. A major tool which seems to be unfit to first-passage percolation is Randomized Algorithms (also known as Decision Trees), a major reason for it being that a lot of edge weights need to be revealed in first-passage percolation to provide a certificate that the travel time is larger than a certain threshold. Details on the study of noise sensitivity using Fourier spectrum can be found in books by O'Donnell and Garban-Steif [ODo14; GS15]. Cordero-Erausquin, Ledoux and Talagrand [CL12; Tal93] were the first to introduce calculus in a Markovian setting to prove influence theory results and Eldan and Gross [EG22] also used differential calculus ideas to deal with noise sensitivity. Van Handel [Han16; RH20] used this setting in lectures in 2016 and 2020, already giving a proof the inequality in which Theorem 1.3 consists. We extend the proof to $p \neq \frac{1}{2}$ and shortcut a martingale argument attributed to Falik-Samoroditsky [FS07] and Rossignol [Ros06], as suggested to the author by Vanneuville.

Theorem 1.2 could maybe be established for exponents larger than $\frac{1}{22}$. The reason behind this upper-bound on the exponent is that bounding the influences relies on estimations of the variance of travel times, which are handled by results of Chatterjee and Dey [CD13]. This is detailed in [AR23] and the analysis of influences for $\tau(n, k_n)$ which is realized in the present paper makes heavy use of the results and methods of this article. It might be possible to improve the exponent by a bit using the same methods, without having to rely on stronger results for the variance of travel times, but even in that case we wouldn't be able to go further than an exponent of $\frac{1}{3}$, which is the maximal exponent for which Chatterjee and Dey managed to prove Gaussian behavior of the flucutations of the travel time. In general, a Tracy-Widom limit distribution is expected for the fluctuations of the unrestricted travel time and good enough bounds for proving noise sensitivity are conjectured from KPZ class of universality arguments.

Further works could be to apply our results in last-passage percolation with geometric weights, which is an exactly solvable model where large deviations concerning the distribution of travel times are known. The main difficulty here would be to extract the probability of belonging to the geodesic from known results in order to provide the correct bound on the influences and use our generalized BKS theorem. Moreover, the present generalization of the BKS theorem would only make it possible for a noise sampled on the hypercube, although we should obtain the same result for any noise generated by a Markov process satisfying hypercontractivity and an exponential decorrelation property. Other problems in first-passage percolation would be to extend the result to distributions that are not binary, in particular [ADS24] gives bounds on quantities named co-influences for much more general distributions and it might be possible to link them to the definition of resampling influence we give here and apply our generalization of the BKS theorem. Finally, another quantity of interest in first-passage percolation is the vertical deviation of the geodesic above the middle point. It would be interesting to study noise sensitivity of this quantity and we believe that our methods and estimates given by Theorem 1.2 of [AR23] would be enough to obtain sensitivity results in this direction, once again for left-right geodesics with bounded vertical fluctuations.

2 Extended definitions of noise sensitivity

The letter ω is very well suited both for elements of a discrete infinite-dimensional hypercube $\{0,1\}^{\mathbb{N}^*}$ and a first-passage percolation process, which will later take values in $\{a,b\}^{E(\mathbb{Z}^2)}$, where $E(\mathbb{Z}^2)$ is the set of edges of the square lattice on $\mathbb{Z}^2$. When dealing with noise sensitivity and the proof of the generalization of the BKS theorem until the end of Section 3, we will work with $(\omega_i^t)_{i\geq 1, t\geq 0}$ as a continuous-time simple random walk on the hypercube $\{0,1\}^{\mathbb{N}^*}$. When dealing with first-passage percolation in Section 4, ω will be a first sample of first-passage percolation, thus taking values in $\{a,b\}^{E(\mathbb{Z}^2)}$, and we will not need to introduce any noise on this process to apply the generalization of the BKS theorem.

Let $\Omega = \{0,1\}^{\mathbb{N}^*}$, endowed with the cylinder topology and σ-algebra. We call $L^2(\Omega)$ the space of real-valued local functions from Ω to $\mathbb{R}$ where local means the functions depend on a finite number of coordinates. In particular, local functions are bounded. We could imagine a setting where Ω is a continuous space such as $\mathbb{R}^{\mathbb{N}}$ and the right functions to consider would be the square-integrable functions with respect to some measure on Ω, hence the notation L^2.

Remark 2.1. We work with an infinite-dimensional hypercube in order to avoid introducing a varying dimension m, which appears in the classical definitions of noise sensitivity since this notion is only relevant for sequence of Boolean functions defined on hypercubes of diverging dimension. Note that functions depending on m coordinates correspond to functions on the m-dimensional hypercube. Local functions are relevant in other contexts in statistical physics, for example to provide a proper definition of infinite-volume measures.

For any $p \in (0,1)$, we define the p-biased continuous-time simple random walk $(\omega^t)_{t\geq 0}$ on Ω as a random variable with values in the space of right-continuous functions from $\mathbb{R}_+$ to Ω, the following way. Let $(X_i^{(n)})_{i\in\mathbb{N}^*, n\in\mathbb{N}}$ be iid random variables of parameter p. Let $(\eta_i^t)_{i\in\mathbb{N}^*}$ be independent Poisson processes of rate 1 on $\mathbb{R}_+$ and $N_i^t = \eta_i^t([0,t])$ for all $i \geq 1$. In other words, for each i, $N_i^0 = 0$ and N_i jumps to the following integer at exponential rate 1. We let $(\omega^t)_{i\in\mathbb{N}^*, t\geq 0}$ be defined by

$$\omega_i^t = X_i^{(N_i^t)}.$$

Thus, at any non-negative time t, the distribution of ω^t is $\prod_{i\geq 1}\big(p\delta_1 + (1-p)\delta_0\big)$. It is said that ω_i is "resampled" when N_t jumps, because ω_i jumps to the outcome of the next Bernoulli random variable, independent of the previous ones. Moreover, letting $(T_i)_{i\geq 1}$ be independent exponential random variables of parameter 1 and $(X_i)_{i\geq 1}$ and $(Y_i)_{i\geq 1}$ be independent vectors of iid Bernoulli random variables of parameter p, for any t, (ω^0, ω^t) has the law of (X, Z^t) where $(Z_i^t)_{i\geq 1}$ is defined by

$$Z_i^t = \begin{cases} X_i & \text{if } T_i > t \\ Y_i & \text{if } T_i \leq t \end{cases}.$$

We call $\mathbb{P}_p$ the probability measure associated to parameter p for all of these notations. For any $f \in L^2(\Omega)$, we also write

$$\mathbb{E}_p[f] := \mathbb{E}_p\big[f(\omega^0)\big] = \mathbb{E}_p\big[f(X)\big],$$

where ω^0 has distribution $\prod_{i\geq 1}(p\delta_1 + (1-p)\delta_0)$, and

$$\mathrm{Var}_p(f) = \mathbb{E}_p\left[\left(f(\omega^0) - \mathbb{E}_p[f]\right)^2\right] = \mathbb{E}_p\left[\left(f(X) - \mathbb{E}_p[f]\right)^2\right],$$

and, likewise, the covariance between two functions which can have as input the values of ω at different times,

$$\mathrm{Cov}_p\left(f(\omega^s), g(\omega^t)\right) = \mathbb{E}_p\left[\left(f(\omega^s) - \mathbb{E}_p[f]\right)\left(g(\omega^t) - \mathbb{E}_p[g]\right)\right].$$

We introduce operators on the spaces Ω and $L^2(\Omega)$. The operators we define on Ω are flipping operators which act by flipping or forcing the value of a coordinate of the hypercube. As such, for $i \geq 1$, let $\sigma_i, \sigma_i^0, \sigma_i^1$ be functions from Ω to Ω defined by

$$\begin{aligned}
\sigma_i(\omega) &:= (\omega_1, \dots, \omega_{i-1}, 1-\omega_i, \omega_{i+1}, \dots),\\
\sigma_i^1(\omega) &:= (\omega_1, \dots, \omega_{i-1}, 1, \omega_{i+1}, \dots),\\
\sigma_i^0(\omega) &:= (\omega_1, \dots, \omega_{i-1}, 0, \omega_{i+1}, \dots).
\end{aligned}$$

So σ_i flips bit i, σ_i^1 replaces the value of bit i by 1 and σ_i^0 replaces it by 0. Note that, as random variables, $\sigma_i^1(X)$ and $\sigma_i^0(X)$ are both independent of X_i for all i.

We now define operators $(P_t)_{t\geq 0}$ and $(D_i)_{i\geq 1}$ on the space $L^2(\Omega)$. Formally, these operators depend on p, which we forget in the notation to make it simpler. We define $(P_t)_{t\geq 0}$ by, for any $t \geq 0, \omega \in \Omega, f \in L^2(\Omega)$,

$$P_t f(\omega) := \mathbb{E}_p\left[f(\omega^t)\middle|\omega^0 = \omega\right]. \tag{5}$$

The quantity $P_t f$ can also be seen as a random variable measurable by ω^0, keeping in mind that the value of p matters in the definition of that variable, which justifies the notation

$$P_t f = \mathbb{E}_p\left[f(\omega^t)\middle|\omega^0\right].$$

The operators $(P_t)_{t\geq 0}$ are often called the Markov semigroup generated by the random walk ω. It satisfies the following semigroup property:

$$\forall s, t \geq 0,\ P_{s+t} f = P_s(P_t f). \tag{6}$$

Then, for $i \geq 1$, let

$$D_i f(\omega) := \mathbb{E}_p\left[f(\sigma_i^{Y_i}(\omega))\right] - f(\omega), \tag{7}$$

where here $\sigma_i^{Y_i}(X)$ is ω where the i^{th} bit has been forced to take the value Y_i instead of ω_i. The operator D_i can be seen as an operator of partial differentiation in the direction i, which is justified by an analogy with differential calculus which we do not detail here. Note that the definition of D_i in the case of $p = \frac{1}{2}$ is

$$D_i f(\omega) = \frac{f \circ \sigma_i(\omega) - f(\omega)}{2}.$$

We also provide the following expressions for $D_i f$, which we might use later in the proofs:

$$D_i f(\omega) = \left((1-p)\omega_i + p(1-\omega_i)\right)\left(f \circ \sigma_i(\omega) - f(\omega)\right) \tag{8}$$

and

$$D_i f(\omega) = p f(\sigma_i^1(\omega)) + (1-p) f(\sigma_i^0(\omega)) - f(\omega). \tag{9}$$

A useful identity, valid for all functions $f \in L^2(\Omega)$ and all $\omega \in \Omega$, which could be used to prove that all of these identities are equivalent, is

$$f(\omega) = \omega_i f(\sigma_i^1(\omega)) + (1 - \omega_i) f(\sigma_i^0(\omega)) .$$

Note that $\mathbb{E}_p[P_t f]$ always equals $\mathbb{E}_p[f]$ by time-invariance of the random walk and $\mathbb{E}_p[D_i f]$ always equals 0. We provide other useful properties of P_t and D_i later.

In the context of Boolean functions, the influence of the coordinate i on a Boolean function f is the probability that changing the state of bit i in ω^0 changes the value of its image by f, when ω^0 is sampled according to $\mathbb{P}_p$. Thus, the influence is also the probability that $|D_i f(\omega^0)| \neq 0$ because f is Boolean so can only take the values 0 or 1. We generalize this definition to functions having values in $\mathbb{R}$ by interpreting this probability as the first moment of $|D_i f|$.

Definition 2.1. Let $i \geq 1$, $p \in (0,1)$ and $f \in L^2(\Omega)$. We let the influence of i on f at parameter p be denoted by $\mathrm{Inf}_i^p(f)$ and defined by

$$\mathrm{Inf}_i^p(f) = \mathbb{E}_p\big[|D_i f|\big] .$$

This definition coincides up to a constant factor with the usual definition of influence in the case where f takes values in $\{0,1\}$. Otherwise, this definition would rather fit the name of "resampling influence".

We now define noise sensitivity in the case of functions from the hypercube to $\mathbb{R}$.

Definition 2.2. A sequence of functions $(f_n)_{n\geq 0}$, each in $L^2(\Omega)$, is said to be noise sensitive at parameter $p \in (0,1)$ if and only if for all $t > 0$,

$$\frac{\mathrm{Cov}_p\big(f_n(\omega^0), f_n(\omega^t)\big)}{\mathrm{Var}(f_n)} \underset{n\to+\infty}{\longrightarrow} 0 . \tag{10}$$

Remark 2.2. Since ω^0 and ω^t have the same law, $f(\omega^0)$ and $f(\omega^t)$ have the same variance and the left-hand side of (10) is the **correlation** between $f(\omega^0)$ and $f(\omega^t)$.

We make extensive use of the semigroup formalism later using the identity, valid for all $f \in L^2$ and all $t > 0$

$$\mathrm{Cov}_p(f(\omega^0), f(\omega^t)) = \mathrm{Var}_p(P_{\frac{t}{2}} f) .$$

which follows from the properties of the semigroup P_t, namely the Markov property, time-translation invariance and time reversibility.

Remark 2.3. The original BKS theorem from [BKS99] states that if a sequence of Boolean functions (f_n) satisfies

$$\sum_{i\geq 1} \mathrm{Inf}_i^p(f_n)^2 \underset{n\to+\infty}{\longrightarrow} 0$$

then the sequence is noise sensitive in the sense of the covariance, i.e.

$$\mathrm{Cov}_p\big(f_n(\omega^0), f_n(\omega^t)\big) \underset{n\to+\infty}{\longrightarrow} 0$$

Then, noise sensitivity in the sense of the covariance and for Boolean functions is the same as noise sensitivity in the sense of correlation as in Definition 2.2 as long as the variances of the functions are uniformly bounded away from 0. Theorem 1.3 thus generalizes the BKS theorem in the sense that influences has been generalized to functions taking values in $\mathbb{R}$ instead of $\{0,1\}$ and the variance is taken into account to allow for a definition of noise sensitivity in terms of correlation instead of covariance.

3 Operator properties and proof of Theorem 1.3

3.1 Markovian properties

Some properties the family of operators (P_t) and (D_i) are summarized in the following proposition which extends Lemma 2 of [RH20] to values of p differing from $\frac{1}{2}$.

Proposition 3.1. Fix $p \in (0,1)$. The following properties hold:

(i) (Dynamical Margulis-Russo formula / Heat equation) For all $f \in L^2(\Omega)$ and all $\omega \in \Omega$, the function $s \mapsto P_s f$ is differentiable on $\mathbb{R}_+$ and satisfies for all $s > 0$,

$$-\frac{d}{ds} P_s f = \sum_{i \geq 1} P_s D_i f \,.$$

(ii) (Commutativity) $P_t \circ D_i = D_i \circ P_t$ for all $i \geq 1$ and all $t > 0$.

(iii) (Integration by parts) For all $f, g \in L^2(\Omega)$ and all $i \geq 1$,

$$\mathbb{E}_p\big[f D_i g\big] = -\mathbb{E}_p\big[D_i f D_i g\big]$$

(iv) (Time-decorrelation) There exists a constant C_p such that for all $f \in L^2(\Omega), i \geq 1$ and $t > 0$,

$$|P_t D_i f| \leq e^{-t} C_p P_t |D_i f| \,.$$

(v) (Decoupling at infinity) $P_t f \to \mathbb{E}_p[f]$ a.s. and in L^1 when $t \to +\infty$.

Remark 3.1. In that proposition, names are given to the listed operator properties. The names of Heat equation, Commutativity and Integration by parts are linked to the parallel with Markovian dynamics and would be valid for any operators P_t corresponding to a Markov semigroup and operators D_i satisfiying differential calculus definitions. More precisely, $\sum_i D_i$ is analogous to a Laplacian Δ and D_i to an operator of partial differentiation in direction i, ∇_i. Then, $\frac{d}{dt} P_t = \Delta \circ P_t = P_t \circ \Delta$ is an expected property as well as $P_t \circ D_i = D_i \circ P_t$ (which is only true here due to the product structure of the measure, it would not be true for a Glauber dynamic associated to the Ising model for example), and $\mathbb{E}[f \nabla_i g] = -\mathbb{E}[\nabla_i f \nabla_i g]$. Time-decorrelation is a consequence of the fact that the dynamics forgets more and more information about the initial configuration. It also can be interpreted as the property that the dynamic interpolates between the initial configuration and an independent copy (Z_t is that interpolation from X to Y). It would be the same with an Ornstein-Uhlenbeck process where $Z_t = e^{-t} X + \sqrt{1 - e^{-2t}} Y$. So, the decoupling at infinity is a consequence of that.

Proof of Proposition 3.1. Let $p \in (0,1)$ be fixed throughout the proof. We first prove (i). Let f be in $L^2(\Omega)$. Using the assumption that f is local, let m be an integer such that f only depends on $(\omega_i)_{1 \leq i \leq m}$. We write, for fixed s, and ds is a positive quantity destined to tend to 0,

$$P_{s+ds} f - P_s f = \mathbb{E}_p\big[f(\omega^{s+ds}) - f(\omega^s) | \omega^0\big] \,.$$

On the event that no coordinate of ω^s is resampled between s and $s+ds$, $f(\omega^s)$ equals $f(\omega^{s+ds})$ and events where more than 1 coordinate is resampled in $\{1, \ldots, m\}$ have probability $o(ds)$. Hence, for any $i \in \{1, \ldots, m\}$, writing as R_i the event "the set of coordinates of index smaller than m resampled between s and $s + ds$ is exactly {i}",

$$P_{s+ds} f - P_s f = \sum_{i=1}^{m} \mathbb{E}_p\big[\mathbb{1}_{R_i}\big(f(\omega^{s+ds}) - f(\omega^s)\big) | \omega^0\big] + o(ds) \,.$$

Given ω^s and R_i, ω^{s+ds} is $\sigma_i^1(\omega^s)$ with probability p and $\sigma_i^0(\omega^s)$ with probability $(1-p)$, so that

$$\mathbb{E}_p\big[f(\omega^{s+ds})|\omega^s, R_i\big] = \big(pf \circ \sigma_i^1(\omega^s) + (1-p)f \circ \sigma_i^0(\omega^s)\big)\ .$$

By independence of ω^s and R_i,

$$\mathbb{E}_p\big[f(\omega^{s+ds})|R_i, \omega^s\big] = \frac{\mathbb{E}_p[f(\omega^{s+ds})\mathbb{1}_{R_i}|\omega^s]}{\mathbb{P}(R_i|\omega^s)}\ ,$$

and substracting $\mathbb{E}_p\big[f(\omega^s)|R_i, \omega^s\big]$ to both sides of that equality, multiplying by $\mathbb{P}_p(R_i|\omega^s)$ and using (7) yields

$$\mathbb{E}_p\big[\mathbb{1}_{R_i}\big(f(\omega^{s+ds}) - f(\omega^s)\big)|\omega^s\big] = \mathbb{P}_p(R_i)D_i f(\omega^s)\,.$$

Taking the conditional expectation with respect to ω^0, it follows that

$$\mathbb{E}_p\big[\mathbb{1}_{R_i}\big(f(\omega^{s+ds}) - f(\omega^s)\big)|\omega^0\big] = \mathbb{P}_p(R_i)\mathbb{E}_p\big[D_i f(\omega^s)|\omega^0\big]\,.$$

By definition of the Poisson point process, the probability that ω_i is the only resampled coordinate among $(\omega_j)_{1\le j\le m}$ between s and $s+ds$ is $(1-e^{-ds})e^{-(m-1)ds} = ds(1+o(1))$. Summing over i yields

$$P_{s+ds}f - P_s f = \sum_{i=1}^m (ds)\mathbb{E}\big[D_i f(\omega^s)|\omega^0\big] + o(ds)\,.$$

As a consequence, $s \mapsto P_s f$ is right-differentiable with the announced expression for the right derivative, and the same method of proof can be used for dealing with the left-derivative.

For the proofs of items (ii) to (iv), let i be a fixed positive integer, $t > 0$ a real number and $f, g \in L^2(\Omega)$. Let also $(X_j)_{j\ge1}, (Y_j)_{j\ge1}, (Y'_j)_{j\ge1}$ denote three sequences of Bernoulli random variables of parameter p.
For item (ii), we use the definitions of P_t and D_i (expressions (5) and (7)) to write on the one hand

$$\begin{aligned} D_i P_t f(\omega) &= \mathbb{E}_p\big[P_t f\big(\sigma_i^{Y_i}(\omega)\big)\big] - P_t f(\omega) \\ &= \mathbb{E}_p\big[\mathbb{E}_p\big[f(\omega^t)\big|\omega^0 = \sigma_i^{Y_i}(\omega)\big]\big] - P_t f(\omega)\,, \end{aligned}$$

and on the other hand

$$\begin{aligned} P_t D_i f(\omega) &= \mathbb{E}_p\big[D_i f(\omega^t)\big|\omega^0 = \omega\big] \\ &= \mathbb{E}_p\big[\mathbb{E}_p\big[f\big(\sigma_i^{Y_i}(\omega^t)\big)\big|\omega^t\big]\big|\omega^0 = \omega\big] - P_t f(\omega) \\ &= \mathbb{E}_p\big[f(\sigma_i^{Y_i}(\omega^t)|\omega^0 = \omega\big] - P_t f(\omega)\,. \end{aligned}$$

The law of ω^t started from $\omega^0 = \sigma_i^{Y_i}(\omega)$ is the same as the law of $\sigma_i^{Y_i}(\omega^t)$ started from $\omega^0 = \omega$. Indeed, ω^t started from $\omega^0 = \sigma_i^{Y_i}(\omega)$ and $\sigma_i^{Y_i}(\omega^t)$ started from $\omega^0 = \omega$ are equal at coordinates other than i and Y_i is a Bernoulli random variable of parameter p, which has time-invariant distribution, so that Y_i has the same distribution as ω_i^t started from $\omega^0 = Y_i$. From this we deduce

$$\mathbb{E}_p\big[\mathbb{E}_p\big[f(\omega^t)\big|\omega^0 = \sigma_i^{Y_i}(\omega)\big]\big] = \mathbb{E}_p\big[f(\sigma_i^{Y_i}(\omega^t)|\omega^0 = \omega\big]$$

and therefore item (ii).

For item (iii), note that the couple $(X, \sigma_i^{Y_i}(X))$ has the same distribution as the couple $(\sigma_i^{Y'_i}(X), \sigma_i^{Y_i}(X))$. As a consequence,

$$\mathbb{E}_p\big[f(\sigma_i^{Y_i}(X))g(X)\big] = \mathbb{E}_p\big[f(\sigma_i^{Y_i}(X))g(\sigma_i^{Y'_i}(X))\big]\,.$$

It follows that

$$\mathbb{E}_p\big[f(\sigma_i^{Y_i}(X))\big(g(\sigma_i^{Y_i'}(X)) - g(X)\big)\big] = 0\,,$$

which rewrites as $\mathbb{E}_p[(f + D_i f) D_i g] = 0$, hence (iii).

For the proof of (iv), let (ω^t) be a p-biased continuous-time simple random walk started from $\omega^0 = \omega$ and let T_i be the first resampling time of coordinate i. In particular, it is independent of ω^0 and $|f(\sigma_i(\omega^t)) - f(\omega^t)|$. First write $P_t D_i f(\omega) = \mathbb{E}_p\big[(\mathbb{1}_{\{T_i<t\}} + \mathbb{1}_{\{T_i \geq t\}}) D_i f(\omega^t)|\omega^0 = \omega\big]$ and note that, given $T_i < t$, ω^t and $\sigma_i^{Y_i}(\omega^t)$ have the same distribution, so that

$$\mathbb{E}_p\big[\mathbb{1}_{\{T_i<t\}} D_i f(\omega^t)\big|\omega^0 = \omega\big] = \mathbb{E}_p\big[\mathbb{1}_{\{T_i<t\}}\big]\mathbb{E}_p\big[D_i f(\omega^t)\big|T_i < t, \omega^0 = \omega\big] = 0\,.$$

Then,

$$|P_t D_i f(\omega)| = |\mathbb{E}_p\big[\mathbb{1}_{\{T_i\geq t\}} D_i f(\omega^t)\big|\omega^0 = \omega\big]\,.$$

By the triangle inequality and writing $D_i f(\omega^t) = \big((1-p)\omega_i^t + p(1-\omega_i^t)\big)\big(f(\sigma_i(\omega^t)) - f(\omega^t)\big)$, it follows that

$$|P_t D_i f(\omega)| \leq \mathbb{E}_p\big[\mathbb{1}_{\{T_i\geq t\}} \max(p, 1-p)|f(\sigma_i(\omega^t)) - f(\omega^t)|\big|\omega^0 = \omega\big]\,,$$

and $|f(\sigma_i(\omega^t) - f(\omega^t)|$ is independent of (ω_i^t), especially it is independent of T_i, so that

$$|P_t D_i f(\omega)| \leq e^{-t} \max(p, 1-p)\mathbb{E}_p\big[|f(\sigma_i(\omega^t)) - f(\omega^t)|\big|\omega^0 = \omega\big]\,.$$

Finally, getting back $(1-p)\omega_i^t + p(1-\omega_i^t)$ inside the expectation up to dividing by $\min(p, 1-p)$, we obtain the desired inequality.

Finally, we prove (v). By properties of the Poisson point process, the probability that at least one coordinate is never resampled among the finite set of coordinates determining f is 0, which implies that $P_t f$ converges to $\mathbb{E}[f]$ almost surely. For the L^1 convergence, use that f is bounded and that $P_t f$ equals $E[f]$ as soon as all the coordinates have been resampled at least once, and that the probability that this has not happened at time t decays to 0 as t goes to infinity.

□

3.2 Hypercontractivity

The remaining properties we need for generalizing the BKS theorem exclusively involve the operators $(P_t)_{t>0}$. This operator satisfies an inequality of **hypercontractivity**, which is a "smoothing" property. To explain this further, the larger a real $\alpha \geq 1$ is, the "rougher" is the α-norm (defined here by $\|f\|_\alpha = \mathbb{E}_p[|f|^\alpha]^{\frac{1}{\alpha}}$), in particular, controlling the L^2-norm of a function is a stronger result than controlling the L^α norm of the same function for some $\alpha < 2$. Hypercontractivity states that the operator P_t has a smoothing effect on functions, which means that a function that might not be that regular, which means that we might only know a bound of its L^α norm for a $\alpha < 2$, can indeed be controlled in L^2 norm using the same estimates after applying P_t for large enough t. We use notations similar to van Handel [RH20], with θ_p standing for the hyperbolic tangent function with a proper scaling depending on p, i.e. let

$$\rho(p) := 2\frac{2p-1}{\log(p) - \log(1-p)}$$

when $p \neq \frac{1}{2}$ and $\rho(\frac{1}{2}) := 1$, and

$$\theta_p(t) := \tanh(2\rho(p)t) = \frac{1 - e^{-2\rho(p)t}}{1 + e^{-2\rho(p)t}}\,. \tag{11}$$

Note that $\rho(p) > 0$ for all $p \in (0,1)$, so that $\theta_p(t) > 0$ for all $t > 0$ and $p \in (0,1)$.

Lemma 3.1. *For all $p \in (0,1)$, $t > 0$ and $f \in L^2(\Omega)$,*

$$\mathbb{E}_p\left[(P_t f)^2\right] \le \mathbb{E}_p\left[|f|^{1+e^{-2\rho t}}\right]^{\frac{2}{1+e^{-2\rho t}}} .$$

Proof. This is a consequence of the logarithmic Sobolev inequalities for product Bernoulli measures on the hypercube, as mentioned in [CL12], page 5. □

Corollary 3.1. *For any $p \in (0,1)$, $t > 0$ and $f \in L^2(\Omega)$,*

$$\mathbb{E}_p\left[(P_t f)^2\right] \le \mathbb{E}_p\left[f^2\right]^{1-\theta_p(t)} \mathbb{E}_p\left[|f|\right]^{2\theta_p(t)} .$$

Proof. Let $p \in (0,1)$, $t > 0$ and $f \in L^2(\Omega)$. Let ρ denote $\rho(p)$. By Lemma 3.1,

$$\mathbb{E}_p\left[(P_t f)^2\right] \le \mathbb{E}_p\left[|f|^{1+e^{-2\rho t}}\right]^{\frac{2}{1+e^{-2\rho t}}} .$$

Applying Hölder's inequality to the functions $|f|^{2e^{-2\rho t}}$, $|f|^{1-e^{-2\rho t}}$ and to the conjugate exponents $e^{2\rho t}$, $(1-e^{-2\rho t})^{-1}$, we obtain

$$\mathbb{E}_p\left[|f|^{1+e^{-2\rho t}}\right] \le \mathbb{E}_p\left[|f|^2\right]^{e^{-2\rho t}} \mathbb{E}_p\left[|f|\right]^{1-e^{-2\rho t}} .$$

Plugging this inequality into the right-hand side of the first one finally yields

$$\mathbb{E}_p\left[(P_t f)^2\right] \le \mathbb{E}_p\left[|f^2|\right]^{\frac{2e^{-2\rho t}}{1+e^{-2\rho t}}} \mathbb{E}_p\left[|f|\right]^{2\frac{1-e^{-2\rho t}}{1+e^{-2\rho t}}}$$

which is the desired inequality. □

3.3 Proof of Theorem 1.3

The following theorem is essentially Theorem 1.3, it is stated using the additional notation that we introduced in Sections 2 and 3. Especially, θ_p is defined by (11) and C_p by item (iv) of Proposition 3.1.

Theorem 3.1. *Let $f \in L^2(\Omega)$. For all $p \in (0,1)$, for all $t > 0$,*

$$\mathrm{Var}_p(P_t f) \le \left(e^{-2t}\mathrm{Var}_p(f)\right)^{1-\theta_p(t)} \left(C_p \sum_{i \ge 1} \mathrm{Inf}_i^p(f)^2\right)^{\theta_p(t)} .$$

Proof. Let $p \in (0,1), t > 0$ and $f \in L^2(\Omega)$. Item (v) of Proposition 3.1 gives $\mathbb{E}_p[P_t f] = \mathbb{E}_p[f] = P_\infty f$, so that we can express $\mathrm{Var}_p(P_t f)$ as

$$\mathrm{Var}_p(P_t f) = \mathbb{E}_p\left[(P_t f)^2\right] - \mathbb{E}_p\left[(P_\infty f)^2\right] .$$

By item (i) of Proposition 3.1, $t \mapsto P_t f$ is differentiable and its derivative expresses as a finite sum of functions $t \mapsto P_t D_i f$. Using (i) again, we see that functions of the form $t \mapsto P_t D_i f$ are also differentiable, thus continuous. This justifies that $t \mapsto (P_t f)$ is of class C^1 and so is $t \mapsto (P_t f)^2$. Finally, by item (v), $\mathbb{E}_p[(P_t f)^2] \to \mathbb{E}_p\left[(P_\infty f)^2\right]$ when $t \to +\infty$. Applying the fundamental theorem of calculus to $t \mapsto (P_t f)^2$, we obtain

$$\mathbb{E}_p\left[(P_t f)^2 - (P_\infty f)^2\right] = -\mathbb{E}_p\left[\int_t^{+\infty} \frac{d}{ds}(P_s f)^2 ds\right] .$$

By the chain rule and item (i) of Proposition 3.1, it follows that

$$\begin{aligned}\mathrm{Var}_p(P_t f) &= -\mathbb{E}_p\left[\int_t^{+\infty} 2\left(\frac{d}{ds}P_s f\right) P_s f\right] ds \\ &\overset{\text{(i)}}{=} -2\mathbb{E}_p\left[\int_t^{+\infty} \sum_{i\geq 1}(P_s D_i f)(P_s f)\right] ds\end{aligned}$$

Since f is local, $P_s f$ and $P_s D_i f$ are local as well for all $i \geq 1$ and $s \geq 0$. In particular they are bounded. Moreover, $D_i f$ is 0 if f doesn't depend on i, so that the above sum is finite. By linearity of the expectation and the integral and Fubini's theorem,

$$\mathrm{Var}_p(P_t f) = -2\sum_{i\geq 1}\int_t^{+\infty} \mathbb{E}_p\left[(P_s D_i f)P_s f\right] .$$

Then, using Proposition 3.1 again and the semigroup property (6),

$$\begin{aligned}\mathrm{Var}_p(P_t f) &\overset{\text{(ii)}}{=} -2\sum_{i\geq 1}\int_t^{+\infty} \mathbb{E}_p\left[(D_i P_s f)P_s f\right] \\ &\overset{\text{(iii)}}{=} 2\sum_{i\geq 1}\int_t^{+\infty} \mathbb{E}_p\left[(D_i P_s f)^2\right] \\ &\overset{\text{(ii)}}{=} 2\sum_{i\geq 1}\int_t^{+\infty} \mathbb{E}_p\left[(P_s D_i f)^2\right] \\ &\overset{(6)}{=} 2\sum_{i\geq 1}\int_t^{+\infty} \mathbb{E}_p\left[(P_t P_{s-t} D_i f)^2\right]\end{aligned}$$

Applying Corollary 3.1 to $P_{s-t}D_i f$ in the last line yields

$$\mathrm{Var}_p(P_t f) \leq \sum_{i\geq 1}\int_t^{+\infty} 2\mathbb{E}_p\left[(P_{s-t}D_i f)^2\right]^{1-\theta_p(t)}\mathbb{E}_p\left[|P_{s-t}D_i f|\right]^{2\theta_p(t)} ds$$

By Hölder's inequality for vector-valued functions, it follows that

$$\mathrm{Var}_p(P_t f) \leq \left(\sum_{i\geq 1}\int_t^{+\infty} 2\mathbb{E}_p\left[(P_{s-t}D_i f)^2\right]\right)^{1-\theta_p(t)} \left(\sum_{i\geq 1}\int_t^{+\infty} 2\mathbb{E}_p\left[|P_{s-t}D_i f|\right]^2 ds\right)^{\theta_p(t)} .$$

Now, apply (iv) of Proposition 3.1 but only to the right factor of the right-hand side, as the other one turns out to be the variance of f by the cascade of computations presented earlier in this proof, in the case where $t = 0$. We obtain

$$\mathrm{Var}_p(P_t f) \leq (\mathrm{Var}_p(f))^{1-\theta_p(t)} \left(\int_t^{+\infty} 2e^{-2(s-t)} C_p \sum_{i\geq 1}\mathbb{E}_p\left[|D_i f|\right]^2\right)^{\theta_p(t)} ds\,.$$

Integrating e^{-2s} over s and rearranging the factors yields the desired inequality. □

4 Noise sensitivity in first-passage percolation

Let $\mathbb{Z}^2$ denote the square lattice and $E(\mathbb{Z}^2)$ its set of edges. Let a, b be two real numbers such that $0 < a < b$, fixed throughout the section. Consider $(\omega_e)_{e\in E(\mathbb{Z}^2)}$ iid random variables, uniformly distributed in $\{a, b\}$. For any path of edges Γ, we let $T(\Gamma)$ be the random variable defined as the total weight picked up by the path, that is, the sum over the edges e belonging to the path of ω_e. First-passage percolation is the study of the random metric defined on $\mathbb{Z}^2$ by

$$T(x, y) = \min_{\text{paths } \Gamma \text{ from } x \text{ to } y} T(\Gamma)\,.$$

We let $\mathcal{L}_n$ be the set of horizontal lines from left to right in the square $[0, n]^2$, so that $\mathcal{L}_n = \{[0, n] \times \{i\}, 0 \le i \le n\}$. We let $\mathcal{P}_k(n)$ be the set of lattice paths from left to right, included in $[0, n]^2$ intersecting at most k horizontal lines in $\mathcal{L}_n$ as geometric closed subsets of $\mathbb{R}^2$. These paths are paths with vertical fluctuation bounded by k, because it means that the downmost vertex of the path and its topmost vertex differ by less than k in ordinate. Note that $\mathcal{L}_n$ has $n+1$ elements so $\mathcal{P}_k(n+1)$ is the set of left-right paths in the square of side n with unbounded fluctuation. Then, define for all positive integers n, k,

$$\tau(n, k) = \min_{\Gamma\in\mathcal{P}_k(n)} T(\Gamma)\,,$$

Any path satisfying the minimum over $\mathcal{P}_k(n)$ above will be called a geodesic and the intersection of all geodesics is denoted by $\pi(n, k)$. We call these minimal total weights over a set of paths travel times because ω can be interpreted as a time to cross an edges, so the minimum over paths from a geometric set to an other one can be interpreted as minimal travel time between these sets.

A natural "observable" in first-passage percolation is the time travel from 0 to a point at distance of order n, typically ne_1 where e_1 is the first vector of the canonical basis. We did not succeed in proving noise sensitivity of the sequence of functions $(T(0, ne_1))_n$, nor of the sequence $(\tau(n, n+1))_n$ (the travel time from left to right in the square with unbounded fluctuations). It appears that upper-bounding the influences of quantities such as $T(x, y)$ or $\tau(n, k)$ can be reduced to upper-bounding the probability of edges to belong to a geodesic. In the square, the geodesic is expected to go straight from left to right, seen from afar, and with close to uniform vertical position, so that edges have roughly probability $\frac{1}{n}$ to belong to a geodesic. Moreover, the variance of the left-right travel time with unbounded fluctuation $\tau(n, n+1)$ is expected to go to infinity like $n^{\frac{2}{3}}$. As a consequence, it is expected that

$$\frac{\sum_{e\in E(\mathbb{Z}^2)} \mathrm{Inf}_e(\tau(n, n+1))^2}{\mathrm{Var}(\tau(n, n+1))} \approx \frac{n^2\left(\frac{1}{n}\right)^2}{n^{\frac{2}{3}}} \approx n^{-\frac{2}{3}}\,.$$

Let us be more precise on the description of the geodesics. The geodesics may be assumed to be self-avoiding paths because erasing a loop can only reduce the total weight of the path. However, it can and will happen that the geodesic cross multiple times the same vertical line. This should not happen too much and we might be able to get better bounds on the influence if we manage to prove that a single vertical line has very low probability to be crossed more than a constant number of times. More specifically, it is easy to show on the full torus (gluing the left side to the right side and the top side to the bottom side) that the influences are of order $\frac{1}{n}$. This bound on the number of crossing of a vertical line could be useful to bound the influences in the initial square by the influences in the torus. So we might be able with the available tools to prove

$$\sum_{e\in E(\mathbb{Z}^2)} \mathrm{Inf}_e(\tau(n, n+1))^2 \le Cn^2\left(\frac{1}{n}\right)^2 \le C\,.$$

However, even in that case, we lack a lower bound on the variance of $\tau(n, n+1)$, the best known lower bound is a constant and does not go to infinity with n, which makes this estimate together with Theorem 1.3 not strong enough to prove noise sensitivity of $\tau(n, n+1)$.

The polynomial bound in Theorem 1.2 is the same as in [AR23] as we use the same intermediate results as in this paper for bounding the influences and variance of $\tau(n, k_n)$ and its variance, in order to apply Theorem 1.3. More precisely, we derive an upper-bound on the sum of influences squared of $\tau(n, k_n)$ and a lower-bound on its variance. The upper-bound directly stems from bounding the fluctuations by k_n and using the near translation symmetry of the square, this is the reason behind the introduction of the square-band and the square-band geodesics in the upcoming proof of Lemma 4.1.

Lemma 4.1. *Let n, k be positive integers. For all $e \in E(\mathbb{Z}^2) \cap [0, n]^2$,*

$$\mathrm{Inf}_e\big(\tau(n,k)\big) \leq 4(b-a)\frac{k}{n}\,.$$

Proof. The influence of an edge being controlled by the probability that this edge belongs to the geodesic, this is what needs to be bounded. We use symmetry between lines of the square with respect to first-passage percolation on the so-called square band and the fact that the geodesic with bounded fluctuations can cross at most k of these lines to bound the probability of belonging to the square-band-geodesic, and a simple union bound trick is sufficient to come back the actual probability of belonging to the geodesic in the regular square.

Let $n, k \in \mathbb{N}^*$ and let e be an edge inside the square $[0, n]^2$. Remember that $\tau(n, k)$ is the minimum of $T(\Gamma)$ over $\Gamma \in \mathcal{P}_k(n)$. Let $\omega \in \{a, b\}^{E(\mathbb{Z}^2)}$ be such that $\omega_e = a$ and let $\tilde{\omega}$ be equal to ω except at e where $\tilde{\omega}_e = b$. We have $|D_e\tau(n,k)(\omega)| = |D_e\tau(n,k)(\tilde{\omega})| = |\tau(n,k)(\omega) - \tau(n,k)(\tilde{\omega})|$. If there is $\Gamma_{\min} \in \mathcal{P}_k(n)$ that is geodesic for $\tau(n,k)$ and such that $e \notin \Gamma_{\min}$, then this path is also of minimal length in $\mathcal{P}_k(n)$ for $\tilde{\omega}$, so that $|D_e\tau(n,k)(\omega)| = 0$. Otherwise, that is, if $e \in \pi(n,k)$, any geodesic path $\Gamma_{\min}$ satisfies $T(\Gamma_{\min})(\tilde{\omega}) = T(\Gamma_{\min})(\omega) + (b-a)$ because the only edge of $\Gamma_{\min}$ which changed value from ω to $\tilde{\omega}$ is e and it went from a to b. Thus, $\tau(n,k)(\tilde{\omega}) \leq \tau(n,k)(\omega) + (b-a)$. It follows that, for all $\omega \in \{a,b\}^{\mathbb{Z}^2}$,

$$|D_e\tau(n,k)(\omega)| \leq (b-a)\big(\mathbb{1}_{\{e\in\pi(n,k)\}}(\omega) + \mathbb{1}_{\{e\in\pi(n,k)\}}(\sigma_e(\omega))\big)\,.$$

Hence,

$$\mathrm{Inf}_e\big(\tau(n,k)\big) \leq 2(b-a)\mathbb{P}(e \in \pi(n,k))\,. \tag{12}$$

Let the **square band** of side n be the graph $\mathbb{G}_n$ with vertex set $\mathbb{Z}^2 \cap [0,n]^2$ and having as edge set $E(\mathbb{G}_n)$ the set of regular edges of the square lattice, to which are added edges between vertices at the top of the square to the bottom ones with respective abscissa. We sample additional iid random variables ω_e attached to the new edges, and $(\omega_e)_{e\in E(\mathbb{G}_n)}$ is thus a family of iid random variables, uniform on $\{a,b\}$, coupled to $(\omega_e)_{e\in E(\mathbb{Z}^2)}$ by identification of the edges inside the square. We also let $\mathcal{P}_k^{\mathbb{G}}(n)$ be the set of left-right paths in $\mathbb{G}_n$ intersecting with at most k elements of $\mathcal{L}_n$, and

$$\tau^{\mathbb{G}}(n,k) = \min_{\Gamma\in\mathcal{P}_k^{\mathbb{G}}(n)} T(\Gamma)\,.$$

Any path satisfying the second minimum is called square-band-geodesic and the intersection of all square-band-geodesics is denoted by $\pi^{\mathbb{G}}(n,k)$.

We use the symmetry of the square using the coupling with first-passage percolation on the square-band $\mathbb{G}_n$ to upper-bound $\mathbb{P}\big(e \in \pi(n,k)\big)$. First, we bound it by twice the probability that the geodesic in $\mathbb{G}_n$ crosses a vertical line, and then we use that the geodesics have bounded fluctuations to complete the proof. For $0 \le j \le n-1$, let E_j be the event that $\pi^{\mathbb{G}}(n,k)$ intersects $[0,n]\times\{j\}$ as closed subsets of $\mathbb{R}^2$. Note that the distribution of $(\omega_e)_{e\in E(\mathbb{G})}$ is invariant under vertical translations by integers, i.e. maps from $\mathbb{G}$ to itself of the form $(i,j)\mapsto(i,(j+z) \bmod n)$ for some $z\in\mathbb{Z}$, so that $\mathbb{P}(E_j)$ is independent of j. The geodesics having fluctuations bounded by k, there can be at most k integers $j\in\{0,1,\dots,n-1\}$ satisfying E_j, so that

$$\sum_{j=0}^{n-1} \mathbb{1}_{E_j} \le k\,.$$

Taking the expectation and using that $\mathbb{P}(E_j)$ doesn't depend on j, it follows that

$$n\mathbb{P}(E_0) \le k\,. \tag{13}$$

If $e\in\pi(n,k)$, then any path in $\mathbb{G}$ that does not go through e nor through the line $[0,n]\times\{0\}$ has a total weight larger than $\tau(n,k)$. As a consequence, $e\in\pi(n,k)$ implies that any geodesic path in $\mathbb{G}$ goes either through e or through the line $[0,n]\times\{0\}$ since its total weight has to be at most $\tau(n,k)$. Hence, letting j be the index of a line in $\mathcal{L}_n$ that e intersects,

$$\mathbb{P}\big(e\in\pi(n,k)\big) \le \mathbb{P}\big(e\in\pi^{\mathbb{G}}(n,k)\big) + \mathbb{P}(E_0) \le \mathbb{P}(E_j)+\mathbb{P}(E_0) = 2\mathbb{P}(E_0)\,. \tag{14}$$

The desired inequality follows from

$$\begin{aligned}\mathrm{Inf}_e\big(\tau(n,k)\big) &\overset{(12)}{\le} 2(b-a)\mathbb{P}\big(e\in\pi^{\mathbb{G}}(n,k)\big)\\ &\overset{(14)}{\le} 4(b-a)\mathbb{P}(E_0)\\ &\overset{(13)}{\le} 4(b-a)\frac{k}{n}\,.\end{aligned}$$

□

Remark 4.1. Lemma 4.1 is clearly not sharp, we would expect $\mathbb{P}\big(e\in\pi(n,k)\big)\le\frac{C}{n}$ with C a constant independent of n, k and e. The main part where we lose sharpness is by bounding the probability that e belongs to the geodesic by the probability that the geodesic crosses a horizontal line where e lies. One would need to keep treating differently edges in the same line in order to obtain the $\frac{C}{n}$ bound.

For bounding the variance, we use the following theorem from Ahlberg and de la Riva [AR23] relying on moderate deviations and results of first-passage percolation crossing times by Chatterjee and Dey [CD13]. The bound on the rate of growth of vertical fluctuations $k_n\le n^\alpha$ with $\alpha<\frac{1}{22}$ is also the upper-bound for which the authors of [AR23] obtain noise sensitivity of being above the median for $\tau(n,k_n)$. We could perhaps obtain the result for an exponent larger than $\frac{1}{22}$, this would mainly require to be more careful on the influence bound. However, a polynomial growth rate for k_n of order $n^{\frac{2}{3}}$ would still be unattainable, the reason being that we cannot provide a good enough lower-bound for the variance of $\tau(n,k_n)$ when k_n is too large.

Theorem 4.1 (Theorem 1.2 of [AR23])**.** *Let $(k_n)_{n\ge1}$ be a sequence of integers such that $k_n=O(n^\alpha)$ for some $\alpha<\frac{1}{22}$. There exists $c>0$ such that*

$$\sup_{x\ge0}\mathbb{P}(\tau(n,k_n)\in[x,x+c]) = o\left(\frac{1}{n^{1/22}}\right)\,.$$

Proof. This is exactly Theorem 1.2 of [AR23]. □

Lemma 4.2. *For any sequence (k_n) such that $k_n = O(n^\alpha)$ for some $\alpha < \frac{1}{22}$,*

$$\frac{\mathrm{Var}\big(\tau(n,k_n)\big)}{n^{\frac{1}{11}}} \underset{n\to+\infty}{\longrightarrow} +\infty\,.$$

Proof. Let (k_n) be a sequence of integers such that $k_n = O(n^\alpha)$ for some $\alpha < \frac{1}{22}$. By Theorem 4.1, let $c > 0$ be such that

$$\sup_{x>0} \mathbb{P}(\tau(n,k_n) \in [x, x+c]) = o(n^{-\frac{1}{22}})\,.$$

Setting $\varepsilon_n = \sqrt{\big(\sup_{x>0}\mathbb{P}(\tau(n,k_n)\in[x,x+c])\big)n^{-\frac{1}{22}}}$, we have $\sup_{x>0}\mathbb{P}(\tau(n,k_n)\in[x,x+c]) = o(\varepsilon_n)$ and $\varepsilon_n = o(n^{-\frac{1}{22}})$. By Chebyshev's inequality,

$$\mathrm{Var}\big(\tau(n,k_n)\big) \geq c^2\varepsilon_n^{-2}\mathbb{P}\left(\left|\tau(n,k_n) - \mathbb{E}\big(\tau(n,k_n)\big)\right| \geq c\varepsilon_n^{-1}\right).$$

Decomposing the interval $\big[\mathbb{E}[\tau(n,k_n)] - cn^{\frac{1}{22}}, \mathbb{E}[\tau(n,k_n)] + cn^{\frac{1}{22}}\big]$ into $\lfloor \frac{2}{\varepsilon_n}\rfloor$ intervals $J_i, 1 \leq i \leq \lfloor 2/\varepsilon_n \rfloor$ of length at most c and using the upper-bound on the probability of $\tau(n,k_n)$ to belong to an interval of length c together with a union bound on the smaller intervals, we obtain

$$\mathrm{Var}\big(\tau(n,k_n)\big) \geq c^2\varepsilon_n^{-2}\big(1 - \mathbb{P}\big(\bigcup_{i=1}^{\lfloor 2/\varepsilon_n\rfloor}\{\tau(n,k_n)\in J_i\}\big)\big) \geq c^2\varepsilon_n^{-2}\big(1 - \lfloor 2/\varepsilon_n\rfloor o(\varepsilon_n)\big)\,.$$

Thus,

$$\frac{\mathrm{Var}(\tau(n,k_n))}{n^{1/11}} \geq c^2\frac{n^{-1/11}}{\varepsilon_n^2}\big(1-o(1)\big)\,,$$

and the divergence of $\frac{\mathrm{Var}(\tau(n,k_n))}{n^{\frac{1}{11}}}$ to $+\infty$ follows from $\varepsilon_n = o(n^{-\frac{1}{22}})$. □

The proof of Theorem 1.2 follows from Lemmas 4.1 and 4.2.

Proof of Theorem 1.2. Let (k_n) be a sequence satisfying $k_n = O(n^\alpha)$ for some $\alpha \in [0, \frac{1}{22})$. For any $n \geq 0$, summing Lemma 4.1 over all the edges with both ends in $[0,n]\times[0,n)$ yields

$$\sum_{e\in[0,n]\times[0,n)} \mathrm{Inf}_e\big(\tau(n,k)\big)^2 \leq 4n^2(b-a)^2\frac{k_n^2}{n^2} = O(n^{\frac{1}{11}})\,.$$

It follows from Lemma 4.2 that $(\tau(n,k_n))_{n\geq 0}$ is a sequence of functions having positive variance and satisfying

$$\frac{\sum_{i=1}^n \mathrm{Inf}_i^p\big(\tau(n,k_n)\big)^2}{\mathrm{Var}_p\big(\tau(n,k_n)\big)} \underset{n\to+\infty}{\rightarrow} 0\,.$$

We deduce from Theorem 1.3 that $(\tau(n,k_n))_{n\geq 0}$ is noise sensitive. □

References

[ADS23] Daniel Ahlberg, Maria Deijfen, and Matteo Sfragara. *Chaos, concentration and multiple valleys in first-passage percolation.* 2023. arXiv: 2302.11367 [math.PR].

[ADS24] Daniel Ahlberg, Maria Deijfen, and Matteo Sfragara. "From stability to chaos in last-passage percolation". English. In: *Bull. Lond. Math. Soc.* 56.1 (2024), pp. 411–422. DOI: 10.1112/blms.12941.

[AR23] Daniel Ahlberg and Daniel de la Riva. *Is 'being above the median' a noise sensitive property?* 2023. arXiv: 2308.16388 [math.PR].

[BKS99] Itai Benjamini, Gil Kalai, and Oded Schramm. "Noise sensitivity of Boolean functions and applications to percolation". In: *Publ. Math., Inst. Hautes Étud. Sci.* 90 (1999). DOI: 10.1007/BF02698830.

[BKS03] Itai Benjamini, Gil Kalai, and Oded Schramm. "First passage percolation has sublinear distance variance." English. In: *Ann. Probab.* 31.4 (2003), pp. 1970–1978. DOI: 10.1214/aop/1068646373.

[Cha08] Sourav Chatterjee. *Chaos, concentration, and multiple valleys.* 2008. arXiv: 0810.4221 [math.PR].

[Cha09] Sourav Chatterjee. *Disorder chaos and multiple valleys in spin glasses.* 2009. arXiv: 0907.3381 [math.PR].

[Cha14] Sourav Chatterjee. *Superconcentration and related topics.* English. Springer Monogr. Math. Cham: Springer, 2014. DOI: 10.1007/978-3-319-03886-5.

[CD13] Sourav Chatterjee and Partha S. Dey. "Central limit theorem for first-passage percolation time across thin cylinders". English. In: *Probab. Theory Relat. Fields* 156.3-4 (2013), pp. 613–663. DOI: 10.1007/s00440-012-0438-z.

[CL12] Dario Cordero-Erausquin and Michel Ledoux. "Hypercontractive measures, Talagrand's inequality, and influences". English. In: *Geometric aspects of functional analysis. Proceedings of the Israel seminar (GAFA) 2006–2010.* Berlin: Springer, 2012, pp. 169–189. DOI: 10.1007/978-3-642-29849-3_10.

[EG22] Ronen Eldan and Renan Gross. "Concentration on the Boolean hypercube via pathwise stochastic analysis". English. In: *Invent. Math.* 230.3 (2022), pp. 935–994. DOI: 10.1007/s00222-022-01135-8.

[FS07] Dvir Falik and Alex Samorodnitsky. "Edge-isoperimetric inequalities and influences". English. In: *Comb. Probab. Comput.* 16.5 (2007), pp. 693–712. DOI: 10.1017/S0963548306008340.

[GPS10] Christophe Garban, Gábor Pete, and Oded Schramm. "The Fourier spectrum of critical percolation". In: *Acta Math.* 205.1 (2010). DOI: 10.1007/s11511-010-0051-x.

[GS15] Christophe Garban and Jeffrey E. Steif. *Noise sensitivity of Boolean functions and percolation.* Vol. 5. IMS Textb. Cambridge: Cambridge University Press, 2015. DOI: 10.1017/CBO9781139924160.

[HW65] J. M. Hammersley and D. J. A. Welsh. *First-passage percolation, subadditive processes, stochastic networks, and generalized renewal theory.* English. Bernoulli-Bayes-Laplace, Anniversary Vol., Proc. Int. Res. Semin., Berkeley 1963, 61-110 (1965). 1965.

[Han16] Ramon van Handel. "Probability in High Dimension". In: 2016. URL: https://web.math.princeton.edu/~rvan/APC550.pdf.

[KK13] Nathan Keller and Guy Kindler. “Quantitative relation between noise sensitivity and influences”. English. In: *Combinatorica* 33.1 (2013), pp. 45–71. DOI: 10.1007/s00493-013-2719-2.

[Led01] Michel Ledoux. *The concentration of measure phenomenon*. English. Vol. 89. Math. Surv. Monogr. Providence, RI: American Mathematical Society (AMS), 2001. ISBN: 0-8218-2864-9.

[ODo14] Ryan O’Donnell. *Analysis of Boolean functions*. English. Cambridge: Cambridge University Press, 2014. DOI: 10.1017/CBO9781139814782.

[RH20] Gregory Rosenthal and Ramon van Handel. “Ramon van Handel’s Remarks on the Discrete Cube”. In: 2020. URL: https://api.semanticscholar.org/CorpusID:236941985.

[Ros06] Raphaël Rossignol. “Threshold for monotone symmetric properties through a logarithmic Sobolev inequality”. English. In: *Ann. Probab.* 34.5 (2006), pp. 1707–1725. DOI: 10.1214/009117906000000287.

[SS10] Oded Schramm and Jeffrey E. Steif. “Quantitative noise sensitivity and exceptional times for percolation”. In: *Ann. Math. (2)* 171.2 (2010). DOI: 10.4007/annals.2010.171.619.

[Tal93] M. Talagrand. “Isoperimetry, logarithmic Sobolev inequalities on the discrete cube, and Margulis’ graph connectivity theorem”. English. In: *Geom. Funct. Anal.* 3.3 (1993), pp. 295–314. DOI: 10.1007/BF01895691.

[TV23] Vincent Tassion and Hugo Vanneuville. “Noise sensitivity of percolation via differential inequalities”. In: *Proceedings of the London Mathematical Society* 126.4 (2023). DOI: https://doi.org/10.1112/plms.12507.