Stockholms
universitet

Last Passage Percolation on a Torus

Ida Wolff

Masteruppsats i matematisk statistik
Master Thesis in Mathematical Statistics

Masteruppsats 2026:10
Matematisk statistik
Juni 2026

www.math.su.se

Matematisk statistik
Matematiska institutionen
Stockholms universitet
106 91 Stockholm

Mathematical Statistics
Stockholm University
Master Thesis **2026:10**
http://www.math.su.se

Last Passage Percolation on a Torus

Ida Wolff*

June 2026

Abstract

We consider the problem of Poissonian last passage percolation on a torus. Given a torus of area n equipped with a Poisson point process, we are interested in the maximal number of points τ_n that can be collected by an oriented path on this torus. We study the asymptotic behavior of τ_n when the number of points on the torus goes to infinity, and we derive upper and lower bounds on the expected value of τ_n.

---

*Postal address: Mathematical Statistics, Stockholm University, SE-106 91, Sweden. E-mail: idwo4428@student.su.se. Supervisor: Daniel Ahlberg.

Innehåll

1 Problem formulation and background **6**

1.1 Historical background . . . 6

1.2 Geometric formulation and link with last passage percolation . . . 6

1.3 Longest increasing circular subsequence and last passage percolation on a torus . . . 7

1.4 Organization of the report . . . 7

2 Technical preliminaries **8**

2.1 Definitions and notations . . . 8

2.2 Some results in last passage percolation . . . 9

2.3 Other useful results . . . 10

3 The maximum of independent crossing times **12**

3.1 Setup . . . 12

3.2 Limit in distribution and variance of $\tilde{\tau}_n$ . . . 12

3.3 Expected value of $\tilde{\tau}_n$ . . . 15

4 Analysis of a simplified problem **16**

4.1 Decoupling argument . . . 17

4.2 Limit in distribution and variance of τ_n^* . . . 18

4.3 Expected value of τ_n^* . . . 20

5 Analysis of the actual problem **21**

5.1 Expected value of τ_n . . . 21

A Proof of Theorem 2.8 **25**

1 Problem formulation and background

1.1 Historical background

Consider a permutation σ of numbers from 1 to n drawn uniformly at random, and denote by $\mathcal{L}_n(\sigma)$ the longest increasing subsequence of this permutation. What is the asymptotic behavior of $\mathcal{L}_n(\sigma)$? This problem is known as the longest increasing subsequence (LIS) problem, and we give here an excerpt of its history as detailed in [8]. The LIS problem was mentioned for the first time in 1961 by the Polish-American matematician Stanisław Ulam who gave it as an example of application for the Monte-Carlo method. Ulam conjectured that $\frac{\mathbb{E}(\mathcal{L}_n)}{\sqrt{n}}$ converges to some constant $c \approx 1.7$ when n goes to infinity. His conjecture was proved by the British mathematician John M.Hammersley who showed that this limit exists, and that $\frac{\mathcal{L}_n}{\sqrt{n}}$ converges in probability. In 1977 Antony Vershik and Sergei Kerov proved this limit is equal to 2, a result which has been independantly proven by Benjamin F. Logan and Lawrence A. Shepp. The longest increasing subsequence problem, hence known as the Ulam-Hammersley problem, was further studied by Jinho Baik, Percy A. Deift, and Kurt Johansson who in 1998 derived the following asymptotics for the expected value of the length of the longest increasing subsequence:

$$\mathbb{E}(\mathcal{L}_n) = 2\sqrt{n} - \gamma n^{\frac{1}{6}} + o(n^{\frac{1}{6}}),$$

where the constant $\gamma = 1.77108...$ is defined using the Painlevé equation of type II.
One of the best known results for the Ulam-Hammersley problem was derived in 1999 by J. Baik, P. Deift and K. Johansson, who proved that, when rescaled correctly, the lenghth of the longest increasing subsequence of a random permutation converges to the Tracy-Widom distribution [2]. More precisely, they showed that

$$\frac{\mathcal{L}_n - 2\sqrt{n}}{n^{\frac{1}{6}}} \to F \quad \text{in distribution},$$

where F is the standard Tracy-Widom distribution defined by

$$F(t) = \exp\left(- \int_t^{\infty} (x-t)u(x)^2 dx \right),$$

with u the solution of the Painlevé II equation.

1.2 Geometric formulation and link with last passage percolation

The longest increasing subsequence problem has an alternative geometric formulation. Consider n points uniformly distributed in a square. We are interested in paths going from the lower left corner to the upper right corner of the square, which are *oriented*, that is they can only go up and to the right, and which collect a maximum number of points along the way. We will call this number of collected points the *length* of the path, sometimes also referred to as the *crossing time* of the square. To see that finding the longest oriented path in this setting is equivalent to finding the longest increasing subsequence of a permutation, note that, since the n points are distributed uniformly at random, almost surely their abscises and ordinates will all be distinct. Denote by $x_1 < x_2 < ... < x_n$ the abscises of the n points in increasing order, and by $y_1, y_2, ..., y_n$ their corresponding ordinates. Let $\sigma \in S_n$ such that

$$y_{\sigma(1)} < ... < y_{\sigma(n)}.$$

Since the points are distributed uniformly on the square, the permutation σ is uniformly distributed on S_n, just like in the Ulam-Hammersley problem.
Notice in addition that, given a subset $i_1 < ... < i_k$ of $\{1, ..., n\}$, there exists an oriented path collecting the points $(x_{i_1}, y_{i_1}), ..., (x_{i_k}, y_{i_k})$ if and only if $y_{i_1} < y_{i_2} < ... < y_{i_k}$, that is if and only if

$$\sigma(i_1) < \sigma(i_2) < ... < \sigma(i_k).$$

Hence, an oriented path in the square corresponds to an increasing subsequence of the associated permutation σ, which justifies the geometric formulation of the Ulam-Hammersley problem. This geometric formulation corresponds to the probabilistic model of Last Passage Percolation (LPP).

1.3 Longest increasing circular subsequence and last passage percolation on a torus

A variant of the LIS problem, the LICS (Longest Increasing Circular Subsequence) problem consists in finding the longest increasing subsequence of a permutation when wrapping around once is allowed. Stated differently, given a permutation we are looking for the longest increasing subsequence of any cyclic rotation of this permutation.

This problem has been studied by M. H. Albert, M. D. Atkinson, D. Nussbaum, J. Sack and N. Santoro [1], who proved that the expected length $\mu(n)$ of a LICS satisfies

$$\frac{\mu(n)}{2\sqrt{n}} \underset{n\to\infty}{\to} 1.$$

The authors also give a Monte-Carlo algorithm to compute the LICS of a given list, and based on their numerical experiments conjecture that $\mu(n)$ may be very close to $2\sqrt{n} + n^{\frac{1}{6}}$.

Just like the LIS problem, the LICS problem also has a geometric formulation, but this time one needs to consider a torus instead of a square. We will allow the paths to make one loop around the torus, as this corresponds to a subsequence of a circular permutation.

In this report we focus on the geometric formulation of the LICS problem. We choose to work in a Poissonian setting, that is we consider that the points on the torus are distributed according to a Poisson point process (defined in Section 2). Although this choice makes us slightly deviate from the original problem, it still provides a reasonably good approximation. Indeed, conditionally to their number, the points of the Poisson point process are distributed uniformly on the torus, just like in the original problem. Moreover, although the number of points is no longer n, but a random variable following the Poisson distribution of parameter n, the concentration of the Poisson distribution around its mean ensures that the number of points will be close to n with high probability. In return, we gain some useful properties, such as the independence of disjoint subsets of the torus, which will simplify our analysis.

Our object of study will thus be the crossing time τ_n of a torus equipped with a Poisson point process, that is the length of the longest oriented path that makes one loop around this torus (a rigorous definition will be given in Section 2). The goal of this report will be to prove the following asymptotic lower and upper bounds on the expected value of τ_n:

$$\mathbb{E}[\tau_n] \geq 2\sqrt{n} + n^{\frac{1}{6}} \left(\left(\frac{1}{8} \log n \right)^{\frac{2}{3}} - \frac{\log s_n}{(\log n)^{\frac{1}{3}}} - \frac{\log(2\pi \log n)}{(\log n)^{\frac{1}{3}}} + \frac{\gamma(1-\epsilon)}{(\log n)^{\frac{1}{3}}} \right), \tag{1.1}$$

and

$$\mathbb{E}[\tau_n] \leq 2\sqrt{n} + n^{\frac{1}{6}} \left(1 + \left(\frac{5}{8} \log n \right)^{\frac{2}{3}} \right) + o(1), \tag{1.2}$$

where γ is the Euler-Mascheroni constant.

1.4 Organization of the report

As a first approach of the problem, in Section 3 we omit the dependence between different paths on the torus and study the maximum of independent crossing times of a square, which we denote by $\tilde{\tau}_n$. We show convergence in distribution for $\tilde{\tau}_n$ after appropriate rescaling, and derive bounds on its variance and expected value.

In Section 4 we consider a simplified version of our problem, where we allow only certain starting positions for the paths on the torus, thus reducing the number of paths. We will see that the maximum length of one of those paths, denoted by τ_n^*, behaves similarly to $\tilde{\tau}_n$, and an analysis similar to the one in Section 3 will give us the limit in distribution as well as bounds on the expected value of τ_n^*.

Finally, in Section 5 we consider the original problem, and prove the lower and upper bounds (1.1) and (1.2) on the expected value of the crossing time τ_n.

2 Technical preliminaries

2.1 Definitions and notations

The setting of our problem will involve a Poisson point process, defined as follows.

Definition 2.1 (Poisson point process)**.** Let $l > 0$, and let Π be a random finite subset of $[0,l]^2$. Denote by μ the standard Lebesgue measure on $\mathbb{R}^2$. We say that Π is a Poisson point process (of intensity 1) on $[0,l]^2$ if and only if:

1. for any Borel set $A \subset [0,l]^2$, $|A \cap \Pi| \sim Pois(\mu(A))$,
2. for all $A_1, A_2 \subset [0,l]^2$ disjoint Borel sets, $A_1 \cap \Pi$ and $A_2 \cap \Pi$ are independent.

For $n \in \mathbb{N}^*$ let $\mathbb{T}_n$ be a $\sqrt{n}$ by $\sqrt{n}$ square. Let Π be a Poisson point process on this square. The following two definitions formalize the notion of oriented paths.

Definition 2.2. Let $a, b \in \mathbb{T}_n$ of respective coordinates (x_a, y_a) and (x_b, y_b), and such that $y_a \leq y_b$. An oriented path from a to b with at most one loop is a subset $\{(x_1, y_1), ...(x_k, y_k)\}$ of Π verifying the following properties:

- If $x_a < x_b$, then
$$x_a \leq x_1 \leq ... \leq x_k \leq x_b$$
and
$$y_a \leq y_1 \leq ... \leq y_k \leq y_b.$$
- If $x_b \leq x_a$, then there exists $i \in \{0, ..., k\}$ such that
$$x_a \leq x_1 \leq ... \leq x_i, \tag{2.1}$$
$$x_{i+1} \leq ... \leq x_k \leq x_b \tag{2.2}$$
and
$$y_a \leq y_1 \leq ... \leq y_k \leq y_b,$$
with the convention that condition (2.1) (resp. condition (2.2)) is trivial when $i = 0$ (resp. when $i = k$).

The integer k is called the length of the path. We denote by $\mathcal{P}_{a\to b}$ the set of all such paths.
We are now ready to define our variables of interest, that is the length of geodesics.

Definition 2.3. Let a and b be two points of $\mathbb{T}_n$ with respective coordinates (x_a, y_a) and (x_b, y_b), and such that $y_a \leq y_b$. Define
$$L_{a\to b} = \max\left\{|p| \middle| p \in \mathcal{P}_{a\to b}\right\}.$$
A path $p \in \mathcal{P}_{a\to b}$ such that $|p| = L_{a\to b}$ is called a geodesic from a to b.

We are particularly interested in geodesics with exactly one loop, that is geodesics of the form $L_{(x,0)\to(x\sqrt{n})}$ with $x \in [0, \sqrt{n})$, which we will call geodesics starting at x. We will use the following notation: for $x \in [0, \sqrt{n})$,
$$L_n^x := L_{(x,0)\to(x\sqrt{n})}.$$
When $x = 0$, L_n^x coincides with the passage time in the regular setting of Poissonian last passage percolation on a square, and we will thus write L_n instead of L_n^0.
Similarly, we can define the length of geodesics with at most two loops:
Note that the random variables $\mathcal{P}_{a\to b}$ and $L_{a\to b}$ are functions of the Poisson point process Π. We will sometimes write $\mathcal{P}_{a\to b}(\Pi)$ and $L_{a\to b}(\Pi)$ but when there is no ambiguity as to the set of points that is considered we will prefer the notations $\mathcal{P}_{a\to b}$ and $L_{a\to b}$.
The passage time around the torus, which we denote by τ_n, is defined as follows:

Definition 2.4 (Passage time)**.**

$$\tau_n = \max_{0 \leq x < \sqrt{n}} L_n^x.$$

When studying crossing times we will also need to consider spatial properties of paths, and in particular their spatial fluctuations which are defined as follows:

Definition 2.5 (Fluctuations)**.** Let $x \in [0, \sqrt{n})$ and let $p = \{(x_1, y_1), ...(x_k, y_k)\} \in \mathcal{P}_{(x,0)\to(x,\sqrt{n})}$. Let $i \in \{0, ..k\}$ such that

$$x_1 \leq ... \leq x_i \quad \text{and} \quad x_{i+1} \leq ... \leq x_k.$$

We define the fluctuations of the path p, denoted by $TF(p)$ as follows:

$$TF(p) := \max\left\{\max_{1\leq j\leq i} \frac{|y_j - (x_j - x)|}{\sqrt{2}}, \max_{i+1\leq j\leq k} \frac{|(y_j - x) - x_j|}{\sqrt{2}}\right\}.$$

We will also use the following notation for $x \in [0, \sqrt{n})$:

$$TF_n^x := \max\left\{TF(p)|p \in \mathcal{P}_{(x,0)\to(x,\sqrt{n})} \quad \text{and} \quad |p| = L_n^x\right\}$$

to denote the maximal fluctuations of a geodesic starting at x, and denote by TF_n the maximal fluctuations of a geodesic starting at 0, which corresponds to a geodesic in the regular setting of Poissonian last passage percolation on a square.

2.2 Some results in last passage percolation

In this section we present some of the known results in Poissonian last passage percolation which will be used in this report.
Firstly, the convergence in distribution derived by J. Baik, P. Deift and K. Johansson in [2] also holds in the Poissonian LPP setting. With the same notations as before, we have

$$\frac{L_n - 2\sqrt{n}}{n^{\frac{1}{6}}} \to F \quad \text{in distribution.}$$

The asymptotic behavior of the upper tail of the length of the geodesic is also quite well understood. T. Sepäläinen proved the following large-devaiation principles for L_n [9] (here reformulated in the setting of this report):

Theorem 2.6. *There exists a convex function $I(x)$ such that, for $x \geq 2$,*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \log \mathbb{P}(L_n \geq \sqrt{n}x) = -I(x).$$

Moreover, $I(x) > 0$ for $x > 2$.

Theorem 2.7. *There exists a function $U(x)$ such that, for all $0 \leq x \leq 2$,*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(L_n \leq \sqrt{n}x) = -U(x)$$

Moreover, $U(x) > 0$ for $0 \leq x < 2$.

T. Sepäläinen gives explicit formulas for I and U, but in this report we simply need to know that $I(x)$ is positive for $x > 2$, and $U(x)$ is positive for $x < 2$.
M. Löwe and F. Merkl proved moderate deviation results for L_n [7]. In this report we will use a special case of their result stated in the following theorem:

Theorem 2.8. *For t_n such that $t_n \underset{n\to\infty}{\to} \infty$ and $t_n \underset{n\to\infty}{=} o(n^{\frac{1}{8}})$, the following asymptotic holds:*

$$\mathbb{P}(L_n > 2\sqrt{n} + n^{\frac{1}{6}}t_n) \underset{n\to\infty}{\sim} \frac{\exp(-\frac{4}{3}t_n^{\frac{3}{2}})}{16\pi t_n^{\frac{3}{2}}}.$$

The proof follows by calculation from Theorem 1.3 of [7], see Appendix for more detail.
It is also known that the longest path, also called the *geodesic*, tends to stay close to the diagonal. In 2000 K. Johansson showed that the typical deviations a the geodesic from the diagonal are of order $n^{\frac{2}{3}}$ [4], in the setting of a Poisson point process of intensity 1 on a n by n square (which corresponds to fluctuations of order $n^{\frac{1}{3}}$ in our setting).
Sharper estimates were derived in 2018 by A. Hammond and S. Sarkar. In this report we will use their Theorem 2.6 from [3], reformulated as follows:

Theorem 2.9. *There exist two positive constants c and C such that, for all $n \in \mathbb{N}$ and for all $s > 1$,*

$$\mathbb{P}(TF_n > sn^{1/3}) < C\exp(-cs^3).$$

2.3 Other useful results

In this section we state some simple properties of our model and lemmas that will be used in the report.

Lemma 2.10. *For all x in $[0, \sqrt{n})$, L_n^x follows the distribution of L_n.*

Bevis. Let x_0 in $[0, \sqrt{n})$. If $x_0 = 0$ than $L_n^{x_0} = L_n$. Suppose $x_0 > 0$. Let $N = |\Pi|$ and denote $\Pi = \{(x_i, y_i)|1 \leq i \leq N\}$ the points of the Poisson point process. Consider the following transformation of the square:

$$f : (x, y) \in [0, \sqrt{n}]^2 \to \begin{cases} (x - x_0, y) & x \geq x_0 \\ (x + \sqrt{n} - x_0, y) & x < x_0 \end{cases}$$

Denote $\Pi' = f(\Pi)$. The transformation f preserves the Lebesgue measure, hence Theorem 5.1 from [6] ensures that Π' is still a Poisson point process of intensity 1. We will show that $L_n^{x_0}(\Pi) = L_n(\Pi')$.
Let $p = \{(x_1, y_1), ..., (x_k, y_k)\} \in \mathcal{P}_{(x_0,0)\to(x_0,\sqrt{n})}(\Pi)$. From the definition of $\mathcal{P}_{(x_0,0)\to(x_0,\sqrt{n})}$, there exists $i \in \{1, ..., k\}$ such that

$$x_0 \leq x_1 \leq ... \leq x_i, \quad x_{i+1} \leq ... \leq x_k \leq x_0, \quad \text{and} \quad 0 \leq y_1 \leq ... \leq y_k.$$

For $i \in \{1, ..., k\}$, denote $(x_i', y_i') = f((x_i, y_i))$. Since $(x_1, y_1), ..., (x_k, y_k)$ are points of a Poisson point process, almost surely $x_1 \neq x_k$, and thus almost surely $x_0 < x_1$ or $x_k < x_0$. Without loss of generality, suppose that $x_k < x_0$. We have

$$x_1' \leq ... \leq x_i' \leq \sqrt{n} - x_0 \leq x_{i+1}' \leq ... \leq x_k' \quad \text{and} \quad y_1' \leq ... \leq y_k',$$

thus $\{(x_1', y_1'), ...(x_k', y_k')\} \in \mathcal{P}_{(0,0)\to(0,\sqrt{n})}(\Pi')$. Hence $\mathcal{P}_{(x_0,0)\to(x_0,\sqrt{n})}(\Pi) \subset \mathcal{P}_{(0,0)\to(0,\sqrt{n})}(\Pi')$ and therefore $L_n^{x_0}(\Pi) \leq L_n(\Pi')$. An analogue argument shows that $L_n(\Pi') \leq L_n^{x_0}(\Pi)$, which yields $L_n^{x_0}(\Pi) = L_n(\Pi')$. Moreover Π and Π' follow the same distribution, hence

$$L_n(\Pi) \sim L_n^{x_0}(\Pi)$$

as stated. □

Remark 2.11. Note that the arguments in the proof above are not specific to the length of oriented paths, and can be extended to other characteristics, such as fluctuations. This proof can be adapted for instance to show that for all x in $[0, \sqrt{n})$, TF_n^x follows the distribution of TF_n.

Notation. For $a, b \in \mathbb{T}_n$ with respective coordinates (x_a, y_a) and (x_b, y_b) such that $y_a \leq y_b$ we define

$$Area_{a,b} = \begin{cases} (x_b - x_a)(y_b - y_a) & \text{if} \quad x_a < x_b \\ (\sqrt{n} - x_a + x_b)(y_b - y_a) & \text{if} \quad x_a \geq x_b. \end{cases}$$

Lemma 2.12. *Let $a, b \in \mathbb{T}_n$ such that their respective coordinates (x_a, y_a) and (x_b, y_b) verify $y_a \leq y_b$. Then the following equality in distribution holds*

$$L_{a\to b} \sim L_{Area_{a,b}}.$$

Bevis. Using the proof of Lemma 2.10, it is sufficient to treat the case where $x_a < x_b$. Denote by $R_{a,b}$ the rectangle spanned by the points a and b, that is

$$R_{a,b} = \{(x, y) \in \mathbb{T} | x_a \leq x \leq x_b, \quad y_a \leq y \leq y_b\}.$$

From the definition of oriented paths it follows that

$$L_{a \to b} := L_{a \to b}(\Pi) = L_{a \to b}(\Pi|_{R_{a,b}}).$$

Consider the following transformation of $R_{a,b}$:

$$g : (x, y) \in R_{a,b} \to \left(\frac{x - x_a}{x_b - x_a} \sqrt{Area_{a,b}}, \frac{y - y_a}{y_b - y_a} \sqrt{Area_{a,b}} \right).$$

The function g maps $R_{a,b}$ to a square of area $Area_{a,b}$. Let $\Pi' = g(\Pi|_{R_{a,b}})$. The function g preserves the order of the points, and thus the oriented paths, hence

$$L_{a \to b}(\Pi|_{R_{a,b}}) = L_{Area_{a,b}}(\Pi').$$

Moreover, since the transformation g preserves the Lebesgue measure, it follows from Theorem 5.1 from [6] that Π' is a Poisson point process of intensity 1, which finishes the proof. □

Throughout the report, we will make use of Fatou's lemma for weakly converging random variables, which we recall below:

Lemma 2.13 (Fatou's lemma for convergence in distribution)**.** *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of non-negative random variables converging in distribution to a random variable X. Then*

$$\mathbb{E}[X] \leq \liminf_{n \to \infty} \mathbb{E}\left[X_n\right].$$

Bevis. Let $(X_n)_{n \in \mathbb{N}}$ and X be such as in the hypotheses of the lemma. Then Skorokhod's representation theorem (see Theorem 3.30 in [5]) ensures that on a suitable probability space, there exist random variables $X' \overset{d}{\sim} X$ and $X'_n \overset{d}{\sim} X_n$ for $n \in \mathbb{N}$ such that $X'_n \underset{n \to \infty}{\to} X'$ almost surely. Hence, Fatou's lemma implies that

$$\mathbb{E}[X'] \leq \liminf_{n \to \infty} \mathbb{E}\left[X'_n\right].$$

Moreover $\mathbb{E}[X'] = \mathbb{E}[X]$ and $\mathbb{E}[X_n] = \mathbb{E}[X'_n]$ for all $n \in \mathbb{N}$, hence the result. □

We will also make use of the FKG inequality, and more precisely of a corollary of this inequality which we will state below. We begin by recalling the FKG inequality. The following definition and theorem are a reformulation of Theorem 20.4 from [6].

Definition 2.14 (Increasing functions)**.** Let N be the space of finite subsets of $\mathbb{T}_n$. A function $f \in \mathbb{R}(\mathbf{N})$ is said to be *increasing* if $f(I \cup \{x\}) \geq f(I)$ for all $I \in \mathbf{N}$ and all $x \in \mathbb{T}_n$. It is said to be *decreasing* if $(-f)$ is increasing.

Theorem 2.15 (FKG inequality)**.** *Let Π be a Poisson point process on $\mathbb{T}_n$, and let $f, g \in L^2(\Pi)$ be increasing functions. Then*

$$\mathbb{E}[f(\Pi)g(\Pi)] \geq \mathbb{E}[f(\Pi)]\mathbb{E}[g(\Pi)].$$

Remark 2.16. Theorem 2.15 still holds if the functions f and g are both decreasing.

Remark 2.17. By induction, Theorem 2.15 can be extended to a product of any finite number of increasing functions.

From the FKG inequality, we deduce the following

Corollary 2.18. *Let $m \in \mathbb{N}$ and let $X_1, ..., X_m$ be a family of crossing times on $\mathbb{T}_n$. For $i \in \{1, ..., m\}$, let X'_i be a copy of X_i, such that the $X'_1, ..., X'_m$ are mutually independent. Then*

$$\mathbb{E}\left[\max_{1 \leq i \leq m} X_i\right] \leq \mathbb{E}\left[\max_{1 \leq i \leq m} X'_i\right].$$

Bevis. Let $x \in \mathbb{R}$. We will show that $\mathbb{P}(\max_{1\le i\le m} X_i \le x) \ge \mathbb{P}(\max_{1\le i\le m} X_i' \le x)$, from which the proposition will follow. We have

$$\mathbb{P}(\max_{1\le i\le m} X_i \le x) = \mathbb{E}\left[\prod_{i=1}^{m} \mathbf{1}_{\{X_i\le x\}}\right].$$

Note that the indicator functions $\mathbf{1}_{\{X_i\le x\}}$ are functions of the Poisson point process Π. Moreover, they are all decreasing since adding a point to the Poisson point process Π can only increase crossing times. Hence the hypotheses of the FKG inequality hold, and we get

$$\mathbb{P}\left(\max_{1\le i\le m} X_i \le x\right) = \mathbb{E}\left[\prod_{i=1}^{m} \mathbf{1}_{\{X_i\le x\}}\right] \ge \prod_{i=1}^{m} \mathbb{E}\left[\mathbf{1}_{\{X_i\le x\}}\right] = \mathbb{E}\left[\prod_{i=1}^{m} \mathbf{1}_{\{X_i'\le x\}}\right] = \mathbb{P}\left(\max_{1\le i\le m} X_i' \le x\right)$$

which finishes the proof. □

3 The maximum of independent crossing times

As a first approach of the problem, we will study the behavior of the maximum of a family of independent crossing times. Although the crossing times on $\mathbb{T}_n$ are not independent, especially for paths with starting points that are close to each other, one can assume that paths with distanced enough starting points will be almost independent (this idea will be developed in the next section). Hence, although it is different from the original problem, taking the maximum over independent crossing times seems like a reasonable first approximation.

3.1 Setup

Let $\alpha, \beta > 0$, and let $m = \lfloor \beta n^{\alpha} \rfloor$. Let $L_n^{(1)}, ..., L_n^{(m)}$ be a family of independent random variables all following the distribution of L_n. We are interested in the maximum of those variables, which we will denote as

$$\tilde{\tau}_n = \max\left\{L_n^{(1)}, ..., L_n^{(m)}\right\}.$$

Remark 3.1. We chose to work with $\beta > 0$ constant for more clarity, however the results proved in this section still hold when β is logarithmic in n.

Take $(t_n)_{n\in\mathbb{N}}$ such that $t_n \underset{n\to\infty}{\to} +\infty$ and $t_n = o(n^{\frac{1}{8}})$. Denote $x := 2\sqrt{n} + n^{\frac{1}{6}} t_n$. Then, using independence and Theorem 2.8, we have

$$\mathbb{P}(\tilde{\tau}_n \le x) = \mathbb{P}(L_n \le x)^m = \left(1 - \frac{\exp(-\frac{4}{3}t_n^{\frac{3}{2}})}{16\pi t_n^{\frac{3}{2}}}(1+o(1))\right)^m = \exp\left(-m\frac{\exp(-\frac{4}{3}t_n^{\frac{3}{2}})}{16\pi t_n^{\frac{3}{2}}}(1+o(1))\right). \tag{3.1}$$

3.2 Limit in distribution and variance of $\tilde{\tau}_n$

We will show the following limit in distribution.

Proposition 3.2. *Let G be a random variable following the standard Gumbel distribution, defined by $\mathbb{P}(G \le x) = e^{-e^{-x}}$ for all $x \in \mathbb{R}$. Then the following limit in distribution holds:*

$$\frac{(6\alpha\log n)^{\frac{1}{3}}}{n^{\frac{1}{6}}}\left[\tilde{\tau}_n - \left(2\sqrt{n} + n^{\frac{1}{6}}\left(\left(\frac{3}{4}\alpha\log n\right)^{\frac{2}{3}} - \frac{\log\log n}{(6\alpha\log n)^{\frac{1}{3}}} + \frac{\log\beta}{(6\alpha\log n)^{\frac{1}{3}}} - \frac{\log(12\pi\alpha)}{(6\alpha\log n)^{\frac{1}{3}}}\right)\right)\right] \xrightarrow{d} G.$$

To prove this proposition we will need the following lemma:

Lemma 3.3. *For γ^* in $(0,1)$ let*

$$\tilde{x}_n(\gamma^*) = \left(\frac{3}{4}\alpha\log n\right)^{\frac{2}{3}} - \frac{\log\log n}{(6\alpha\log n)^{\frac{1}{3}}} + \frac{\log\beta}{(6\alpha\log n)^{\frac{1}{3}}} - \frac{\log 12\pi\alpha}{(6\alpha\log n)^{\frac{1}{3}}} - \frac{\log\log\frac{1}{1-\gamma^*}}{(6\alpha\log n)^{\frac{1}{3}}}.$$

Then

$$\lim_{n\to\infty}\mathbb{P}(\tilde{\tau}_n > 2\sqrt{n} + n^{\frac{1}{6}}\tilde{x}_n(\gamma^*)) = \gamma^*.$$

Bevis. Let γ^* in $(0,1)$. It follows from (3.1) that

$$\mathbb{P}(\tilde{\tau}_n \le 2\sqrt{n} + n^{\frac{1}{6}}\tilde{x}_n(\gamma^*)) \sim \exp\left(-m\frac{\exp(-\frac{4}{3}\tilde{x}_n(\gamma^*)^{\frac{3}{2}})}{16\pi\tilde{x}_n(\gamma^*)^{\frac{3}{2}}}\right). \tag{3.2}$$

To show that this quantity is equivalent to $1-\gamma^*$ we will begin by computing the asymptotics of $\tilde{x}_n(\gamma^*)$. We have

$$\tilde{x}_n(\gamma^*) = \left(\frac{3}{4}\alpha\log n\right)^{\frac{2}{3}}\left[1 - \frac{1}{\frac{3}{2}\alpha\log n}\left(\log\log n - \log\beta + \log 12\pi\alpha + \log\log\frac{1}{1-\gamma^*}\right)\right],$$

thus

$$\begin{aligned}
\tilde{x}_n(\gamma^*)^{\frac{3}{2}} &= \frac{3}{4}\alpha\log n\left[1 - \frac{1}{\alpha\log n}\left(\log\log n - \log\beta + \log 12\pi\alpha + \log\log\frac{1}{1-\gamma^*}\right) + O\left(\left(\frac{\log\log n}{\log n}\right)^2\right)\right] \\
&= \frac{3}{4}\left(\alpha\log n - \left(\log\log n - \log\beta + \log 12\pi\alpha + \log\log\frac{1}{1-\gamma^*}\right)\right) + o(1) \\
&\sim \frac{3}{4}\alpha\log n.
\end{aligned}$$

It follows that

$$\exp(-\frac{4}{3}\tilde{x}_n(\gamma^*))^{\frac{3}{2}}) \sim \exp\left(-\alpha\log n + \log\log n - \log\beta - \log 12\pi\alpha + \log\log\frac{1}{1-\gamma^*}\right) = \frac{12\pi\alpha\log n}{\beta n^\alpha}\log\frac{1}{1-\gamma^*}$$

and

$$16\pi\tilde{x}_n(\gamma^*)^{\frac{3}{2}} \sim 12\pi\alpha\log n.$$

Using the fact that $m \sim \beta n^\alpha$, we finally get

$$-m\frac{\exp(-\frac{4}{3}\tilde{x}_n(\gamma^*))^{\frac{3}{2}})}{16\pi\tilde{x}_n(\gamma^*)^{\frac{3}{2}}} \to -\log\frac{1}{1-\gamma^*} = \log(1-\gamma^*)$$

and thus

$$\lim_{n\to\infty}\exp\left(-m\frac{\exp(-\frac{4}{3}\tilde{x}_n(\gamma^*))^{\frac{3}{2}})}{16\pi\tilde{x}_n(\gamma^*)^{\frac{3}{2}}}\right) = 1-\gamma^*$$

which concludes thanks to (3.2) ☐

We will now prove Proposition 3.2.

Proof of Proposition 3.1. Let $x \in \mathbb{R}$. It follows from Lemma 3.3 that

$$\mathbb{P}\left(\tilde{\tau}_n \le 2\sqrt{n} + n^{\frac{1}{6}}\left(\left(\frac{3}{4}\alpha\log n\right)^{\frac{2}{3}} - \frac{\log\log n}{(6\alpha\log n)^{\frac{1}{3}}} + \frac{\log\beta}{(6\alpha\log n)^{\frac{1}{3}}} - \frac{\log 12\pi\alpha}{(6\alpha\log n)^{\frac{1}{3}}} + \frac{x}{(6\alpha\log n)^{\frac{1}{3}}}\right)\right) \underset{n\to\infty}{\to}$$
$$\underset{n\to\infty}{\to} e^{-e^{-x}} = \mathbb{P}(G \le x)$$

Moreover

$$
\begin{aligned}
&\mathbb{P}\left(\tilde{\tau}_n \leq 2\sqrt{n} + n^{\frac{1}{6}}\left(\left(\frac{3}{4}\alpha\log n\right)^{\frac{2}{3}} - \frac{\log\log n}{(6\alpha\log n)^{\frac{1}{3}}} + \frac{\log\beta}{(6\alpha\log n)^{\frac{1}{3}}} - \frac{\log(12\pi\alpha)}{(6\alpha\log n)^{\frac{1}{3}}} + \frac{x}{(6\alpha\log n)^{\frac{1}{3}}}\right)\right) = \\
&= \mathbb{P}\left(\tilde{\tau}_n - \left(2\sqrt{n} + n^{\frac{1}{6}}\left(\left(\frac{3}{4}\alpha\log n\right)^{\frac{2}{3}} - \frac{\log\log n}{(6\alpha\log n)^{\frac{1}{3}}} + \frac{\log\beta}{(6\alpha\log n)^{\frac{1}{3}}} - \frac{\log(12\pi\alpha)}{(6\alpha\log n)^{\frac{1}{3}}}\right)\right) \leq \frac{n^{\frac{1}{6}}}{(6\alpha\log n)^{\frac{1}{3}}}x\right) \\
&= \mathbb{P}\left(\frac{(6\alpha\log n)^{\frac{1}{3}}}{n^{\frac{1}{6}}}\left[\tilde{\tau}_n - \left(2\sqrt{n} + n^{\frac{1}{6}}\left(\left(\frac{3}{4}\alpha\log n\right)^{\frac{2}{3}} - \frac{\log\log n}{(6\alpha\log n)^{\frac{1}{3}}} + \frac{\log\beta}{(6\alpha\log n)^{\frac{1}{3}}} - \frac{\log(12\pi\alpha)}{(6\alpha\log n)^{\frac{1}{3}}}\right)\right)\right] \leq x\right)
\end{aligned}
$$

which finishes the proof. □

Lemma 3.3 can also be used to derive a lower bound on the variance of $\tilde{\tau}_n$. We will show the following

Proposition 3.4. *There exists a positive constant D such that*

$$
\mathrm{Var}(\tilde{\tau}_n) \geq D\frac{n^{\frac{1}{3}}}{(\log n)^{\frac{2}{3}}}.
$$

In other words, the variance of $\tilde{\tau}_n$ is at least of order $\frac{n^{\frac{1}{3}}}{(\log n)^{\frac{2}{3}}}$.

Bevis. Let $\tilde{\tau}_n'$ be an independent copy of $\tilde{\tau}_n$. Let γ^* and $\gamma^{*\prime}$ in $(0,1)$ with $\gamma^* > \gamma^{*\prime}$. Let $u_n = 2\sqrt{n} + n^{\frac{1}{6}}\tilde{x}_n(\gamma^*)$, and $u_n' = 2\sqrt{n} + n^{\frac{1}{6}}\tilde{x}_n(\gamma^{*\prime})$. We have

$$
\mathrm{Var}(\tilde{\tau}_n) = \frac{1}{2}\mathbb{E}(|\tilde{\tau}_n - \tilde{\tau}_n'|^2)
$$

and thus

$$
\begin{aligned}
\mathrm{Var}(\tilde{\tau}_n) &\geq \frac{1}{2}\mathbb{E}\left(|\tilde{\tau}_n - \tilde{\tau}_n'|^2\mathbf{1}_{\{\tilde{\tau}_n \leq u_n\}}\mathbf{1}_{\{\tilde{\tau}_n' > u_n'\}}\right) \\
&\geq \frac{1}{2}|u_n - u_n'|^2\mathbb{P}(\tilde{\tau}_n \leq u_n)\mathbb{P}(\tilde{\tau}_n > u_n').
\end{aligned}
$$

Moreover

$$
u_n - u_n' = n^{\frac{1}{6}}(\tilde{x}_n(\gamma^*) - \tilde{x}_n(\gamma^{*\prime})) \sim \frac{n^{\frac{1}{6}}}{(6\alpha\log n)^{\frac{1}{3}}}\log\left(\frac{\log\frac{1}{1-\gamma^{*\prime}}}{\log\frac{1}{1-\gamma^*}}\right) = \frac{n^{\frac{1}{6}}}{(6\alpha\log n)^{\frac{1}{3}}}\log\left(\frac{\log(1-\gamma^{*\prime})}{\log(1-\gamma^*)}\right), \tag{3.3}
$$

and from Lemma 3.3 it follows that

$$
\mathbb{P}(\tilde{\tau}_n \leq u_n)\mathbb{P}(\tilde{\tau}_n > u_n') \underset{n\to\infty}{\sim} (1-\gamma^*)\gamma^{*\prime}.
$$

Finally,

$$
\frac{1}{2}|u_\theta - u_{\theta'}|^2\mathbb{P}(\tilde{\tau}_n \leq u_\theta)\mathbb{P}(\tilde{\tau}_n > u_{\theta'}) \underset{n\to\infty}{\sim} \frac{1}{2}n^{\frac{1}{3}}\left(\frac{\log\left(\frac{\log(1-\gamma^{*\prime})}{\log(1-\gamma^*)}\right)}{(6\alpha\log n)^{\frac{1}{3}}}\right)^2(1-\gamma^*)\gamma^{*\prime},
$$

which shows that the variance of $\tilde{\tau}_n$ is at least of order $\frac{n^{\frac{1}{3}}}{(\log n)^{\frac{2}{3}}}$. □

3.3 Expected value of $\tilde{\tau}_n$

We will now derive an upper and lower bound for the expected value of $\tilde{\tau}_n$. We start with the upper bound. Define

$$\tilde{z}_n := \left[\frac{3}{4}\left(\left(\alpha+\frac{1}{2}\right)\log n\right)\right]^{\frac{2}{3}} + \frac{\log\beta}{(6(\alpha+\frac{1}{2})\log n)^{\frac{1}{3}}}$$

Then we have the following

Proposition 3.5. *The following asymptotic upper bound holds:*

$$\mathbb{E}(\tilde{\tau}_n) \leq 2\sqrt{n} + n^{\frac{1}{6}}\tilde{z}_n + o(1).$$

Bevis. We begin by splitting the expected value of $\tilde{\tau}_n$ into three terms:

$$\begin{aligned}\mathbb{E}(\tilde{\tau}_n) &= \mathbb{E}(\tilde{\tau}_n\mathbf{1}_{\{\tilde{\tau}_n\leq 2\sqrt{n}+n^{\frac{1}{6}}\tilde{z}_n\}}) + \mathbb{E}(\tilde{\tau}_n\mathbf{1}_{\{3\sqrt{n}\geq\tilde{\tau}_n>2\sqrt{n}+n^{\frac{1}{6}}\tilde{z}_n\}}) + \mathbb{E}(\tilde{\tau}_n\mathbf{1}_{\{\tilde{\tau}_n>3\sqrt{n}\}})\\ &\leq 2\sqrt{n} + n^{\frac{1}{6}}\tilde{z}_n + O\left(\sqrt{n}\mathbb{P}(\tilde{\tau}_n > 2\sqrt{n}+n^{\frac{1}{6}}\tilde{z}_n)\right) + \mathbb{E}(\tilde{\tau}_n\mathbf{1}_{\{\tilde{\tau}_n>3\sqrt{n}\}}).\end{aligned}$$

We will now show that the two last terms of the equation above tend to zero when n goes to infinity. For the second term we use (3.1) to write

$$\mathbb{P}(\tilde{\tau}_n > 2\sqrt{n}+n^{\frac{1}{6}}\tilde{z}_n) = 1-\exp\left(-m\frac{\exp(-\frac{4}{3}\tilde{z}_n^{\frac{3}{2}})}{16\pi\tilde{z}_n^{\frac{3}{2}}}(1+o(1))\right).$$

It follows by computation that

$$\tilde{z}_n^{\frac{3}{2}} = \frac{3}{4}(\alpha+\frac{1}{2})\log n + \frac{3}{4}\log\beta + o(1)$$

hence

$$m\frac{\exp(-\frac{4}{3}\tilde{z}_n^{\frac{3}{2}})}{16\pi\tilde{z}_n^{\frac{3}{2}}} \sim \frac{1}{\sqrt{n}12\pi(\alpha+\frac{1}{2})\log n} = o\left(\frac{1}{\sqrt{n}}\right),$$

which yields

$$\mathbb{P}(\tilde{\tau}_n > 2\sqrt{n}+n^{\frac{1}{6}}\tilde{z}_n) = 1-(1+o\left(\frac{1}{\sqrt{n}}\right)) = o\left(\frac{1}{\sqrt{n}}\right).$$

For the third we use the Cauchy-Schwarz inequality which yields

$$\mathbb{E}(\tilde{\tau}_n\mathbf{1}_{\{\tilde{\tau}_n>3\sqrt{n}\}}) \leq \mathbb{E}(\tilde{\tau}_n^2)^{\frac{1}{2}}\mathbb{P}(\tilde{\tau}_n > 3\sqrt{n})^{\frac{1}{2}}. \tag{3.4}$$

It follows from the definition of $\tilde{\tau}_n$ and from the fact that the crossing times $L_n^{(i)}$ are non negative that

$$\tilde{\tau}_n^2 = \max\left\{(L_n^{(1)})^2, ..., (L_n^{(m)})^2\right\} \leq \sum_{i=1}^{m}(L_n^{(i)})^2$$

hence

$$\mathbb{E}(\tilde{\tau}_n^2) \leq m\mathbb{E}(L_n^2). \tag{3.5}$$

Let $N = |\Pi|$. Since Π is a Poisson point process on $\mathbb{T}_n$, N follows a Poisson distribution of parameter n, and the length of the geodesic L_n is at most N, hence

$$\mathbb{E}(L_n^2) \leq \mathbb{E}(N^2) = n^2+n. \tag{3.6}$$

Moreover the large deviation result from Theorem 2.6 yields

$$\mathbb{P}(\tilde{\tau}_n > 3\sqrt{n}) = \mathbb{P}\left(\bigcup_{i=1}^{m}\{L_n^i > 3\sqrt{n}\}\right) \leq m\mathbb{P}(L_n > 3\sqrt{n}) = m\exp\left(-\sqrt{n}I(3)(1+o(1))\right), \tag{3.7}$$

where I is the convex function given in Theorem 2.6 (recall that we know in particular that $I(3) > 0$). Putting together equations (3.4), (3.5), (3.6) and (3.7), we get

$$\mathbb{E}(\tilde{\tau}_n\mathbf{1}_{\{\tilde{\tau}_n>3\sqrt{n}\}}) \leq (mn^2+mn)^{\frac{1}{2}}m^{\frac{1}{2}}e^{-\sqrt{n}\frac{I(3)}{2}(1+o(1))} = o(1)$$

which finishes the proof. $\square$

We will now show the following asymptotic lower bound on the expected value of $\tilde{\tau}_n$.

Proposition 3.6. *Let γ be the Euler-Mascheroni constant. For any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$,*

$$\mathbb{E}[\tilde{\tau}_n] \geq 2\sqrt{n} + n^{\frac{1}{6}} \left(\left(\frac{3}{4} \alpha \log n \right)^{\frac{2}{3}} - \frac{\log \log n}{(6\alpha \log n)^{\frac{1}{3}}} + \frac{\log \beta}{(6\alpha \log n)^{\frac{1}{3}}} - \frac{\log(12\pi\alpha)}{(6\alpha \log n)^{\frac{1}{3}}} + \frac{\gamma(1-\epsilon)}{(6\alpha \log n)^{\frac{1}{3}}} \right).$$

Bevis. Let $X_n = \frac{(6\alpha \log n)^{\frac{1}{3}}}{n^{\frac{1}{6}}} \left[\tilde{\tau}_n - \left(2\sqrt{n} + n^{\frac{1}{6}} \left(\left(\frac{3}{4} \alpha \log n \right)^{\frac{2}{3}} - \frac{\log \log n}{(6\alpha \log n)^{\frac{1}{3}}} + \frac{\log \beta}{(6\alpha \log n)^{\frac{1}{3}}} - \frac{\log(12\pi\alpha)}{(6\alpha \log n)^{\frac{1}{3}}} \right) \right) \right]$.
Proposition 3.2 gives us

$$X_n \xrightarrow{d} G,$$

thus using Lemma 2.13 we have

$$\liminf \mathbb{E}[X_n] \geq \mathbb{E}[G] = \gamma.$$

This yields

$$\liminf \frac{(6\alpha \log n)^{\frac{1}{3}}}{n^{\frac{1}{6}}} \left[\mathbb{E}[\tilde{\tau}_n] - \left(2\sqrt{n} + n^{\frac{1}{6}} \left(\left(\frac{3}{4} \alpha \log n \right)^{\frac{2}{3}} - \frac{\log \log n}{(6\alpha \log n)^{\frac{1}{3}}} + \frac{\log \beta}{(6\alpha \log n)^{\frac{1}{3}}} - \frac{\log(12\pi\alpha)}{(6\alpha \log n)^{\frac{1}{3}}} \right) \right) \right] \geq \gamma$$

and thus asymptotically

$$\mathbb{E}[\tilde{\tau}_n] \geq 2\sqrt{n} + n^{\frac{1}{6}} \left(\left(\frac{3}{4} \alpha \log n \right)^{\frac{2}{3}} - \frac{\log \log n}{(6\alpha \log n)^{\frac{1}{3}}} + \frac{\log \beta}{(6\alpha \log n)^{\frac{1}{3}}} - \frac{\log(12\pi\alpha)}{(6\alpha \log n)^{\frac{1}{3}}} + \frac{\gamma(1-\epsilon)}{(6\alpha \log n)^{\frac{1}{3}}} \right)$$

for any $\epsilon > 0$. ☐

Both the lower and upper bound on the expectation of $\tilde{\tau}_n$ are of the form

$$2\sqrt{n} + Bn^{\frac{1}{6}} (\log n)^{\frac{2}{3}} (1 + o(1)),$$

with $B = (\frac{3}{4}\alpha)^{\frac{2}{3}}$ for the lower bound, and $B = \left(\frac{3}{4}(\alpha + \frac{1}{2})\right)^{\frac{2}{3}}$ for the upper bound. This suggests that the expectation of $\tilde{\tau}_n$ should be of the form

$$\mathbb{E}(\tilde{\tau}_n) = 2\sqrt{n} + Bn^{\frac{1}{6}} (\log n)^{\frac{2}{3}} (1 + o(1))$$

with B in $\left[(\frac{3}{4}\alpha)^{\frac{2}{3}}, \left(\frac{3}{4}(\alpha + \frac{1}{2})\right)^{\frac{2}{3}} \right]$.

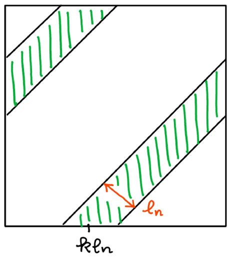
4 Analysis of a simplified problem

In the previous section we have seen that supposing independence of crossing times greatly simplifies the problem and we can understand quite well the behavior of $\tilde{\tau}_n$. We would thus like to introduce independence in our original problem. To do so, we will analyze a simplified version of the problem where, instead of taking the maximum over all possible starting positions in $[0, \sqrt{n})$ we only consider a subset of starting positions, which we choose at sufficiently large distances from each other for the paths to be roughly independent. More precisely, set

$$l_n = s_n n^{\frac{1}{3}}, \quad \text{where} \quad s_n = 2 \left(\frac{(1 + \frac{1}{6}) \log n}{c} \right)^{\frac{1}{3}},$$

and c is the positive constant given by Theorem 2.9. We will consider a subset of points of $[0, \sqrt{n})$ at distance l_n from each other, namely $\{0, l_n, ..., (m-1)l_n\}$ where $m = \lfloor \frac{n^{\frac{1}{6}}}{s_n} \rfloor$, and denote by τ_n^* the maximum length of a path that starts at one of these points, that is

$$\tau_n^* = \max\{L_n^{kl_n} | 0 \leq k \leq m-1\}.$$



Figur 1: Strip S_k

4.1 Decoupling argument

To clarify what we mean by paths being roughly independent, let us introduce the following notations and events. For x in $[0, \sqrt{n})$ let A_x be the event that the transversal fluctuations of a geodesic starting at x do not exceed $\frac{l_n}{2}$:

$$A_x = \{TF_n^x < \frac{l_n}{2}\}.$$

Moreover, let A be the event that the transversal fluctuations of any geodesic whose starting point belongs to the subset $\{0, l_n, ..., (m-1)l_n\}$ do not exceed l_n:

$$A = \bigcap_{i=0}^{m-1} A_{il_n}.$$

Finally, for k in $\{0, 1, ...m-1\}$ let

$$S_k = \left\{(x,y) \in \mathbb{T}_n : |(x - kl_n) - y| \leq \frac{l_n}{\sqrt{2}}\right\} \bigcup \left\{(x,y) \in \mathbb{T}_n : |x - (y - kl_n)| \leq \frac{l_n}{\sqrt{2}}\right\}.$$

Intuitively, when seeing $\mathbb{T}_n$ as a torus, for k in $\{0, .., m\}$, S_k is a strip of width l_n making one loop around this torus (see Figure 1). It is therefore immediate to verify that supposing that the fluctuations of a geodesic starting at kl_n do not exceed $\frac{l_n}{2}$ is equivalent to supposing that this geodesic stays inside the strip S_k. In other words, under the event A_{kl_n}, $L_n^{kl_n} := L_n^{kl_n}(\Pi) = L_n^{kl_n}(\Pi|_{S_k})$.

The following observations justify the choice of the event A:

Remark 4.1.

- Under the event A, for all k in $\{0, ..., m-1\}$, $L_n^{kl_n} = L_n^{kl_n}(\Pi|_{S_k})$.
- Since the strips S_k are disjoint for different values of k, and Π is a Poisson process, it follows that the variables $L_n^{kl_n}(\Pi|_{S_k})$ are mutually independent.
- The probability of A^c is very small. This point will be made more precise in the following lemma.

Lemma 4.2. *Let C be the positive constant from Theorem 2.9. Then*

$$\mathbb{P}(A_0^c) \leq \frac{C}{n^{1+\frac{1}{6}}} \text{ and } \mathbb{P}(A^c) = o\left(\frac{1}{n}\right).$$

Bevis. The first inequality follows by direct computation from Theorem 2.9, since $\mathbb{P}(A_0^c) = \mathbb{P}(TF_n^0 \geq \frac{l_n}{2}) \leq \frac{C}{n^{1+\frac{1}{6}}}$. For the second inequality we use a union bound to write

$$\mathbb{P}(A^c) = \mathbb{P}(\bigcup_{i=0}^{m-1} \{TF_n^{il_n} \geq \frac{l_n}{2}\}) \leq \sum_{i=0}^{m-1} \mathbb{P}(TF_n^{il_n} \geq \frac{l_n}{2}).$$

From Remark 2.11 it follows that $\mathbb{P}(TF_n^{il_n} \geq \frac{l_n}{2}) = \mathbb{P}(TF_n \geq \frac{l_n}{2})$ for all i in $\{0, ..., m-1\}$, hence

$$\mathbb{P}(A^c) \leq \frac{n^{\frac{1}{6}}}{s_n} \mathbb{P}(TF_n \geq \frac{l_n}{2}) \leq \frac{C}{ns_n} = o\left(\frac{1}{n}\right)$$

which finishes the proof. □

4.2 Limit in distribution and variance of τ_n^*

We are now ready to begin our study of τ_n^*. We will show the following convergence in distribution:

Proposition 4.3. *Let G be a random variable following the standard Gumbel distribution. Then the following limit in distribution holds:*

$$\frac{(\log n)^{\frac{1}{3}}}{n^{\frac{1}{6}}}\left[\tau_n^* - \left(2\sqrt{n} + n^{\frac{1}{6}}\left(\left(\frac{1}{8}\log n\right)^{\frac{2}{3}} - \frac{\log s_n}{(\log n)^{\frac{1}{3}}} - \frac{\log(2\pi\log n)}{(\log n)^{\frac{1}{3}}}\right)\right)\right] \xrightarrow{d} G.$$

To prove this proposition we will need the following lemmas:
For n in $\mathbb{N}^*$ and γ^* in $(0,1)$ let

$$x_n^*(\gamma^*) = \left(\frac{1}{8}\log n\right)^{\frac{2}{3}} - \frac{\log s_n}{(\log n)^{\frac{1}{3}}} - \frac{\log(2\pi\log n)}{(\log n)^{\frac{1}{3}}} - \frac{\log\log\frac{1}{1-\gamma^*}}{(\log n)^{\frac{1}{3}}}.$$

Lemma 4.4. *For γ^* in $(0,1)$ the following asymptotics hold:*

$$\begin{aligned}\mathbb{P}(L_n > 2\sqrt{n} + n^{\frac{1}{6}}x_n^*(\gamma^*)) &\underset{n\to+\infty}{\sim} \frac{s_n}{n^{\frac{1}{6}}}\log\frac{1}{1-\gamma^*}\\ &\underset{n\to\infty}{\sim} \frac{1}{m}\log\frac{1}{1-\gamma^*}.\end{aligned}$$

Bevis. Let γ^* in $(0,1)$. We have $x_n^*(\gamma^*) \underset{n\to\infty}{\to} +\infty$ and $x_n^* = o(n^{\frac{1}{8}})$, hence Theorem 2.8 yields

$$\mathbb{P}(L_n > 2\sqrt{n} + n^{\frac{1}{6}}x_n^*(\gamma^*)) \underset{n\to\infty}{\sim} \frac{\exp(-\frac{4}{3}x_n^*(\gamma^*)^{\frac{3}{2}})}{16\pi x_n^*(\gamma^*)^{\frac{3}{2}}}.$$

We have

$$x_n^*(\gamma^*) = \left(\frac{1}{8}\log n\right)^{\frac{2}{3}}\left[1 - \frac{4}{\log n}(\log s_n + \log(2\pi\log n) + \log\log\frac{1}{1-\gamma^*})\right],$$

hence

$$\begin{aligned}\frac{4}{3}x_n^*(\gamma^*)^{\frac{3}{2}} &= \frac{1}{6}\log n\left[1 - \frac{3}{2}\frac{4}{\log n}(\log s_n + \log(2\pi\log n) + \log\log\frac{1}{1-\gamma^*})\right] + O\left(\frac{(\log\log n)^2}{\log n}\right)\\ &= \frac{1}{6}\log n - (\log s_n + \log(2\pi\log n) + \log\log\frac{1}{1-\gamma^*}) + o(1)\end{aligned}$$

and

$$16\pi x_n^*(\gamma^*)^{\frac{3}{2}} \sim 2\pi\log n,$$

thus

$$\frac{\exp(-\frac{4}{3}x_n^*(\gamma^*)^{\frac{3}{2}})}{16\pi x_n^*(\gamma^*)^{\frac{3}{2}}} \sim \frac{s_n}{n^{\frac{1}{6}}}\frac{2\pi\log n}{2\pi\log n}\log\frac{1}{1-\gamma^*} = \frac{s_n}{n^{\frac{1}{6}}}\log\frac{1}{1-\gamma^*} \sim \frac{1}{m}\log\frac{1}{1-\gamma^*}$$

as stated. □

Lemma 4.5. *For γ^* in $(0,1)$*

$$\lim_{n\to\infty}\mathbb{P}(\tau_n^* > 2\sqrt{n} + n^{\frac{1}{6}}x_n^*(\gamma^*)) = \gamma^*$$

Bevis. Let γ^* in $(0,1)$. To simplify the notations, in this proof we will denote

$$x = 2\sqrt{n} + n^{\frac{1}{6}}x_n^*(\gamma^*).$$

With this notation, we need to show that

$$\lim_{n\to\infty}\mathbb{P}(\tau_n^* > x) = \gamma^*.$$

Throughout the proof we will make an extensive use of Lemma 4.2 to justify that, for an event B,

$$\mathbb{P}(B \cap A^c) \leq \mathbb{P}(A^c) = o\left(\frac{1}{n}\right).$$

Using the total probability formula,

$$\mathbb{P}(\tau_n^* \leq x) = \mathbb{P}((\tau_n^* \leq x) \cap A) + \mathbb{P}((\tau_n^* \leq x) \cap A^c) = \mathbb{P}((\tau_n^* \leq x) \cap A) + o\left(\frac{1}{n}\right). \tag{4.1}$$

Moreover, using the fact that, under the event A, $L_n^{kl_n} = L_n^{kl_n}(\Pi|_{S_k})$ fo all k in $\{0, ..., m-1\}$,

$$\begin{aligned}
\mathbb{P}((\tau_n^* \leq x) \cap A) &= \mathbb{P}((\max_{0 \leq k \leq m-1} L_n^{kl_n} \leq x) \cap A) \\
&= \mathbb{P}((\max_{0 \leq k \leq m-1} L_n^{kl_n}(\Pi|_{S_k}) \leq x) \cap A) \\
&= \mathbb{P}(\max_{0 \leq k \leq m-1} L_n^{kl_n}(\Pi|_{S_k}) \leq x) - \mathbb{P}((\max_{0 \leq k \leq m-1} L_n^{kl_n}(\Pi|_{S_k}) \leq x) \cap A^c) \\
&= \mathbb{P}(\max_{0 \leq k \leq m-1} L_n^{kl_n}(\Pi|_{S_k}) \leq x)) + o\left(\frac{1}{n}\right).
\end{aligned} \tag{4.2}$$

From Remark 4.1 we know that the variables $L_n^{kl_n}(\Pi|_{S_k})$ are mutually independent, hence

$$\begin{aligned}
\mathbb{P}(\max_{0 \leq k \leq m-1} L_n^{kl_n}(\Pi|_{S_k}) \leq x) &= \mathbb{P}(\bigcap_{k=0}^{m-1} (L_n^{kl_n}(\Pi|_{S_k}) \leq x)) \\
&= \prod_{k=0}^{m-1} \mathbb{P}(L_n^{kl_n}(\Pi|_{S_k}) \leq x) \\
&= \mathbb{P}(L_n(\Pi|_{S_0}) \leq x)^m.
\end{aligned} \tag{4.3}$$

Furthermore,

$$\begin{aligned}
\mathbb{P}(L_n(\Pi|_{S_0}) \leq x) &= \mathbb{P}((L_n(\Pi|_{S_0}) \leq x) \cap A) + \mathbb{P}((L_n(\Pi|_{S_0}) \leq x) \cap A^c) \\
&= \mathbb{P}((L_n \leq x) \cap A) + o\left(\frac{1}{n}\right) \\
&= \mathbb{P}(L_n \leq x) - \mathbb{P}((L_n \leq x) \cap A^c) + o\left(\frac{1}{n}\right) \\
&= \mathbb{P}(L_n \leq x) + o\left(\frac{1}{n}\right) \\
&= 1 - \mathbb{P}(L_n > x) + o\left(\frac{1}{n}\right) \\
&= 1 - \frac{1}{m} \log \frac{1}{1-\gamma^*} + o\left(\frac{1}{m}\right).
\end{aligned} \tag{4.4}$$

The last equality follows from Lemma 4.4 and the fact that $\frac{1}{n} = o(\frac{1}{m})$. Putting together (4.1), (4.2), (4.3) and (4.4), we obtain

$$\mathbb{P}(\tau_n^* \leq x) = \left(1 - \frac{1}{m} \log \frac{1}{1-\gamma^*} + o\left(\frac{1}{m}\right)\right)^m + o\left(\frac{1}{n}\right) \underset{n \to \infty}{\to} 1 - \gamma^*,$$

and thus

$$\mathbb{P}(\tau_n^* > x) \underset{n \to \infty}{\to} \gamma^*$$

which finishes the proof. □

We will now prove Proposition 4.3.

Proof of Proposition 4.3. Let $x \in \mathbb{R}$. It follows from Lemma 4.5 that

$$\mathbb{P}\left(\tau_n^* \leq 2\sqrt{n} + n^{\frac{1}{6}}\left(\left(\frac{1}{8}\log n\right)^{\frac{2}{3}} - \frac{\log s_n}{(\log n)^{\frac{1}{3}}} - \frac{\log(2\pi \log n)}{(\log n)^{\frac{1}{3}}} + \frac{x}{(\log n)^{\frac{1}{3}}}\right)\right) \underset{n\to\infty}{\to} e^{-e^{-x}} = \mathbb{P}(G \leq x).$$

Moreover,

$$\begin{aligned}
&\mathbb{P}\left(\tau_n^* \leq 2\sqrt{n} + n^{\frac{1}{6}}\left(\left(\frac{1}{8}\log n\right)^{\frac{2}{3}} - \frac{\log s_n}{(\log n)^{\frac{1}{3}}} - \frac{\log(2\pi \log n)}{(\log n)^{\frac{1}{3}}} + \frac{x}{(\log n)^{\frac{1}{3}}}\right)\right)\\
&= \mathbb{P}\left(\tau_n^* - \left(2\sqrt{n} + n^{\frac{1}{6}}\left(\left(\frac{1}{8}\log n\right)^{\frac{2}{3}} - \frac{\log s_n}{(\log n)^{\frac{1}{3}}} - \frac{\log(2\pi \log n)}{(\log n)^{\frac{1}{3}}}\right)\right) \leq \frac{n^{\frac{1}{6}}}{(\log n)^{\frac{1}{3}}}x\right)\\
&= \mathbb{P}\left(\frac{(\log n)^{\frac{1}{3}}}{n^{\frac{1}{6}}}\left[\tau_n^* - \left(2\sqrt{n} + n^{\frac{1}{6}}\left(\left(\frac{1}{8}\log n\right)^{\frac{2}{3}} - \frac{\log s_n}{(\log n)^{\frac{1}{3}}} - \frac{\log(2\pi \log n)}{(\log n)^{\frac{1}{3}}}\right)\right)\right] \leq x\right)
\end{aligned}$$

which finishes the proof. ☐

Just like in Section 3, we can use Lemma 4.5 in order to get a lower bound on the variance of τ_n^*, as stated in the following proposition.

Proposition 4.6. *The crossing time* τ_n^* *has a variance of order at least* $\frac{n^{\frac{1}{3}}}{(\log n)^{\frac{1}{3}}}$ *when* n *goes to infinity.*

4.3 Expected value of τ_n^*

We will use the result about the convergence in distribution of τ_n^* to show the following asymptotic lower bound on its expected value.

Proposition 4.7. *Let* γ *be the Euler-Mascheroni constant. For any* $\epsilon > 0$*, there exists* $n_0 \in \mathbb{N}$*, such that for all* $n \geq n_n$*,*

$$\mathbb{E}[\tau_n^*] \geq 2\sqrt{n} + n^{\frac{1}{6}}\left(\left(\frac{1}{8}\log n\right)^{\frac{2}{3}} - \frac{\log s_n}{(\log n)^{\frac{1}{3}}} - \frac{\log(2\pi \log n)}{(\log n)^{\frac{1}{3}}} + \frac{\gamma(1-\epsilon)}{(\log n)^{\frac{1}{3}}}\right).$$

Bevis. Let $X_n = \frac{(\log n)^{\frac{1}{3}}}{n^{\frac{1}{6}}}\left[\tau_n^* - \left(2\sqrt{n} + n^{\frac{1}{6}}\left(\left(\frac{1}{8}\log n\right)^{\frac{2}{3}} - \frac{\log s_n}{(\log n)^{\frac{1}{3}}} - \frac{\log(2\pi \log n)}{(\log n)^{\frac{1}{3}}}\right)\right)\right]$. Proposition 4.3 gives us

$$X_n \xrightarrow{d} G$$

and thus by Lemma 2.13

$$\liminf \mathbb{E}[X_n] \geq \mathbb{E}[G] = \gamma.$$

This yields

$$\liminf \frac{(\log n)^{\frac{1}{3}}}{n^{\frac{1}{6}}}\left[\mathbb{E}[\tau_n^*] - \left(2\sqrt{n} + n^{\frac{1}{6}}\left(\left(\frac{1}{8}\log n\right)^{\frac{2}{3}} - \frac{\log s_n}{(\log n)^{\frac{1}{3}}} - \frac{\log(2\pi \log n)}{(\log n)^{\frac{1}{3}}}\right)\right)\right] \geq \gamma,$$

and thus asymptotically

$$\mathbb{E}[\tau_n^*] \geq 2\sqrt{n} + n^{\frac{1}{6}}\left(\left(\frac{1}{8}\log n\right)^{\frac{2}{3}} - \frac{\log s_n}{(\log n)^{\frac{1}{3}}} - \frac{\log(2\pi \log n)}{(\log n)^{\frac{1}{3}}} + \frac{\gamma(1-\epsilon)}{(\log n)^{\frac{1}{3}}}\right)$$

for any $\epsilon > 0$. ☐

We will now derive an upper bound on $\mathbb{E}(\tau_n^*)$ using the results from Section 3.

Proposition 4.8. *The following asymptotic upper bound holds:*

$$\mathbb{E}[\tau_n^*] \leq 2\sqrt{n} + n^{\frac{1}{6}} \left(\left(\frac{1}{2} \log n \right)^{\frac{2}{3}} - \frac{\log s_n}{(4 \log n)^{\frac{1}{3}}} \right) + o(1).$$

Bevis. Let $L_n^{(1)}, ..., L_n^{(m)}$ be m independent copies of L_n. Lemma 2.10 ensures that $L_n^{kl_n} \sim L_n$ for all $k \in \{0, ..., m-1\}$, thus it follows from Corollary 2.18 that

$$\mathbb{E}[\tau_n^*] = \mathbb{E}\left[\max_{0 \leq k \leq m-1} L_n^{kl_n} \right] \leq \mathbb{E}\left[\max_{1 \leq i \leq m} L_n^{(i)} \right].$$

Recall that $m \sim \frac{n^{\frac{1}{6}}}{s_n}$ with s_n logarithmic in n. Using Remark 3.1 and Proposition 3.5 we get

$$\begin{aligned} \mathbb{E}[\tau_n^*] &\leq 2\sqrt{n} + n^{\frac{1}{6}} \left(\left[\frac{3}{4} \left(\left(\frac{1}{6} + \frac{1}{2} \right) \log n \right) \right]^{\frac{2}{3}} - \frac{\log s_n}{(6(\frac{1}{6} + \frac{1}{2}) \log n)^{\frac{1}{3}}} \right) + o(1) \\ &= 2\sqrt{n} + n^{\frac{1}{6}} \left(\left(\frac{1}{2} \log n \right)^{\frac{2}{3}} - \frac{\log s_n}{(4 \log n)^{\frac{1}{3}}} \right) + o(1) \end{aligned}$$

which finishes the proof. □

5 Analysis of the actual problem

5.1 Expected value of τ_n

A first remark we can make is that, since τ_n is the maximum over all oriented paths on the torus, while τ_n^* is the maximum over only a subset of paths, τ_n is necessarily larger than τ_n^*. It follows that $\mathbb{E}(\tau_n) \geq \mathbb{E}(\tau_n^*)$, and so from the lower bound on $\mathbb{E}(\tau_n^*)$ which we derived in Proposition 4.7, we get the same lower bound on $\mathbb{E}(\tau_n)$.

Corollary 5.1. *Let γ be the Euler-Mascheroni constant. For any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_n$,*

$$\mathbb{E}[\tau_n] \geq 2\sqrt{n} + n^{\frac{1}{6}} \left(\left(\frac{1}{8} \log n \right)^{\frac{2}{3}} - \frac{\log s_n}{(\log n)^{\frac{1}{3}}} - \frac{\log(2\pi \log n)}{(\log n)^{\frac{1}{3}}} + \frac{\gamma(1-\epsilon)}{(\log n)^{\frac{1}{3}}} \right).$$

We will now derive the following upper bound on the expectation of τ_n.

Proposition 5.2. *The following asymptotic upper bound holds*

$$\mathbb{E}[\tau_n] \leq 2\sqrt{n} + n^{\frac{1}{6}} \left(1 + \left(\frac{5}{8} \log n \right)^{\frac{2}{3}} \right) + o(1).$$

For the proof of this propositions we will need the following two definitions:

Definition 5.3. Let $a, b \in \mathbb{T}_n$ of respective coordinates (x_a, y_a) and (x_b, y_b), and such that $y_a \leq y_b$. An oriented path from a to b with at most two loops is a subset $\{(x_1, y_1), ...(x_k, y_k)\}$ of Π verifying the following properties:

- If $x_a < x_b$, then there exists $i \in \{0, ..., k\}$ such that

  $$x_a \leq x_1 \leq ... \leq x_i,$$

  $$x_{i+1} \leq ... \leq x_k \leq x_b$$

  and

  $$y_a \leq y_1 \leq ... \leq y_k \leq y_b.$$

- If $x_b \leq x_a$, then there exist $i, j \in \{0, ..., k\}$ such that
$$x_a \leq x_1 \leq ... \leq x_i$$
$$x_{i+1} \leq ... \leq x_j$$
$$x_{j+1} \leq ... \leq x_k \leq x_b$$
  and
$$y_a \leq y_1 \leq ... \leq y_k \leq y_b,$$

with the same conventions as in Definition 2.2. The integer k is called the length of the path. We denote by $\mathcal{P}^2_{a\to b}$ the set of all such paths.

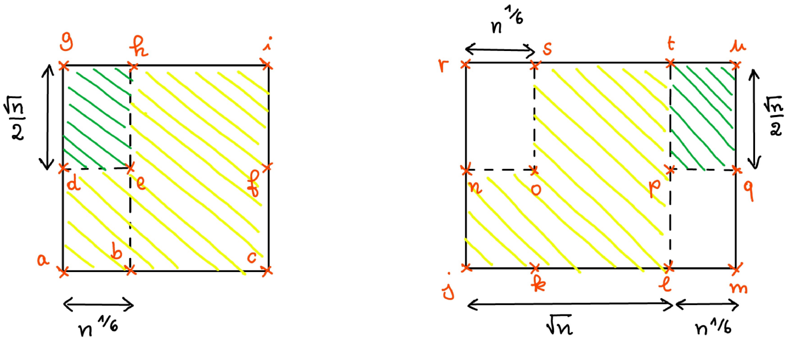
Definition 5.4. Let $a, b \in \mathbb{T}_n$ with respective coordinates (x_a, y_a) and (x_b, y_b), and such that $y_a \leq y_b$. Define
$$R_{a\to b} = \max\left\{|p| | p \in \mathcal{P}^2_{a\to b}\right\}.$$
A path $p \in \mathcal{P}^2_{a\to b}$ such that $|p| = R_{a\to b}$ is called a geodesic with at most two loops from a to b.

Proof of Proposition 5.2. Let $m = \lceil n^{\frac{1}{3}} \rceil$. For i in $\{1, ..., m-1\}$ let
$$M_i = \max_{x\in[(i-1)n^{\frac{1}{6}}, in^{\frac{1}{6}}]} L^x_n,$$
and let
$$M_m = \max_{x\in[(m-1)n^{\frac{1}{6}}, \sqrt{n}]} L^x_n.$$
It follows from the definition of τ_n that $\tau_n = \max_{1\leq i\leq m} M_i$. We define also, for i in $\{1, ..., m-1\}$,
$$R_i = R_{((i-1)n^{\frac{1}{6}},0)\to(in^{\frac{1}{6}},\sqrt{n})},$$
and
$$R_m = R_{((m-1)n^{\frac{1}{6}},0)\to(\sqrt{n},\sqrt{n})}$$
It follows from the definition of oriented paths that
$$\bigcup_{x\in[(i-1)n^{\frac{1}{6}}, in^{\frac{1}{6}}]} \mathcal{P}_{(x,0)\to(x,\sqrt{n})} \subset \mathcal{P}^2_{((i-1)n^{\frac{1}{6}},0)\to(in^{\frac{1}{6}},\sqrt{n})},$$
hence $R_i \geq M_i$ for all i in $\{1, ..., m\}$. Let $R'_1, ..., R'_m$ be mutually independent copies of $R_1, ..., R_m$ ($R'_1, ..., R'_{m-1}$ are iid following the distribution of R_1, and R'_m is an independent copy of R_m). Hence, using Corollary 2.18 we have
$$\mathbb{E}(\tau_n) = \mathbb{E}(\max_{1\leq i\leq m} M_i) \leq \mathbb{E}(\max_{1\leq i\leq m} R_i) \leq \mathbb{E}(\max_{1\leq i\leq m} R'_i). \tag{5.1}$$

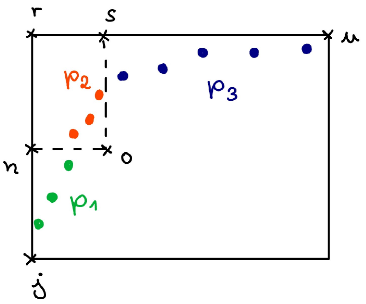
Note that since $\sqrt{n} - (m-1)n^{\frac{1}{6}} \leq n^{\frac{1}{6}}$, R'_m is stochastically dominated by R_1, and hence the inequality above still holds if we assume that R'_m follows the distribution of R_1, rather than R_m. We will make this assumption from now on.

We will now study the law of R_1, and link it to the law of L'_n with $n' = n + n^{\frac{2}{3}}$. Recall that $\mathbb{T}_n$ is a $\sqrt{n}$ by $\sqrt{n}$ square, and Π a Poisson point process with intensity one on $\mathbb{T}_n$. Let $\mathbb{T}'_n$ be a $\sqrt{n}$ by $(\sqrt{n} + n^{\frac{1}{6}})$ rectangle. Note that $\mathbb{T}'_n$ has area n'. Let a, b, c, d, e, f, g, h and i be points on $\mathbb{T}_n$ and $j, k, l, m, n, o, p, q, r, s, t$ and u be points on $\mathbb{T}'_n$, placed like on Figure 2. We consider Π' a Poisson point process with intensity one on $\mathbb{T}'$ coupled with Π in the following way:

- Let Π_1 be a Poisson point process with intensity one on $rson$ and Π_2 be a Poisson point process with intensity one on $pqml$.
- We define $\Pi'|_{rson} = \Pi_1$, $\Pi'|_{pqml} = \Pi_2$, $\Pi'|_{stljno} = \Pi|_{hicade}$ (see area hatched in yellow on Figure 2) and $\Pi'|_{tuqp} = \Pi|_{ghed}$ (see area hatched in green on Figure 2).



Figur 2: Coupling between Π and Π'



Figur 3: Path p

The independence property of Poisson processes ensures that Π' constructed as above is indeed a Poisson point process with intensity one on $\mathbb{T}_n'$. We thus have the following equalities in distribution:

$$R_{a\to h} \sim R_1 \quad \text{and} \quad L_{j\to u} \sim L_{n'}.$$

Let $\epsilon \in (0, \frac{2-\sqrt{2}}{4})$. We consider the following event:

$$G = \left\{ L_{x\to y}(\Pi') \leq 2\sqrt{Area_{x,y}} + \epsilon\sqrt{n} \quad \forall (x,y) \in P \right\} \bigcap \left\{ L_{j\to u}(\Pi') \geq 2\sqrt{Area_{j,u}} - \epsilon\sqrt{n} \right\}$$

where $P = \{(j,o),(n,s),(o,u),(j,p),(l,q),(p,u)\}$. The following two lemmas show that on the event G, $R_{a\to h} = L_{j\to u}$, and that G^c has small probability.

Lemma 5.5. *For n large enough*

$$\mathbf{1}_G L_{j\to u} = \mathbf{1}_G R_{a\to h}.$$

Bevis. We will first show that, on the event G, a geodesic from j to u does not intersect the rectangles $rson$ and $pqml$. Suppose event G occurs. Let p in $\mathcal{P}_{j\to u}(\Pi')$. Suppose that p intersects $rson$. Then we can partition p into three sub-paths $p = p_1 \cup p_2 \cup p_3$ such that $p_1 \in \mathcal{P}_{j\to o}$, $p_2 \in \mathcal{P}_{n\to s}$ and $p_3 \in \mathcal{P}_{o\to u}$ (see Figure 3).

From the definition of G we know that

- $|p_1| \leq L_{j\to o}(\Pi') \leq 2\left(\frac{\sqrt{n}}{2}n^{\frac{1}{6}}\right)^{\frac{1}{2}} + \epsilon\sqrt{n} = \sqrt{2}n^{\frac{1}{3}} + \epsilon\sqrt{n}$,
- $|p_2| \leq L_{n\to s}(\Pi') \leq \sqrt{2}n^{\frac{1}{3}} + \epsilon\sqrt{n}$,
- $|p_3| \leq L_{o\to u}(\Pi') \leq 2\left(\frac{\sqrt{n}}{2}\sqrt{n}\right)^{\frac{1}{2}} + \epsilon\sqrt{n} = (\sqrt{2}+\epsilon)\sqrt{n}$,

hence $|p| \leq (\sqrt{2}+3\epsilon)\sqrt{n} + o(\sqrt{n})$.
Moreover, it follows from the definition of the event G that $L_{j\to u}(\Pi') \geq 2(n+n^{\frac{2}{3}})^{\frac{1}{2}} - \epsilon\sqrt{n} = (2-\epsilon)\sqrt{n} + o(\sqrt{n})$. Hence, for n large enough $|p| < L_{j\to u}$ and thus p is not a geodesic from j to u. We conclude that, under the event G, a geodesic from j to u cannot intersect $rson$. An analogue argument shows that under the event G a geodesic from j to u cannot intersect $pqml$. Note that, given the coupling between Π and Π', for every oriented path from j to u on $\mathbb{T}'_n$ that does not intersect the rectangles $rson$ and $pqml$, one can find an oriented path with at most two loops from a to h on $\mathbb{T}_n$. Consequently, under the event G, $R_{a\to h} \geq L_{j\to u}$. A similar argument shows that, under the event G, for any geodesic with at most two loops from a to h on $\mathbb{T}'_n$, one can find an oriented path from j to u on $\mathbb{T}_n$ which has the same length. This shows that, under the event G, $R_{a\to h} = L_{j\to u}$ as stated. ☐

Lemma 5.6. *The event G satisfies $Q(n)\mathbb{P}(G^c) \underset{n\to\infty}{\to} 0$ for any polynomial Q.*

Bevis. It follows from Lemma 2.12 that

$$L_{x\to y} \sim L_{\frac{1}{2}n^{\frac{2}{3}}} \quad \text{for} \quad (x,y) \in \{(j,o),(n,s),(l,q),(p,u)\},$$

and

$$L_{x\to y} \sim L_{\frac{1}{2}(n+n^{\frac{2}{3}})} \quad \text{for} \quad (x,y) \in \{(o,u),(j,p)\}.$$

For n large enough and for $(x,y) \in \{(j,o),(n,s),(l,q),(p,u)\}$,

$$\mathbb{P}\left(L_{x\to y} > 2\sqrt{Area_{x,y}} + \sqrt{n}\epsilon\right) = \mathbb{P}\left(L_{\frac{1}{2}n^{2/3}} > 2\sqrt{\frac{n^{\frac{2}{3}}}{2}} + \epsilon\sqrt{n}\right) \overset{\text{for large } n}{\leq} \mathbb{P}\left(L_{\frac{1}{2}n^{2/3}} > 3\sqrt{\frac{n^{\frac{2}{3}}}{2}}\right).$$

Moreover, Theorem 2.6 ensures that for large n

$$\mathbb{P}\left(L_{\frac{1}{2}n^{2/3}} > 3\sqrt{\frac{n^{\frac{2}{3}}}{2}}\right) \leq \exp\left(-\frac{n^{\frac{1}{3}}}{2}I(3)\right),$$

hence for all (x,y) in $\{(j,o),(n,s),(l,q),(p,u)\}$, $\mathbb{P}\left(L_{x\to y} > 2\sqrt{Area_{x,y}} + \sqrt{n}\epsilon\right)$ decays faster than any polynomial in n. Analogue computations show that this also holds for (x,y) in $\{(o,u),(j,p)\}$, while similar arguments applied with Theorem 2.7 show that $\mathbb{P}(L_{j\to u}(\Pi') < 2\sqrt{Area_{j,u}} - \epsilon\sqrt{n})$ decays faster than any polynomial. Applying an union bound to $\mathbb{P}(G^c)$ concludes. ☐

Let $(\Pi^{(1)},\Pi'^{(1)}),...,(\Pi^{(m)},\Pi'^{(m)})$ be independent copies of the pair (Π,Π') of Poisson point processes, on $\mathbb{T}$ and $\mathbb{T}'$. Denote by $(R^{(1)},L^{(1)}),...(R^{(m)},L^{(m)})$ the corresponding crossing times. For i in $\{1,...,m\}$ let $G^{(i)}$ be the analogue of the event G for the couple $(\Pi^{(i)},\Pi'^{(i)})$. Recall in particular that

$$\mathbf{1}_{G^{(i)}}R^{(i)} = \mathbf{1}_{G^{(i)}}L^{(i)}. \tag{5.2}$$

Let $G_m = \bigcap_{i=1}^m G^{(i)}$. Note that since m is polynomial in n, Lemma 5.6 ensures that $\mathbb{P}(G_m^c)$ decays faster than any polynomial. Note also that

$$(R'_1,...,R'_m) \sim (R^{(1)},...,R^{(m)}),$$

and thus by (5.1) we have

$$\mathbb{E}[\tau_n] \leq \mathbb{E}\left[\max_{1\leq i\leq m} R^{(i)}\right] = \mathbb{E}\left[\left(\max_{1\leq i\leq m} R^{(i)}\right)\mathbf{1}_{G_m}\right] + \mathbb{E}\left[\left(\max_{1\leq i\leq m} R^{(i)}\right)\mathbf{1}_{G_m^c}\right]. \tag{5.3}$$

Using (5.2) we have

$$\mathbb{E}\left[\left(\max_{1\leq i\leq m} R^{(i)}\right)\mathbf{1}_{G_m}\right] = \mathbb{E}\left[\left(\max_{1\leq i\leq m} L^{(i)}\right)\mathbf{1}_{G_m}\right] \leq \mathbb{E}\left[\max_{1\leq i\leq m} L^{(i)}\right] \leq 2\sqrt{n'} + n'^{\frac{1}{6}}\tilde{z}_{n'} + o(1)$$

where the second inequality comes from Proposition 3.5. Plugging in $n' = n\left(1+\frac{1}{n^{\frac{1}{3}}}\right)$ to the formula for $\tilde{z}_n$, with $\alpha = \frac{1}{3}$ and $\beta = 1$, we

$$\tilde{z}_{n'} = \left(\frac{5}{8}\log n\right)^{\frac{2}{3}} + o\left(\frac{1}{n^{\frac{1}{3}}}\right).$$

Moreover

$$2\sqrt{n'} = 2\sqrt{n} + n^{\frac{1}{6}} + o\left(\frac{1}{n^{\frac{1}{6}}}\right) \quad \text{and} \quad n'^{\frac{1}{6}} = n^{\frac{1}{6}} + O\left(\frac{1}{n^{\frac{1}{6}}}\right) + o(1)$$

hence

$$\mathbb{E}\left[\left(\max_{1\leq i\leq m} R^{(i)}\right)\mathbf{1}_{G_m}\right] \leq 2\sqrt{n} + n^{\frac{1}{6}} + n^{\frac{1}{6}}\left(\frac{5}{8}\log n\right)^{\frac{2}{3}} + o(1).$$

We bound the second term of (5.3) using the Cauchy-Schwarz inequality

$$\mathbb{E}\left[\left(\max_{1\leq i\leq m} R^{(i)}\right)\mathbf{1}_{G_m^c}\right] \leq \mathbb{E}\left[\left(\max_{1\leq i\leq m} R^{(i)}\right)^2\right]^{\frac{1}{2}} \mathbb{P}(G_m^c)^{\frac{1}{2}}.$$

Using the the fact that the crossing times $R^{(i)}$ are non negative we have

$$\mathbb{E}\left[\left(\max_{1\leq i\leq m} R^{(i)}\right)^2\right] = \mathbb{E}\left[\max_{1\leq i\leq m} R^{(i)^2}\right] \leq m\mathbb{E}\left[R^{(1)^2}\right] \leq m(n^2+n),$$

where the last inequality follows from the fact that $R^{(i)}$ is bounded by $|\Pi^{(1)}|$, and $|\Pi^{(1)}| \sim Pois(n)$. The upper bound $m(n+n^2)$ is polynomial in n, yet $\mathbb{P}(G_m^c)$ decays faster then any polynomial, thus $\mathbb{E}\left[\left(\max_{1\leq i\leq m} R^{(i)}\right)\mathbf{1}_{G_m^c}\right] = o(1)$ which finishes the proof.

□

The two bounds we have for the expected value of τ_n are of the form

$$2\sqrt{n} + Cn^{\frac{1}{6}}(\log n)^{\frac{2}{3}}(1+o(1)),$$

with $B = B_1 := (\frac{1}{8})^{\frac{2}{3}} = \frac{1}{4}$ for the lower bound, and $B = B_2 := \left(\frac{5}{8}\right)^{\frac{2}{3}} = \frac{5^{\frac{2}{3}}}{4}$ for the upper bound. This suggests that the expectation of τ_n is of the form

$$\mathbb{E}(\tau_n) = 2\sqrt{n} + Bn^{\frac{1}{6}}(\log n)^{\frac{2}{3}} + o(n^{\frac{1}{6}}(\log n)^{\frac{2}{3}})$$

with B in $\left[\frac{1}{4}, \frac{5^{\frac{2}{3}}}{4}\right]$.

This is confirms the conjecture made by M. H. Albert, M. D. Atkinson, D. Nussbaum, J. Sack and N. Santoro in [1] that the expected value should be close to $2\sqrt{n} + n^{\frac{1}{6}}$.

A Proof of Theorem 2.8

Theorem 2.8 is a special case of Theorem 1.3 stated by M. Löwe and F. Merkl in [7]. In the notations of this report, the theorem states the following:

Theorem A.1. *For every $n \in \mathbb{N}^*$, $l \in \mathbb{N}$ with $2\sqrt{n} < l$, we have the following asymptotics, uniformly in $\gamma_{l,n}$:*

$$\mathbb{P}(L_n > l) \sim \frac{(\gamma_{l,n} + 2\sqrt{1-\gamma_{l,n}^2})\gamma_{l,n}^2 e^{2lw_0(\gamma_{l,n})}}{8\pi l(1-\gamma_{l,n}^2)^{\frac{3}{2}}(1+\sqrt{1-\gamma_{l,n}^2})} \quad \textit{as } M_{l,n} \to +\infty, \tag{A.1}$$

where

$$\gamma_{l,n} := \frac{2\sqrt{n}}{l}, \quad M_{l,n} := \frac{l-2\sqrt{n}}{l^{\frac{1}{3}}}, \quad \textit{and} \quad w_0(\gamma) := \sqrt{1-\gamma^2} - \operatorname{arcosh}\frac{1}{\gamma}.$$

Let t_n be as in Lemma 2.8. We apply the above theorem with

$$l := 2\sqrt{n} + n^{\frac{1}{6}} t_n.$$

We have $l > 2\sqrt{n}$ and $M_{l,n} \sim \frac{t_n}{2^{\frac{1}{3}}} \to +\infty$ so the conditions of Theorem 1.3 hold. Note that in that case $\gamma_{l,n} \sim 1$, so (A.1) can be simplified as follows:

$$\mathbb{P}(L_n > 2\sqrt{n} + n^{\frac{1}{6}} t_n) \sim \frac{e^{2lw_0(\gamma_{l,n})}}{8\pi l (1-\gamma_{l,n}^2)^{\frac{3}{2}}} \quad \text{as } M_{l,n} \to +\infty. \tag{A.2}$$

To further simplify this expression, we compute the asymptotics of $w_0(\gamma_{l,n}) = \sqrt{1-\gamma_{l,n}^2} - \text{arcosh}\frac{1}{\gamma_{l,n}}$. Let

$$u_n := \frac{t_n}{n^{\frac{1}{3}}}.$$

We then have

$$l = 2\sqrt{n} + n^{\frac{1}{6}} t_n = 2\sqrt{n}(1 + \frac{u_n}{2}).$$

In order to get an equivalent of $e^{2lw_0(\gamma_{l,n})}$, given that $l \sim 2\sqrt{n}$, we need to compute the asymptotics of $w_0(\gamma_{l,n})$ to the order $o(\frac{1}{\sqrt{n}})$. Yet, recalling that $t_n = o(n^{\frac{1}{8}})$, we have

$$u_n^2\sqrt{u_n} = \frac{t_n^{\frac{5}{2}}}{n^{\frac{5}{6}}} = o(\frac{n^{\frac{5}{16}}}{n^{\frac{5}{6}}}), \quad \text{and} \quad \sqrt{n}\frac{n^{\frac{5}{16}}}{n^{\frac{5}{6}}} = \frac{n^{\frac{5}{16}}}{n^{\frac{1}{3}}} = o(1), \quad \text{so} \quad u_n^2\sqrt{u_n} = o(\frac{1}{\sqrt{n}})$$

and it is therefore enough to compute the asymptotics of $w_n(\gamma_{l,n})$ to the order $O(u_n^2\sqrt{u_n})$. We have

$$\begin{aligned} 1-\gamma_{l,n}^2 &= \frac{l^2-4n}{l^2} \\ &= \frac{4n(1+\frac{u_n}{2})^2 - 4n}{4n(1+\frac{u_n}{2})^2} \\ &= \frac{u_n + \frac{u_n^2}{4}}{(1+\frac{u_n}{2})^2}, \end{aligned} \tag{A.3}$$

so

$$\begin{aligned} \sqrt{1-\gamma_{l,n}^2} &= \sqrt{u_n}\frac{\sqrt{1+\frac{u_n}{4}}}{1+\frac{u_n}{2}} \\ &= \sqrt{u_n}(1+\frac{u_n}{8} + O(u_n^2))(1-\frac{u_n}{2} + O(u_n^2)) \\ &= \sqrt{u_n}(1-\frac{3}{8}u_n + O(u_n^2)). \end{aligned} \tag{A.4}$$

We have

$$\frac{1}{\gamma_{l,n}} = \frac{2\sqrt{n}(1+\frac{u_n}{2})}{2\sqrt{n}} = 1 + \frac{u_n}{2}.$$

Recall the asymptotic development of Arcosh around 1:

$$\text{arcosh}(1+x) = \sqrt{2x} - \frac{\sqrt{2x}}{12}x + O(x^2\sqrt{x}).$$

We thus have

$$\text{arcosh}(\frac{1}{\gamma_{l,n}}) = \sqrt{u_n}(1 - \frac{1}{24}u_n + O(u_n^2)). \tag{A.5}$$

Subtracting (A.5) to (A.4) yields

$$\begin{aligned} w_0(\gamma_{l,n}) &= \sqrt{u_n}(-\frac{3}{8} + \frac{1}{24})u_n + O(u_n^2\sqrt{u_n}) \\ &= -\frac{1}{3}\frac{t_n^{\frac{3}{2}}}{\sqrt{n}} + o(\frac{1}{\sqrt{n}}). \end{aligned}$$

Hence,

$$\begin{aligned}\exp(2lw_0(\gamma_{l,n})) &= \exp((4\sqrt{n} + 2n^{\frac{1}{6}}t_n)(-\frac{1}{3}\frac{t_n^{\frac{3}{2}}}{\sqrt{n}} + o(\frac{1}{\sqrt{n}})) \\ &= \exp(-\frac{4}{3}t_n^{\frac{2}{3}} + o(1)) \sim \exp(-\frac{4}{3}t_n^{\frac{2}{3}}).\end{aligned}$$

In the second equality we used the fact that

$$n^{\frac{1}{6}}t_n\frac{t_n^{\frac{3}{2}}}{\sqrt{n}} = \frac{t_n^{\frac{5}{2}}}{n^{\frac{1}{3}}} = o(n^{\frac{5}{16}-\frac{1}{3}}) = o(1).$$

We will now compute an equivalent of the denominator in (A.2).
We know from (A.3) that

$$1 - \gamma_{l,n}^2 = \frac{u_n + \frac{u_n^2}{4}}{(1 + \frac{u_n}{2})^2} \sim u_n,$$

thus

$$(1 - \gamma_{l,n}^2)^{\frac{3}{2}} \sim u_n^{\frac{3}{2}} = \frac{t_n^{\frac{3}{2}}}{\sqrt{n}}.$$

Finally, using the fact that $l \sim 2\sqrt{n}$, we get

$$\frac{e^{2lw_0(\gamma_l,n)}}{8\pi l(1 - \gamma_{l,n}^2)^{\frac{3}{2}}} \sim \frac{e^{-\frac{4}{3}t_n^{\frac{3}{2}}}}{16\pi t_n^{\frac{3}{2}}}$$

which finishes the proof of Lemma 2.8.

Erratum

- The proof of Proposition 3.6 in Section 3 is incorrect. We apply Fatou's lemma to the family of random variables $(X_n)_{n\in\mathbb{N}}$, where

$$X_n = \frac{(6\alpha\log n)^{\frac{1}{3}}}{n^{\frac{1}{6}}}\left[\tilde{\tau}_n - \left(2\sqrt{n} + n^{\frac{1}{6}}\left(\left(\frac{3}{4}\alpha\log n\right)^{\frac{2}{3}} - \frac{\log\log n}{(6\alpha\log n)^{\frac{1}{3}}} + \frac{\log\beta}{(6\alpha\log n)^{\frac{1}{3}}} - \frac{\log(12\pi\alpha)}{(6\alpha\log n)^{\frac{1}{3}}}\right)\right)\right],$$

  but these random variables are not non-negative. Although the proof is incorrect, Proposition 3.6 still holds. Indeed, it follows by long but standard computations from Theorem 1.3 from [7] that the family of random variables $(X_n)_{n\in\mathbb{N}}$ is uniformly integrable. We thus get

$$\mathbb{E}[\tilde{\tau}_n] = 2\sqrt{n} + n^{\frac{1}{6}}\tilde{x}_n(\gamma) + o\left(\frac{n^{\frac{1}{6}}}{(\log n)^{\frac{1}{3}}}\right)$$

  which in particular implies the lower bound from Proposition 3.6.

- Similarly, the proof of Proposition 4.7 in Section 4 is also incorrect as Fatou's lemma is applied to a family of random variables which are not non-negative. The result still holds as this family is uniformly integrable. We thus get

$$\mathbb{E}[\tau_n^*] = 2\sqrt{n} + n^{\frac{1}{6}}x_n^*(\gamma) + o\left(\frac{n^{\frac{1}{6}}}{(\log n)^{\frac{1}{3}}}\right),$$

  which in particular implies the lower bound from Proposition 4.7.

Referenser

[1] Michael Albert, M. Atkinson, Doron Nussbaum, and Jörg-Rüdiger Sack. On the longest increasing subsequence of a circular list. *Inf. Process. Lett.*, 101:55–59, 01 2007. https://doi.org/10.1016/j.ipl.2006.08.003 doi:10.1016/j.ipl.2006.08.003.

[2] Jinho Baik, Percy Deift, and Kurt Johansson. On the distribution of the length of the longest increasing subsequence of random permutations. *Journal of the American Mathematical Society*, 12(4):1119–1178, 1999. URL: http://www.jstor.org/stable/2646100.

[3] Alan Hammond and Sourav Sarkar. Modulus of continuity for polymer fluctuations and weight profiles in poissonian last passage percolation. *Electronic Journal of Probability*, 25, 01 2020. https://doi.org/10.1214/20-EJP430 doi:10.1214/20-EJP430.

[4] Kurt Johansson. Transversal fluctuations for increasing subsequences on the plane. *Probability Theory and Related Fields*, 116, 11 1999. https://doi.org/10.1007/s004400050258 doi:10.1007/s004400050258.

[5] Olav Kallenberg. *Foundations of modern probability / Olav Kallenberg*. Springer, Jan 1997. URL: https://research.ebsco.com/linkprocessor/plink?id=694470cf-f94c-3c32-be95-be352053ffe6.

[6] Günter Last and Mathew Penrose. *Lectures on the Poisson Process*. Institute of Mathematical Statistics Textbooks. Cambridge University Press, 2017.

[7] Matthias Löwe and Franz Merkl. Moderate deviations for longest increasing subsequences: The upper tail. *Communications on Pure and Applied Mathematics*, 54(12):1488–1519, 2001. URL: https://onlinelibrary.wiley.com/doi/abs/10.1002/cpa.10010, https://arxiv.org/abs/https://onlinelibrary.wiley.com/doi/pdf/10.1002/cpa.10010 arXiv:https://onlinelibrary.wiley.com/doi/pdf/10.1002/cpa.10010, https://doi.org/10.1002/cpa.10010 doi:10.1002/cpa.10010.

[8] Dan Romik. *The Surprising Mathematics of Longest Increasing Subsequences*. 10 2014. https://doi.org/10.1017/CBO9781139872003 doi:10.1017/CBO9781139872003.

[9] Timo Seppäläinen. Large deviations for increasing sequences on the plane. *Probab. Theory Relat. Fields*, 112(2):221–244, October 1998.