STOCKHOLMS UNIVERSITET
MATEMATISKA INSTITUTIONEN
Avd. Matematisk statistik

MT5011 – Part TEOR
EXAM
May 27, 2020

Suggested solutions

Exam in Basic Insurance Mathematics, 7.5 credits

May 27, 2020 – time: 9–17

Problem 1

We can start by noting that we are given an *incremental claim amounts* claims triangle, but the Chain Ladder model is defined in terms of aggregated amounts, and the corresponding claims triangle with *accumulated* amounts is given by

	1	2	3
1	1523	3843	4030
2	785	1706	$C_{2,3}$
3	343	$C_{3,2}$	$C_{3,3}$

where we have added the future accumulated amounts $C_{i,j}$ for $i+j > 4$, and due to that there are only two development years to be predicted, the Chain Ladder model only has two development factors to be estimated; f_1, f_2:

$$\widehat{f}_1 = \frac{3843 + 1706}{1523 + 785}, \text{ and } \widehat{f}_2 = \frac{4030}{3843}.$$

The **a)**-part amounts to calculating the reserves for accident year 2 and 3, which are given by

$$\widehat{R}_2 = c_{2,2}(\widehat{f}_2 - 1) = 83.01, \text{ and } \widehat{R}_3 = c_{3,1}(\widehat{f}_1\widehat{f}_2 - 1) = 521.78.$$

The **b)**-part corresponds to calculating the expected values of the incremental $I_{i,j}$s in the original data table

	1	2	3
1	1523	3843	4030
2	785	1706	$I_{2,3}$
3	343	$I_{3,2}$	$i_{3,3}$

using the Chain Ladder model. To start off, note that $\widehat{I}_{2,3} = \widehat{R}_2$. Further, from the relation between incremental and accumulated amounts, it follows that

$$\widehat{I}_{3,2} = \widehat{C}_{3,2} - c_{3,1}, \text{ and } \widehat{I}_{3,3} = \widehat{C}_{3,3} - \widehat{C}_{3,2},$$

which from the lecture notes gives us that

$$\widehat{I}_{3,2} = c_{3,1}(\widehat{f}_1 - 1) = 481.66, \text{ and } \widehat{I}_{3,3} = c_{3,1}\widehat{f}_1(\widehat{f}_2 - 1) = 40.13,$$

where you can check that $\widehat{R}_3 = \widehat{I}_{3,2} + \widehat{I}_{3,3}$ as it should.

Problem 2

The problem can be thought of as a deferred annuity with at most one payment, two years into the future as seen from today of a today 65 year old individual. That is, the individual will receive 1 unit of money if the individual is alive at its 67th birthday, which corresponds to the random discounted payment (or total cost)

$$L = d(2)1_{\{\text{alive at 67, given 65 today}\}}$$

where $d(2) = \exp\{-2r_2\}$, and

$$\mathbb{E}[L] = d(2)\mathbb{P}(T_0 > 67 \mid T_0 > 65) = d(2)p_{65}p_{66},$$

where the last equality follows from the lecture notes, where $p_x = 1 - q_x$. That is,

$$\mathbb{E}[L] = 0.958$$

which is equal to the discounted fair premium. This answers part **a)**.

Part **b)** can be answered by noting that the stochastic indicator $1_{\{\text{alive at 67, given 65 today}\}}$ is Bernoulli(p) distributed, where $p = p_{65}p_{66}$, which gives us that

$$\text{Var}(1_{\{\text{alive at 67, given 65 today}\}}) = p(1-p),$$

and that

$$\text{Var}(L) = d(2)^2 \,\text{Var}(1_{\{\text{alive at 67, given 65 today}\}}) = d(2)^2 p(1-p) = 0.012.$$

Problem 3

The idea with this problem is to use the thinning property of Poisson processes described in the lecture notes: Let $N^{(i)}$, $i = 0, 1$, denote the number of small ($i = 0$) and large ($i = 1$) claims. It then holds that

$$N^{(0)} \sim \mathrm{Po}(p\lambda),\ N^{(1)} \sim \mathrm{Po}((1-p)\lambda),\ \text{and } N := N^{(0)} + N^{(1)} \sim \mathrm{Po}(\lambda),$$

and in particular $\mathbb{E}[N^{(0)}] = \mathrm{Var}(N^{(0)}) = p\lambda$ and $\mathbb{E}[N^{(1)}] = \mathrm{Var}(N^{(1)}) = (1-p)\lambda$, which is needed to be able to solve the problem.

Introduce the total claim costs for next year's small and large claims, which are given by

$$S_{N^{(i)}} = \sum_{j=1}^{N^{(i)}} X_j^{(i)},\ i = 0, 1.$$

From the lecture notes we know that

$$\mathbb{E}[S_N^{(i)}] = \mathbb{E}[N^{(i)}]\mathbb{E}[X_1^{(i)}] = \mathbb{E}[N^{(i)}]\mu^{(i)},$$

where we have used that $\mathbb{E}[X_j^{(i)}] = \mu^{(i)}$ for all j, and

$$\begin{aligned}\mathrm{Var}(S_N^{(i)}) &= \mathbb{E}[N^{(i)}]\,\mathrm{Var}(X_1^{(i)}) + \mathrm{Var}(N^{(i)})\mathbb{E}[X_1^{(i)}]^2 \\ &= \mathbb{E}[N^{(i)}](\sigma^{(i)})^2 + \mathrm{Var}(N^{(i)})(\mu^{(i)})^2\end{aligned}$$

where we have used that $\mathrm{Var}(X_j^{(i)}) = (\sigma^{(i)})^2$ for all j. We have now essentially answered part **b)**, which we do by simplifying the above giving us the following expressions for large claims

$$\mathbb{E}[S_N^{(1)}] = (1-p)\lambda\mu^{(1)}, \text{and } \mathrm{Var}(S_N^{(1)}) = (1-p)\lambda((\sigma^{(1)})^2 + (\mu^{(1)})^2).$$

Analogously, for small claims, we arrive at

$$\mathbb{E}[S_N^{(0)}] = p\lambda\mu^{(0)}, \text{and } \mathrm{Var}(S_N^{(0)}) = p\lambda((\sigma^{(0)})^2 + (\mu^{(0)})^2).$$

Thus, if we introduce the total claim cost for next year

$$S_N := S_N^{(0)} + S_N^{(1)},$$

we can answer part **a)**, since

$$\mathbb{E}[S_N] = \mathbb{E}[S_N^{(0)}] + \mathbb{E}[S_N^{(1)}] = \lambda(p\mu^{(0)} + (1-p)\mu^{(1)}),$$

and by using that $S_N^{(0)}$ and $S_N^{(1)}$ are independent by construction, it follows that

$$\begin{aligned}\mathrm{Var}(S_N) &= \mathrm{Var}(S_N^{(0)}) + \mathrm{Var}(S_N^{(1)}) \\ &= \lambda(p((\sigma^{(0)})^2 + (\mu^{(0)})^2) + (1-p)((\sigma^{(1)})^2 + (\mu^{(1)})^2)).\end{aligned}$$

Alternative solution. For part **a)**, introduce $Z_j = \delta_j X_j^{(0)} + (1-\delta_j)X_j^{(1)}$, where $\delta_j \sim \mathrm{Be}(p)$, independent of all $X_j^{(i)}$s and N, and it holds that

$$S_N := \sum_{j=1}^{N} Z_j.$$

Further, note that

$$\begin{aligned}\mathrm{Var}(S_N) &= \mathbb{E}[N]\,\mathrm{Var}(Z_1) + \mathrm{Var}(N)\mathbb{E}[Z_1]^2 \\ &= \lambda(\mathrm{Var}(Z_1) + \mathbb{E}[Z_1]^2) \\ &= \lambda\mathbb{E}[Z_1^2] \\ &= \lambda\mathbb{E}[(\delta_j X_j^{(0)} + (1-\delta_j)X_j^{(1)})^2],\end{aligned}$$

but due to independence between the $X_j^{(i)}$s and the δ_js it follows that

$$\mathbb{E}[(\delta_j X_j^{(0)} + (1-\delta_j)X_j^{(1)})^2] = p\mathbb{E}[(X_j^{(0)})^2] + (1-p)\mathbb{E}[(X_j^{(1)})^2]$$

where we have used that $\mathbb{E}[\delta_j^2] = p$ and that $\mathbb{E}[\delta_j(1-\delta_j)] = 0$. Thus, it follows that

$$\mathrm{Var}(S_N) = \lambda(p((\sigma^{(0)})^2 + (\mu^{(0)})^2) + (1-p)((\sigma^{(1)})^2 + (\mu^{(1)})^2)),$$

as above. To answer part **b)**, repeat the last arguments using $Z_j^{(1)} = (1-\delta_j)X_j^{(i)}$ and

$$S_{N^{(1)}} := \sum_{j=1}^{N} Z_j^{(1)}.$$

Problem 4

In the problem formulation we are asked to calculate **a)** the density function and in **b)** the survival function based on $A(x)$ from above. The perhaps easiest way is to start with part **b)**, since

$$S(x) = \exp\{-A(x)\} = \exp\{-\frac{a}{b}(\mathrm{e}^{bx} - 1)\},\ x > 0,$$

and **a)** follows from **b)** using the relation that

$$\alpha(x) := \frac{\mathrm{d}}{\mathrm{d}\,x}A(x) = a\,\mathrm{e}^{bx} \geq 0.$$

together with that

$$f(x) = \alpha(x)S(x) = a\exp\{bx - \frac{a}{b}(\mathrm{e}^{bx} - 1)\}, \quad x > 0.$$

To see which values of a and b that are permissible, we know that $\alpha(x) \geq 0$, which implies that $A(x)$ must be non-decreasing and positive, and in order for this to induce a proper distribution, we need that $S(x) \to 0$ (or $F(x) = 1 - S(x) \to 1$) as $x \to \infty$. Combining these observations it follows that $a \geq 0$ and $b > 0$, where one can note that $b \to 0$ gives us the Exponential distribution as a boundary case (e.g. $A(x) \to ax$ as $b \to 0$).

Problem 5

The basic question concerns how the number of insurance contracts will affect your portfolio risk. This is hard unless you only consider the liability side, which is the situation that we will consider. If you consider the asset side as well, you need to clearly explain the assumptions that you make. We will not go into these details here.

Let $Y^{(i)}$ denote the total one year *insurance loss* for line of business $i = 1, 2$, where we assume that

$$Y^{(i)} = -\sum_{j=1} X_j^{(i)} = \sum_{j=1} L_j^{(i)},$$

that is $Y^{(i)} > 0$ corresponds to a loss and $X_j^{(i)} = A_j^{(i)} - L_j^{(i)} = -L_j^{(i)}$, where $\mathbb{E}[X_1^{(i)}] = \mu^{(i)}$, which gives us that $\mathbb{E}[Y^{(i)}] = -n_i\mu^{(i)}$, and $\mathrm{Var}(X_1^{(i)}) = (\sigma^{(i)})^2$, which together with that all contracts are independent gives us that $\mathrm{Var}(Y^{(i)}) = n_i(\sigma^{(i)})^2$. To conclude, it follows that

$$Y^{(i)} \sim \mathrm{N}(-n_i\mu^{(i)}, n_i(\sigma^{(i)})^2), \ i = 1, 2.$$

In part **a)** we are supposed to calculate the Value-at-Risk for each line of business separately, but we can treat them identically: From the definition of VaR at level p is given by

$$\mathrm{VaR}_p(-Y^{(i)}) = F^{-1}_{-d(1)(-Y^{(i)})}(1 - p),$$

and by using that $Y^{(i)}$ follows a normal distribution, together with no discounting ($d(1) = 1$), following the steps of the lecture notes, we arrive at

$$\mathrm{VaR}_p(-Y^{(i)}) = -n_i\mu^{(i)} + \sqrt{n_i}\sigma^{(i)}\Phi^{-1}(1-p),\ i = 1, 2,$$

where $\phi^{-1}(1-p)$ denotes the $1-p$ percentile of a standard $\mathrm{N}(0, 1)$-distribution, which answers part **a)**.

N.B. All of these minus signs are a bit confusing, and it is OK to argue directly for the correct sign using that VaR is a percentile, and that you calculate this percentile based on the correct tail of the distribution.

In part **b)**, from the lecture notes we know that

$$\mathrm{SCR}^{(i)} := \mathrm{VaR}_p(-(Y^{(i)} - \mathbb{E}[Y^{(i)}]))$$

and that

$$\mathrm{SCR} := \sqrt{(\mathrm{SCR}^{(1)})^2 + (\mathrm{SCR}^{(2)})^2 + 2\rho\,\mathrm{SCR}^{(1)}\,\mathrm{SCR}^{(2)}}.$$

In order to ease notation, let $z_{1-p} := \Phi^{-1}(1-p)$, which gives us that

$$\mathrm{SCR}^{(i)} = \sqrt{n_i}\sigma^{(i)}z_{1-p}$$

and

$$\mathrm{SCR} = z_{1-p}\sqrt{n_1(\sigma^{(1)})^2 + n_2(\sigma^{(2)})^2 + 2\rho\sqrt{n_1 n_2}\sigma^{(1)}\sigma^{(2)}}.$$

Further, by repeating the above calculations when doubling the number of contracts gives us that

$$\mathrm{SCR}_{2\times\#\text{ of contracts}} = \sqrt{2}\,\mathrm{SCR},$$

whereas if you double the insured amount (by repeating the above or using positive homogeneity of VaR) we get

$$\mathrm{SCR}_{2\times\text{insured amount}} = 2\,\mathrm{SCR},$$

where SCR corresponds to the original SCR-value.