STOCKHOLMS UNIVERSITET
MATEMATISKA INSTITUTIONEN
Avd. Matematisk statistik

MT5011 – Part TEOR
EXAM
August 21, 2020

Suggested solutions

Exam in Basic Insurance Mathematics, 7.5 credits

August 21, 2020 – time: 9–17

Problem 1

For part **a)** and **b)** we will make use of that each individual has a Be(p)-distributed number of claims, meaning that the total number of claims, N, is Bin(n, p), since we have n i.i.d. contracts. If we let X_i denote the ith claim cost, we know that all of these are i.i.d. as well. This results in that **a)** is given by

$$\mathbb{E}[S_N] = \mathbb{E}[\sum_{i=1}^{N} X_i] = \mathbb{E}[N]\mathbb{E}[X_i] = np\mathbb{E}[X_1],$$

and **b)** is given by

$$\begin{aligned}\operatorname{Var}(S_N) &= \mathbb{E}[\operatorname{Var}(S_N \mid N)] + \operatorname{Var}(\mathbb{E}[S_N \mid N]) \\ &= \mathbb{E}[N]\operatorname{Var}(X_1) + \operatorname{Var}(N)(\mathbb{E}[X_1])^2 \\ &= np(\operatorname{Var}(X_1) + (1-p)(\mathbb{E}[X_1])^2).\end{aligned}$$

Concerning **c)**, by repeating the previous arguments we now have $N_1 \sim$ Bin(n, p_1) claims of type one, and $N_2 \sim$ Bin(n, p_2), where N_1 and N_2 are independent, and we assume that all individual claim costs of type one are i.i.d., and independent of all claim costs of type 2 which are i.i.d. That is, if we let $X_i^{(j)}$ denote the claim cost of the ith claim of type j, $j = 1, 2$, it follows that

$$\operatorname{Var}(S_{N_1} + S_{N_2}) = \operatorname{Var}(S_{N_1}) + \operatorname{Var}(S_{N_2}),$$

due to independence, where

$$S_{N_j} = \sum_{i=1}^{N_j} X_i^{(j)}, \ j = 1, 2.$$

which gives us that the standard deviation asked for in part **c)** is given by

$$\sqrt{\mathrm{Var}(S_{N_1}) + \mathrm{Var}(S_{N_2})},$$

where the individual variances are given by part **b)**.
Alternative setup. Note that you could also have used the construction

$$S_n := \sum_{i=1}^{n} X_i \delta_i,$$

where the X_is are as above, but where $\delta_i \sim \mathrm{Be}(p)$, where all X_is and δ_is are independent. In particular

$$\mathbb{E}[X_i\delta_i] = \mathbb{E}[X_i]\mathbb{E}[\delta_i], \ \text{and } \mathbb{E}[(X_i\delta_i)^2] = \mathbb{E}[X_i^2\delta_i],$$

which follows from independence and that the square of a (random) indicator is the same as the original (random) indicator.

Problem 2

From the problem description we are given the *incremental claim amounts* (left) which gives us the *cumulative claim amounts* (right):

	1	2	3
1	1430	2020	50
2	681	1005	
3	259		

	1	2	3
1	1430	3450	3500
2	681	1686	$C_{2,3}$
3	259	$C_{3,2}$	$C_{3,3}$

where we have added the future which gives us the Chain-Ladder development factors

$$\widehat{f}_1 = \frac{3450 + 1686}{1430 + 681}, \ \text{and } \widehat{f}_2 = \frac{3500}{3450},$$

and the reserves in part **a)**, corresponding to accident year 2 and 3, are given by

$$\widehat{R}_2 = c_{2,2}(\widehat{f}_2 - 1) = 24.43, \ \text{and } \widehat{R}_3 = c_{3,1}(\widehat{f}_1\widehat{f}_2 - 1) = 380.27.$$

In part **b)** we shall calculate the discounted (expected) cash flow, which consists of two future years, b_1 and b_2, where

$$\widehat{b}_1 := \mathrm{e}^{-1\cdot r_1}(\widehat{I}_{3,2} + \widehat{I}_{2,3}), \text{ and } \widehat{b}_2 := \mathrm{e}^{-2\cdot r_2}\,\widehat{I}_{3,3},$$

where $\widehat{I}_{i,j}$ corresponds to the expected incremental amount paid for accident year i during development year j, obtained using the Chain-Ladder model. By expressing the expected incremental amounts in terms of differences of accumulated amounts it follows that

$$\begin{aligned} \widehat{I}_{2,3} &= \widehat{C}_{2,3} - c_{2,2} = \widehat{R}_2 = 24.43, \\ \widehat{I}_{3,2} &= \widehat{C}_{3,2} - c_{3,1} = c_{3,1}(\widehat{f}_1 - 1) = 371.14 \\ \widehat{I}_{3,3} &= \widehat{C}_{3,3} - \widehat{C}_{3,2} = c_{3,1}\widehat{f}_1(\widehat{f}_2 - 1) = 9.13, \end{aligned}$$

which gives us the discounted discounted cashflow $(b_1, b_2) = (390.86, 8.88)$.

In **c)** we are asked to calculate $\widehat{I}_{i,j}$ for $i = 1, 2, 3$ and $j = 2, 3$, which we do analogously as in **b)** using

$$\widehat{I}_{i,2} = c_{i,1}(\widehat{f}_1 - 1), \text{ and } \widehat{I}_{i,3} = c_{i,1}\widehat{f}_1(\widehat{f}_2 - 1), \; i = 1, 2, 3,$$

resulting in the following table with incremental amounts (note first column as before)

	1	2	3
1	1430	2049.15	50.42
2	681	975.85	24.01
3	259	371.14	9.13

Problem 3

Let L denote the total cost for the described contract, which can be expressed as

$$L = 0.5 \cdot 1_{\{\text{dies before 67, given 65 today}\}} + 1 \cdot 1_{\{\text{alive at 67, given 65 today}\}},$$

which gives us that

$$\mathbb{E}[L] = 0.5 \cdot \mathbb{P}(T_0 \leq 67 \mid T_0 > 65) + 1 \cdot \mathbb{P}(T_0 > 67 \mid T_0 > 65),$$

where

$$\mathbb{P}(T_0 \leq 67 \mid T_0 > 65) = 1 - \frac{S(67)}{S(65)} = 1 - \frac{S(66)}{S(65)} \cdot \frac{S(67)}{S(66)} = 1 - (1 - q_{65})(1 - q_{66}),$$

that is, $\mathbb{P}(T_0 \leq 67 \mid T_0 > 65) = 0.013$, and consequently $\mathbb{P}(T_0 > 67 \mid T_0 > 65) = 1 - \mathbb{P}(T_0 \leq 67 \mid T_0 > 65) = 0.987$, which gives us that

$$\mathbb{E}[L] = 0.994,$$

which is equal to the single fair premium.

Problem 4

Assume that you have received 10 000 units of money, i.e. premium income, to cover all of next years costs for n i.i.d. contracts, where the mean cost per contract is 0.1 units of money, and where the standard deviation of the cost per contract is 0.03. Determine the approximate size of n so that the probability that the total claim cost for next year will not exceed your premium income is 0.05.

The idea here is to use a normal approximation: Let X_i denote the cost of the ith contract, let $\mu = \mathbb{E}[X_1] = 0.1$ and $\sigma = \sqrt{\mathrm{Var}(X_1)} = 0.03$, and let

$$S_n := \sum_{i=1}^{n} X_i.$$

We are searching for n such that

$$\mathbb{P}(S_n > 10\ 000) \approx 0.05,$$

and we will make use of that all contracts are i.i.d. which means that

$$\begin{aligned}\mathbb{P}(S_n > 10\ 000) &= \mathbb{P}\Big(\frac{S_n - \mathbb{E}[S_n]}{\sqrt{\mathrm{Var}(S_n)}} > \frac{10\ 000 - \mathbb{E}[S_n]}{\sqrt{\mathrm{Var}(S_n)}}\Big) \\ &= 1 - \mathbb{P}\Big(\frac{S_n - \mathbb{E}[S_n]}{\sqrt{\mathrm{Var}(S_n)}} \leq \frac{10\ 000 - \mathbb{E}[S_n]}{\sqrt{\mathrm{Var}(S_n)}}\Big) \\ &\approx 1 - \Phi\Big(\frac{10\ 000 - \mathbb{E}[S_n]}{\sqrt{\mathrm{Var}(S_n)}}\Big),\end{aligned}$$

where $\Phi(z) := \mathbb{P}(Z \leq z),\ Z \sim N(0,1)$. That is, we want to find n such that

$$\frac{10\ 000 - \mathbb{E}[S_n]}{\sqrt{\mathrm{Var}(S_n)}} = z_{0.95},$$

where $\Phi(z_{0.95}) = 0.95$, i.e. $z_{0.95} = 1.644$. Since, $\mathbb{E}[S_n] = n\mu$ and $\sqrt{\mathrm{Var}(S_n)} = \sqrt{n}\sigma$ (due to independence), and it follows that

$$n \approx \left(-\frac{z_{0.95}\sigma}{2\mu} + \sqrt{\frac{10\ 000}{\mu} + (\frac{z_{0.95}\sigma}{2\mu})^2}\right)^2 \approx 99\ 845.$$

Problem 5

The **a)** -part corresponds to the expected total portfolio cost, that is $\pi_{\text{tot}} = n\mu = 9\ 984.5$.
The **b)** -part corresponds to that the total premium income using the standard deviation principle should be

$$\pi_{\text{sd tot}} := \pi_{\text{tot}} + \sqrt{\text{Var}(S_n)} = 9\ 993.98,$$

but, since we know that the choice of $n = 99\ 845$ should give us that

$$n\mu + z_{0.95}\sigma\sqrt{n} \approx 10\ 000,$$

it follows that the safety loading a such that

$$n\mu + \sigma\sqrt{n} + a = 10\ 000,$$

approximately satisfies $a = (z_{0.95} - 1)\sigma\sqrt{n}$, or $a = 6.02$.

N.B. If you use $n = 5\ 000$ and premium income 500, the standard deviation principle will be greater than 500, so there is no additional safety loading.