STOCKHOLMS UNIVERSITET
MATEMATISKA INSTITUTIONEN
Avd. Matematisk statistik

MT5011 – Part TEOR
EXAM
June 1, 2021

Suggested solutions
Exam in Basic Insurance Mathematics, 7.5 credits

June 1, 2021 – time: 9–17

Problem 1

The problem formulation provides us with the survival function

$$S(t) := \exp\{-(t/\lambda)^k\},$$

where we in **a)** make use of that $\alpha(t) = f(t)/S(t)$, together with that $S(t) = 1 - F(t)$. That is, direct differentiation of $S(t)$ yields

$$f(t) = -\frac{\mathrm{d}}{\mathrm{d}t}S(t) = \frac{k}{\lambda}(\frac{t}{\lambda})^{k-1}S(t),$$

which gives us

$$\alpha(t) = \frac{k}{\lambda}(\frac{t}{\lambda})^{k-1}.$$

Alternatively, use that $\alpha(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\log S(t)$.
When it comes to **b)**, ignoring constants not depending on λ gives us that

$$\log f(t) \propto -k\log\lambda - (\frac{t}{\lambda})^k,$$

i.e., if we have $t_i, i = 1, \ldots, n$, it follows that

$$\log\prod_{i=1}^{n} f(t_i) \propto -nk\log\lambda - (\frac{1}{\lambda})^k \sum_{i=1}^{n} t_i^k,$$

and
$$\frac{\partial}{\partial \lambda}\log\prod_{i=1}^{n} f(t_i) = -\frac{nk}{\lambda} - k(\frac{1}{\lambda})^{k+1}\sum_{i=1}^{n} t_i^k,$$
where the MLE $\widehat{\lambda} > 0$ is the solution to $\frac{\partial}{\partial \lambda}\log\prod_{i=1}^{n} f(t_i) = 0$, which is given by
$$\widehat{\lambda} = (\frac{1}{n}\sum_{i=1}^{n} t_i^k)^{1/k}.$$
From the problem formulation we know that $k = 2, n = 100$ and we are given $\sum_{i=1}^{n} t_i^k$, which combined results in the estimate
$$\widehat{\lambda} = 4.57.$$

Concerning **c)**, the idea is that you ignore right-censoring and use the estimator of λ from **b)**. This means that some of the observed time points corresponds to censoring times t_i^* that are smaller than the true event time t_i. That is, if we let $t_i^* = t_i$ if time i is an event time, and $t_i^* < t_i$ if observation i is censored, it follows that
$$\widehat{\lambda}^* = (\frac{1}{n}\sum_{i=1}^{n}(t_i^*)^k)^{1/k} \leq (\frac{1}{n}\sum_{i=1}^{n} t_i^k)^{1/k} = \widehat{\lambda},$$
which means that not taking censoring into account will lead to a smaller estimate of λ, which together with **a)** implies that you will over-estimate the hazard rate.

Problem 2

In the problem formulation you are given incremental claim amounts, which corresponds to the following cumulative claims trianlge

	1	2	3
1	100	127	132
2	173	224	
3	206		

and by using Mack's chain-ladder model you get the following development factor estimates
$$\widehat{f}_1 = \frac{c_{12} + c_{22}}{c_{11} + c_{21}} = 1.29 \quad \text{and} \quad \widehat{f}_2 = \frac{c_{13}}{c_{12}} = 1.04,$$

and the future incremental payments asked for in **a)** are given by

$$\begin{aligned}
\widehat{I}_{23} &= \widehat{C}_{23} - c_{22} = c_{22}(\widehat{f}_2 - 1) = 8.82,\\
\widehat{I}_{32} &= \widehat{C}_{32} - c_{31} = c_{31}(\widehat{f}_1 - 1) = 58.86,\\
\widehat{I}_{33} &= \widehat{C}_{33} - \widehat{C}_{32} = c_{31}\widehat{f}_1(\widehat{f}_2 - 1) = 10.43.
\end{aligned}$$

Based on the future incremental payments, the future cash-flow is

$$\widehat{\mathbf{c}}' = (\widehat{I}_{23} + \widehat{I}_{32}, \widehat{I}_{33}) = (67.68, 10.43),$$

which is the answer in **b)**.
Regarding **c)**, we know that $I_{32} = C_{32} - C_{31}$, and, hence,

$$\mathrm{Var}(I_{32} \mid C_{31}) = \mathrm{Var}(C_{32} - C_{31} \mid C_{31}) = \mathrm{Var}(C_{32} \mid C_{31}),$$

where the last equality follows from that C_{31} is a constant given the conditioning, which together with Mack's model assumptions gives us that

$$\mathrm{Var}(I_{32} \mid C_{31}) = \sigma_1^2 C_{31}.$$

Further, from the lecture notes we know that

$$\widehat{(\sigma_1^2)} = c_{11}(\frac{c_{12}}{c_{11}} - \widehat{f}_1)^2 + c_{21}(\frac{c_{22}}{c_{21}} - \widehat{f}_1)^2 \approx 0.039,$$

which finally gives us the (plug-in) estimate

$$\sqrt{\widehat{\mathrm{Var}}(I_{32} \mid C_{31} = c_{31})} = \sqrt{\widehat{(\sigma_1^2)}c_{31}} = 2.83.$$

Problem 3

By the construction described in the problem formulation it follows that, N_i, the number of claims of type $i = 1, 2$, is $\mathrm{Bin}(n, p_i)$-distributed. Further, let X_{ij}denote the claim cost of the jth claim of type i, which are assumed to be i.i.d. with $\mathbb{E}[X_{ij}] = \mu_i, \mathrm{Var}(X_{ij}) = \sigma_j^2$, and independent of N_1, N_2. Moreover, let

$$S = \sum_{j=1}^{N_1} X_{1j} + \sum_{k=1}^{N_2} X_{2k}.$$

It then follows that **a)**, the expected value of S is given by

$$\mathbb{E}[S] = \mathbb{E}[\sum_{j=1}^{N_1} X_{1j}] + \mathbb{E}[\sum_{k=1}^{N_2} X_{2k}] = n(p_1\mu_1 + p_2\mu_2),$$

and by independence between the N_is and all X_ijs, **b)** becomes

$$\begin{aligned}\sqrt{\mathrm{Var}(S)} &= \sqrt{\mathrm{Var}(\sum_{j=1}^{N_1} X_{1j}) + \mathrm{Var}(\sum_{k=1}^{N_2} X_{2k})} \\ &= \sqrt{n(p_1\sigma_1^2 + p_1(1-p_1)\mu_1^2 + p_2\sigma_2^2 + p_2(1-p_2)\mu_2^2)}.\end{aligned}$$

All details of the calculations can be found in the lecture notes. If you haven't noted the Binomial structure, you will arrive at the above expression by direct calculations using indicators.

Problem 4

The presented disability compensation corresponds to an annuity payment structure. Based on the problem formulation it follows that **a)**, is given by

$$\mathbb{P}(\{\text{no further payments}\}) = \mathbb{P}(T \leq 66 \mid T > 65) = q_{65},$$

b) is given by

$$\mathbb{P}(\{\text{receive maximum remaining amount}\}) = \mathbb{P}(T > 67 \mid T > 65) = p_{65}p_{66},$$

where the details are given in the lecture notes.
Concerning **c)**, start off by noting that

$$\mathbb{P}(\text{receive last payment}) = \mathbb{P}(\{\text{receive maximum remaining amount}\}),$$

together with that if we let B denote the last payment, it follows that $B \sim \mathrm{Be}(\pi)$, where $\pi = p_{65}p_{66}$, and, hence,

$$\sqrt{\mathrm{Var}(B)} = \sqrt{\pi(1-\pi)}.$$

Problem 5

The problem formulation tells us that the total premium Π, for n identical contracts, satisfies

$$\Pi = n\pi_{\text{fair}} + na,$$

where $\pi_{\text{fair}} = \mathbb{E}[L_1]$, L_1 corresponding to the cost of a single contract, and π_{fair}, hence, corresponds to the individual fair premium, and a is the individual safety loading. From the problem we know that $na = 215$, which gives us that

$$a = \frac{215}{800}, \quad \text{and} \quad \pi_{\text{fair}} = \frac{1000 - 215}{800},$$

which answers **a)**).

When it comes to **b)**, we are told that the standard premium principle is used, i.e.

$$\Pi = n\mathbb{E}[L_1] + n\frac{\mathrm{Cov}(L_1, L)}{\sqrt{\mathrm{Var}(L)}},$$

where $L = L_1 + \ldots + L_{800}$, where all L_i are identically distributed. Further, if all contracts would be independent, this would mean that

$$\mathrm{Cov}(L_1, L) = \mathrm{Var}(L_1), \quad \text{and} \quad \mathrm{Var}(L) = n\,\mathrm{Var}(L_1),$$

i.e.

$$n\frac{\mathrm{Cov}(L_1, L)}{\mathrm{Var}(L)} = \sqrt{n\,\mathrm{Var}(L1)}.$$

Moreover, from the problem formulation we know that $\mathrm{Var}(L_1) = 52$, which gives us

$$\sqrt{n\,\mathrm{Var}(L_1)} \approx 203 \neq 215,$$

and it follows that the contracts are not independent.