STOCKHOLMS UNIVERSITET
MATEMATISKA INSTITUTIONEN
Avd. Matematisk statistik

MT5011 – Part TEOR
EXAM
May 31, 2022

Suggested solutions
Exam in Basic Insurance Mathematics, 7.5 credits

May 31, 2022 – time: 14–19

Problem 1

Let c_i denote the expected payments during future year i, i = 1, 2. Based on the problem formulation it follows that

$$c_1 := (\widehat{C}_{2,3} - c_{2,3}) + (\widehat{C}_{3,2} - c_{3,1}) = 68.70, \quad \text{and} \quad c_2 := \widehat{C}_{3,3} - \widehat{C}_{3,2} = 10.63,$$

where you can check that $c_1 + c_2$ equals the total expected future payments (or reserve), which solves part **a)**.

In part **b)** we should determine the implied development factors if the predictions would have been made using Mack's C-L model. This means that

$$\begin{aligned}\widehat{C}_{3,2} &= c_{3,1}\widehat{f}_1,\\ \widehat{C}_{2,3} &= c_{2,2}\widehat{f}_2,\\ \widehat{C}_{3,3} &= \widehat{C}_{3,2}\widehat{f}_2 = c_{3,1}\widehat{f}_1\widehat{f}_2,\end{aligned}$$

which gives us that

$$\widehat{f}_1 := \frac{\widehat{C}_{3,2}}{c_{3,1}} = 1.29, \quad \text{and} \quad \widehat{f}_2 := \frac{\widehat{C}_{2,3} + \widehat{C}_{3,3}}{c_{2,2} + \widehat{C}_{3,2}} = 1.04.$$

One can also check that these development factors actually are the ones that have been used to generate the data, since $\widehat{C}_{2,3} = c_{2,2}\widehat{f}_2$ and $\widehat{C}_{3,3} = \widehat{C}_{3,2}\widehat{f}_2$ hold separately.

Given the results from above, part **c)** follows from that

$$\begin{cases} \widehat{C}_{1,3} &= \widehat{C}_{1,2}\widehat{f}_2 = \widehat{C}_{1,1}\widehat{f}_1\widehat{f}_2 \\ \widehat{C}_{1,2} &= \widehat{C}_{1,1}\widehat{f}_1 \\ \widehat{C}_{2,2} &= \widehat{C}_{2,2}\widehat{f}_1 \end{cases} \implies \begin{cases} \widehat{C}_{1,2} &= 126.92 \\ \widehat{C}_{1,1} &= 98.39 \\ \widehat{C}_{2,1} &= 173.64 \end{cases},$$

where $\widehat{I}_{1,1} = \widehat{C}_{1,1}$, $\widehat{I}_{2,1} = \widehat{C}_{2,1}$, and $\widehat{I}_{1,2} = \widehat{C}_{1,2} - \widehat{C}_{1,1} = 28.53$.

Problem 2

From the problem formulation we know that

$$\mathbb{E}[N_i \mid Z_i] = Z_i = \mathrm{Var}(N_i \mid Z_i),$$

which gives us that

$$\mathbb{E}[N] = \mathbb{E}[N_0] + \mathbb{E}[N_1] = \mathbb{E}[\mathbb{E}[N_0 \mid Z_0]] + \mathbb{E}[\mathbb{E}[N_1 \mid Z_1]] = \mathbb{E}[Z_0] + \mathbb{E}[Z_1],$$

where the second equality uses the tower property, hence,

$$\mathbb{E}[N] = \lambda_0 + \lambda_1.$$

The variance of N can be handled similarly using the “hint”, which states the general variance decomposition formula, together with that

$$\mathrm{Var}(N) = \mathrm{Var}(N_0) + \mathrm{Var}(N_1),$$

due to independence, and where

$$\mathrm{Var}(N_i) = \mathbb{E}[\mathrm{Var}(N_i \mid Z_i)] + \mathrm{Var}(\mathbb{E}[N_i \mid Z_i]) = \mathbb{E}[Z_i] + \mathrm{Var}(Z_i) = \lambda_i + \sigma_i^2,$$

which gives us that $\mathrm{Var}(N) = \lambda_0 + \lambda_1 + \sigma_0^2 + \sigma_1^2$. This ends part **a)**.
Note that the expectation in part **b)** again can be solved using the tower property, since

$$\mathbb{E}[N] = \mathbb{E}[WN_0] + \mathbb{E}[(1-W)N_1] = \mathbb{E}[\mathbb{E}[WN_0 \mid W]] + \mathbb{E}[\mathbb{E}[(1-W)N_1 \mid W]],$$

where it holds that $\mathbb{E}[WN_0 \mid W] = W\mathbb{E}[N_0]$ and $\mathbb{E}[(1-W)N_1 \mid W] = (1-W)\mathbb{E}[N_1]$, which results in that

$$\mathbb{E}[N] = \mathbb{E}[W]\mathbb{E}[N_0] + \mathbb{E}[1-W]\mathbb{E}[N_1] = p\mathbb{E}[N_0] + (1-p)\mathbb{E}[N_1],$$

with $\mathbb{E}[N_i]$ from part **a)**. This is one way of doing it, but you could just as well have conditioned on the N_is instead of W, or just directly used that W is independent of everything:

$$\mathbb{E}[N] = \mathbb{E}[WN_0] + \mathbb{E}[(1-W)N_1] = \mathbb{E}[W]\mathbb{E}[N_0] + \mathbb{E}[1-W]\mathbb{E}[N_1].$$

When it comes to the variance, note that $\mathbb{E}[W^2] = \mathbb{E}[W]$ and $\mathbb{E}[(1-W)^2] = \mathbb{E}[1-W]$, since $W \in \{0,1\}$, and that $\mathbb{E}[W(1-W)] = 0$. If we use variance decomposition it is convenient to condition on W, since conditional on W, then WN_0 and $(1-W)N_1$ will be independent:

$$\begin{aligned}
\operatorname{Var}(N) &= \mathbb{E}[\operatorname{Var}(WN_0 + (1-W)N_1 \mid W)] + \operatorname{Var}(\mathbb{E}[WN_0 + (1-W)N_1 \mid])\\
&= \mathbb{E}[W^2 \operatorname{Var}(N_0) + (1-W)^2 \operatorname{Var}(N_1)] + \operatorname{Var}(W\mathbb{E}[N_0] + (1-W)\mathbb{E}[N_1])\\
&= \mathbb{E}[W^2] \operatorname{Var}(N_0) + \mathbb{E}[(1-W)^2] \operatorname{Var}(N_1)] + (\mathbb{E}[N_0] - \mathbb{E}[N_1])^2 \operatorname{Var}(W)\\
&= p \operatorname{Var}(N_0) + (1-p) \operatorname{Var}(N_1)] + (\mathbb{E}[N_0] - \mathbb{E}[N_1])^2 p(1-p).
\end{aligned}$$

Alternatively, you could directly calculate the variance of a sum of *dependent* random variables, since unless conditioning on W, then WN_0 and $(1-W)N_1$ will be dependent:

$$\operatorname{Var}(N) = \operatorname{Var}(WN_0) + \operatorname{Var}((1-W)N_1) + 2\operatorname{Cov}(WN_0, (1-W)N_1),$$

where

$$\begin{aligned}
\operatorname{Var}(WN_0) &= \mathbb{E}[(WN_0)^2] - (\mathbb{E}[W]\mathbb{E}[N_0])^2\\
&= \mathbb{E}[W^2]\mathbb{E}[N_0^2] - (\mathbb{E}[W]\mathbb{E}[N_0])^2\\
&= p(\operatorname{Var}(N_0) + \mathbb{E}[N_0]^2) - p^2\mathbb{E}[N_0]^2,
\end{aligned}$$

where the second equality is due to independence, and analogously it holds that

$$\operatorname{Var}((1-W)N_1) = (1-p)(\operatorname{Var}(N_1) + \mathbb{E}[N_1]^2) - (1-p)^2\mathbb{E}[N_1]^2,$$

and

$$\begin{aligned}
\operatorname{Cov}(WN_0, (1-W)N_1) &= \mathbb{E}[W(1-W)N_0N_1] - \mathbb{E}[WN_0]\mathbb{E}[(1-W)N_1]\\
&= (\mathbb{E}[W(1-W)] - \mathbb{E}[W]\mathbb{E}[1-W])\mathbb{E}[N_0]\mathbb{E}[N_1]\\
&= -\mathbb{E}[W]\mathbb{E}[1-W])\mathbb{E}[N_0]\mathbb{E}[N_1]\\
&= -p(1-p)\mathbb{E}[N_0]\mathbb{E}[N_1].
\end{aligned}$$

By combining the variance and covariance terms you arrive at the same expressions as obtained above using variance decomposition.

Problem 3

Since we are given $S(t)$ and know that $S(t) = 1 - F(t)$, it follows that the density is given by

$$f(t) = -\frac{\mathrm{d}}{\mathrm{d}t} S(t) = \beta \exp\{-\beta t\},$$

i.e. $T \sim \mathrm{Exp}(\beta)$ s.t. $\mathbb{E}[T] = 1/\beta$, which finishes part **a)**.
Part **b)** follows from

$$\begin{aligned}
\mathbb{P}(T \leq t \mid T > s) &= \frac{\mathbb{P}(s < T \leq t)}{\mathbb{P}(T > s)} \\
&= \frac{F(t) - F(s)}{S(s)} \\
&= \frac{S(s) - S(t)}{S(s)} = 1 - \exp\{-\beta(t-s)\} = \mathbb{P}(T \leq t - s),
\end{aligned}$$

as expected, given part **a)**.
Part **c)** follows from a combination of **a)** and the problem formulation, since

$$-\frac{\mathrm{d}}{\mathrm{d}t} S(t) = f(t) = \alpha(t) S(t) \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} \log(S(t)) = -\alpha(t),$$

which together with that $S(0) = 1$ yields the desired result.

Problem 4

From the problem formulation we know that

$$L = \mathbb{1}_{T_{64} > 3},$$

which means that

$$\mathbb{E}[L] = \mathbb{P}(T_{64} > 3) = \mathbb{P}(T > 67 \mid T > 64) = \frac{\mathbb{P}(T > 65)}{\mathbb{P}(T > 64)} \cdot \frac{\mathbb{P}(T > 66)}{\mathbb{P}(T > 65)} \cdot \frac{\mathbb{P}(T > 67)}{\mathbb{P}(T > 66)},$$

where

$$q_x := \mathbb{P}(T_x \leq 1) = 1 - \frac{\mathbb{P}(T_0 > x + 1)}{\mathbb{P}(T_0 > x)} = 1 - p_x.$$

That is,

$$\mathbb{E}[L] = p_{64} p_{65} p_{66} = 0.9832,$$

and the fair premium is given by $\pi := \mathbb{E}[L]$.

Concerning **b)**, given that it is reasonable that you should make a payment when entering the contract and the last payment if you survive your 66th birthday, the *random* total premium payment becomes

$$\Pi := b(\mathbb{1}_{T_{64}>0} + \mathbb{1}_{T_{64}>1} + \mathbb{1}_{T_{64}>2}),$$

where b is the fixed amount to be determined. This premium payment process is fair *when the contract is written* if

$$\mathbb{E}[\Pi] = b(1 + p_{64} + p_{64}p_{65}) = \mathbb{E}[L] \quad \Rightarrow \quad b = \frac{p_{64}p_{65}p_{66}}{1 + p_{64} + p_{64}p_{65}} = 0.3293,$$

which answers part **b)**.

Problem 5

See Example 5.1 in the lecture notes, use translation invariance, positive homogeneity and that if $Z \sim \mathrm{N}(0,1)$, then Z and $-Z$ are equal in distribution.