Suggested solutions
Exam in Basic Insurance Mathematics, 7.5 credits

May 24, 2023 – time: 14–19

Problem 1

First note that the provided claims triangle is <u>incremental</u> and should be cumulatively summed row wise. In order to calculate the reserve we need to calculate the development factors:

$$\widehat{f}_1 := \frac{49+53}{37+48} = 1.200 \text{ and } \widehat{f}_2 := \frac{51}{49} = 1.041,$$

which gives us that the expected payments i years into the future, c_i, $i = 1, 2$, is given by

$$c_1 := (\widehat{C}_{2,3} - c_{2,2}) + (\widehat{C}_{3,2} - c_{3,1}) = 2.163 + 14.400 = 16.563,$$

and

$$c_2 := (\widehat{C}_{3,3} - \widehat{C}_{3,2}) = 3.527,$$

where $\widehat{C}_{i,j}$ corresponds to Mack's Chain-Ladder predictions. Thus, the undiscounted reserve is given by

$$\widehat{R} := c_1 + c_2 = 20.090.$$

This answers **a)**. Given the problem formulation it is OK to answer discounted or undiscounted.
If we let $d(i) = \exp\{-ir_i\}$, the expected discounted cash flow is given by

$$(c_1 d(1), c_2 d(2)) = (15.914, 3.223),$$

which is the answer to **b)**.

Problem 2

We will assume that time is continuous. Following the lecture notes, the answer to **a)** results in

$$\mathbb{P}(T_{48} \in [2,3] \mid T_{48} > 1) = \mathbb{P}(T_0 \in [50,51] \mid T_0 > 49) = \frac{S(50) - S(51)}{S(49)},$$

where $S(x) := \mathbb{P}(T_0 > x)$. This corresponds to the probability of a today 48 year old dying in the age interval $[50, 51]$, given that the individual reaches age at least 49.

Part **b)** follows analogously and results in

$$\mathbb{P}(T_{48} > 5 \mid T_{48} > 2) = \frac{S(53)}{S(50)},$$

which corresponds to that a today 48 year old reaches at least age 53, given that the individual at least reaches age 50.

Part **c)** is already given above.

Problem 3

This question can be interpreted in different ways. Let $b_{\text{dead}} := 100\ 000$, $b_{\text{alive}} := 40\ 000$ and let $d(i) := \exp\{ir_i\}$. The following two are regarded as reasonable alternatives: (i) the payment to family is made at the end of the year the individual dies no later than on its 67th birthday; (ii) if the individual dies no later than on its 67th birthday, the payment is made at the time point when the individual would have turned 67. We will focus on alternative (i), since this alternative is the most realistic, and corresponds to a standard term insurance. The expected discounted payment for death benefits i years into the future, $c_i^{\text{dead}}, i = 1, 2, 3$, is, hence, given by

$$c_i^{\text{dead}} := d(i) b_{\text{dead}} \mathbb{P}(T_x \in (i-1, i]) = d(i) b_{\text{dead}} \left(\prod_{j=0}^{i-2} p_{x+j} \right) q_{x+i-1},$$

see the lecture notes, and for the corresponding survivor benefit we have

$$c_3^{\text{alive}} := d(i) b_{\text{alive}} \mathbb{P}(T_x > 3) = d(i) b_{\text{alive}} p_{x+2}.$$

By using the supplied life table with $x = 64$, this gives us the following expected discounted cash flow

$$\boldsymbol{c} := (c_1^{\text{dead}}, c_2^{\text{dead}}, c_3^{\text{dead}} + c_3^{\text{alive}}) = (406.894, 559.232, 552.303 + 33647.110)$$

which completes **a)**.

Continuing, the actuarially fair single premium paid at the 64th birthday, π, is given by

$$\pi := \mathbb{E}[L] = c_3^{\text{alive}} + \sum_{i=1}^{3} c_i^{\text{dead}} = 35165.540.$$

This answers part **b)**.

Problem 4

In part **a)** we are asked to consider the coefficient of variation (CV) for the two LoBs separately. By analysing these ratios, either w.r.t. the claim sizes (when we get 0.1 for LoB 1 and 0.3 for LoB 2) or the full claim costs (see below), the CV is lower for LoB 1 than for LoB 2. This means that we expect greater dispersion in LoB 2 relative to its expected value, compared with LoB 1. This implies a higher relative risk for LoB 2.

Part **b)** we have that all claims within a LoB are i.i.d. and that the two LoBs are independent. Let $L^{(i)}$ denote the total cost for LoB, $N^{(i)}$ denote the corresponding claim counts, and $X_j^{(i)}$ denote the corresponding claim size for claim $j = 1, \ldots, N^{(i)}$, $i, i = 1, 2$. By using the hint, this gives us that

$$\mathbb{E}[L^{(i)}] = \mathbb{E}[N^{(i)}]\mathbb{E}[X_1^{(i)}] = w^{(i)}\lambda^{(i)}\mu^{(i)},$$

where we have used the independence between $N^{(i)}$ and the $X_j^{(i)}$s, and together with

$$\operatorname{Var}(L^{(i)}) = \mathbb{E}[N^{(i)}]\operatorname{Var}(X_1^{(i)}) + \operatorname{Var}(N^{(i)})(\mathbb{E}[X_1^{(i)}]^2) = w^{(i)}\lambda^{(i)}(\theta^{(i)} + (\mu^{(i)})^2)$$

again making use of the independence between $N^{(i)}$ and the $X_j^{(i)}$s, see the lecture notes for details.

Next, if we let $L := L^{(1)} + L^{(2)}$ we directly get that

$$\mathbb{E}[L] = \sum_{i=1}^{2} \mathbb{E}[L^{(i)}] = 1\ 000\ 000 + 500\ 000 = 1\ 500\ 000,$$

and due to independence between the LoBs it holds that

$$\operatorname{Var}(L) = \sum_{i=1}^{2} \operatorname{Var}(L^{(i)}) = 2.020 \cdot 10^9 + 5.450 \cdot 10^8 = 2.565 \cdot 10^9,$$

which answers part **b)**.

When it comes to part **c)**, it is not possible to model the total loss as a compound Poisson distribution as in the sense of the lecture notes, since the claim size distributions differ between the LoBs. It is, however, possible to describe the total loss distribution as a compound Poisson distribution, but w.r.t. to a mixture claim size distribution, see e.g. Proposition 3.3.4 in [1].

Problem 5

In the problem formulation it is not clearly stated whether the A and L are independent or not. If we assume that these are dependent it is sufficient to introduce an additional correlation parameter θ, say. It is also OK if you clearly state that you assume that A and L are independent (i.e. you assume that $\theta = 0$). Further, since the sum of two normally distributed random variables is again normally distributed with the expected value given by the expected value of the sum of the individual random variables, and a variance given by the variance of the sum of the random variables, we get that $X \sim \mathrm{N}(\mu_X, \sigma_X^2)$, where

$$\mu_X := \mu_A - \mu_L, \text{ and } \sigma_X^2 := \sigma_A^2 + \sigma_L^2 + 2\theta\sigma_A\sigma_L.$$

Thus, we can represent

$$X \overset{d}{=} \mu_X + \sigma_X Z,$$

where $Z \sim \mathrm{N}(0,1)$. Once this has been established, part **a)** follows by repeating the steps from Example 5.1 in the lecture notes; use translation invariance, positive homogeneity and that if $Z \sim \mathrm{N}(0,1)$, then Z and $-Z$ are equal in distribution. This results in that

$$\mathrm{VaR}_p(X) = d(0,1)(-\mu_X + \sigma_X F_Z^{-1}(1-p)).$$

When it comes to part **b)**, recall translation invariance, i.e.

$$\begin{aligned} \mathrm{VaR}_p(X) &= \mathrm{VaR}_p(X - (a_0 - l_0)/d(0,1) + (a_0 - l_0)/d(0,1)) \\ &= \mathrm{VaR}_p(X - (a_0 - l_0)/d(0,1)) - (a_0 - l_0) \\ &= 100 - 500 = -400, \end{aligned}$$

and the statement from the problem formulation is true.

Finally, in part **c)**, by using part **b)** we know that

$$\mathrm{VaR}_p(X - (a_0 - l_0)/d(0,1)) - (a_0 - l_0) < 0,$$

which can be rewritten according to

$$a_0 > l_0 + \mathrm{VaR}_p(X - (a_0 - l_0)/d(0,1)).$$

Thus, the initial capital is sufficient in order to cover the one-year risk in assets net of liabilities measured in this particular sense.

References

[1] Mikosch T. (2009) *Non-Life Insurance Mathematics*, Springer.