Department of Mathematics, Stockholm University

Exam in MT7027, Risk models and reserving in non-life insurance, March 18, 2025, 14:00–19:00.

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*Allowed aids*: None.

*Return*: Communicated via course forum.

Arguments and computations should be clear and easy to follow.

Possibly the following approximations can be useful $1 - \Phi(1) \approx 0.159$, $1 - \Phi(2) \approx 0.023$, $1 - \Phi(3) \approx 0.001$, $\Phi^{-1}(0.95) \approx 1.64$, $\Phi^{-1}(0.975) \approx 1.96$, $\Phi^{-1}(0.995) \approx 2.58$, $\Phi$ denotes the standard normal distribution function.

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## Problem 1

For a certain insurance product the number of claims during one year is modeled as a Poisson(1000)-distributed random variable, independent of the independent and identically distributed claim sizes. The common claim size distribution is modeled as a mixture distribution. With probability 0.9 a claim is a standard claim with claim size following a normal distribution with mean 100 and variance $40^2$. With probability 0.1 a claim is a more serious claim with claim size following a normal distribution with mean 600 and variance $30^2$. A reinsurance company offers to sell a one-year XL reinsurance with a lower cutoff point 500 and no upper limit. Let $R$ denote the claims cost for the reinsurance company in case the insurance company buys the XL reinsurance contract. Determine, as accurately as possible, $\mathrm{E}[R]$ and $\mathrm{Var}(R)$. (10 p)

## Problem 2

Consider an insurance company with a set of insurance contracts for a certain product for which the number of claims during a year depends on weather conditions. Suppose that the weather conditions during a year can be classified into either normal or severe, and that the weather conditions for the next year may be any of the two, with a probability of 1/2 for each. Suppose that, given normal weather conditions, the number of claims is $\mathrm{Pois}(\lambda_n)$ distributed. Suppose that, given severe weather conditions, the number of claims is $\mathrm{Pois}(\lambda_s)$ distributed, $\lambda_s > \lambda_n$. Suppose that, seen over many historical years, the mean and standard deviation of the time series of number of claims per year has been estimated to $\mu$ and $\sigma$. Determine an equation system that can be used to determine $\lambda_n$ and $\lambda_s$ in terms of $\mu$ and $\sigma$. (10 p)

### Problem 3

For $i \in \{0, \dots, 4\}$ and $j \in \{1, \dots, 4\}$, let $C_{i,j}$ be the cumulative claim amount for the first $j$ development years due to accidents during accident year $i$. Assume that $\mathcal{D} = \{C_{i,j} : i + j \leq 5\}$ are known whereas $\{C_{i,j} : i + j > 5\}$ are yet unknown. Assume Mack's distribution-free chain ladder.

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 0 | $C_{0,1}$ | $C_{0,2}$ | $C_{0,3}$ | $C_{0,4}$ |
| 1 | $C_{1,1}$ | $C_{1,2}$ | $C_{1,3}$ | $C_{1,4}$ |
| 2 | $C_{2,1}$ | $C_{2,2}$ | $C_{2,3}$ | $\mathbf{C_{2,4}}$ |
| 3 | $C_{3,1}$ | $C_{3,2}$ | $\mathbf{C_{3,3}}$ | $\mathbf{C_{3,4}}$ |
| 4 | $C_{4,1}$ | $\mathbf{C_{4,2}}$ | $\mathbf{C_{4,3}}$ | $\mathbf{C_{4,4}}$ |

Table 1: Cumulative claim amounts $C_{i,j}$.

(a) Express the predictor $\widehat{C}_{3,4}$ in terms of elements in $\mathcal{D}$. (5 p)

(b) Compute $\mathrm{Var}(C_{3,4} \mid \mathcal{D})$. (5 p)

### Problem 4

Consider the same setting as in Problem 3. Derive, as explicit as possible, an expression for the conditional mean squared error of prediction $\mathrm{E}[(C_{3,4} - \widehat{C}_{3,4})^2 \mid \mathcal{D}]$ and explain why it is difficult to estimate this quantity. (10 p)

### Problem 5

Suppose that the current aggregate liabilities for an insurance company corresponds to a simple single-year cashflow $C_1$ at the end of the year, which is assumed normally distributed with mean $\mu$ and variance $\sigma^2$. Determine the current value of the liabilities according to cost-of-capital valuation with unlimited liability for the capital provider. Solvency corresponds to $\mathrm{VaR}_{0.005}(X) \leq 0$, where $X$ denotes a generic net value one year from today. Suppose that the one-year discount factor is $d(0,1)$ and the cost-of-capital rate is $\eta$. (10 p)

## Problem 1

The probability that a severe claim takes a value below 500 is less than the probability of a "negative three standard deviation event" which is approximately 0.001. The probability that a standard claim takes a value above 500 is the probability of a "positive ten standard deviation event" which is approximately 0. Hence, the probability that any given claim size exceeds 500 is 0.1 (approximately, but with high accuracy). Hence, the number of claims causing costs for the reinsurer is Pois(100):

$$N_R = \sum_{k=1}^{N} I\{X_k > 500\} \sim \mathrm{Pois}(1000 \cdot \mathrm{P}(X_1 > 500))$$

and

$$R = \sum_{k=1}^{N} \max(X_k - 500, 0) = \sum_{k=1}^{N_R} Z_k,$$

where $Z_k \sim N(100, 30^2)$ (an accurate approximation) and the $Z_k$ are independent of $N_R$. Therefore, $\mathrm{E}[R] = \mathrm{E}[N_R]\,\mathrm{E}[Z_1] = 100^2$ and

$$\begin{aligned}\mathrm{Var}(R) &= \mathrm{E}[N_R]\,\mathrm{Var}[Z_1] + \mathrm{Var}(N_R)\,\mathrm{E}[Z_1]^2 = \mathrm{E}[N_R]\,\mathrm{E}[Z_1^2] = \mathrm{E}[N_R](\mathrm{Var}(Z_1) + \mathrm{E}[Z_1]^2)\\ &= 100(30^2 + 100^2) = 100 \cdot 10900 = 100^2 \cdot 109\end{aligned}$$

## Problem 2

Let $\Lambda$ be a 2-point random variable that takes either value $\lambda_n$ or $\lambda_s$, each with probability 1/2. Note that

$$\begin{aligned}\mathrm{E}[N] &= \mathrm{E}[\mathrm{E}[N \mid W]] = \mathrm{E}[\Lambda] = \frac{1}{2}(\lambda_n + \lambda_s),\\ \mathrm{Var}(N) &= \mathrm{E}[\mathrm{Var}(N \mid W)] + \mathrm{Var}(\mathrm{E}[N \mid W]) = \mathrm{E}[\Lambda] + \mathrm{Var}(\Lambda) = \mathrm{E}[\Lambda] + \frac{1}{2}(\lambda_n^2 + \lambda_s^2) - \mathrm{E}[\Lambda]^2\end{aligned}$$

Hence,

$$\begin{aligned}\mu &= \frac{1}{2}(\lambda_n + \lambda_s),\\ \sigma^2 &= \frac{1}{2}(\lambda_n + \lambda_s)\left(1 - \frac{1}{2}(\lambda_n + \lambda_s)\right) + \frac{1}{2}(\lambda_n^2 + \lambda_s^2)\end{aligned}$$

## Problem 3

(a) The predictor is $\widehat{C}_{3,4} = C_{3,2}\widehat{f}_2\widehat{f}_3$ with

$$\widehat{f}_2 = \frac{C_{0,3} + C_{1,3} + C_{2,3}}{C_{0,2} + C_{1,2} + C_{2,2}}, \quad \widehat{f}_3 = \frac{C_{0,4} + C_{1,4}}{C_{0,3} + C_{1,3}}$$

(b) Mack's model says that

$$\begin{aligned}\mathrm{E}[C_{i,k+1} \mid C_{i,1}, \ldots, C_{i,k}] &= f_k C_{i,k},\\ \mathrm{Var}(C_{i,k+1} \mid C_{i,1}, \ldots, C_{i,k}) &= \sigma_k^2 C_{i,k}\end{aligned}$$

and independence across accident years. Hence,

$$\begin{aligned}
\operatorname{Var}(C_{3,4} \mid \mathcal{D}) &= \operatorname{Var}(C_{3,4} \mid C_{3,1}, C_{3,2}) \\
&= \operatorname{E}[\operatorname{Var}(C_{3,4} \mid C_{3,1}, C_{3,2}, C_{3,3}) \mid C_{3,1}, C_{3,2}] \\
&\quad + \operatorname{Var}(\operatorname{E}[C_{3,4} \mid C_{3,1}, C_{3,2}, C_{3,3}] \mid C_{3,1}, C_{3,2}) \\
&= \sigma_3^2 \operatorname{E}[C_{3,3} \mid C_{3,1}, C_{3,2}] + f_3^2 \operatorname{Var}(C_{3,3} \mid C_{3,1}, C_{3,2}) \\
&= \sigma_3^2 f_2 C_{3,2} + f_3^2 \sigma_2^2 C_{3,2}
\end{aligned}$$

## Problem 4

$$\begin{aligned}
\operatorname{E}[(C_{3,4} - \widehat{C}_{3,4})^2 \mid \mathcal{D}] &= \operatorname{E}[(C_{3,4} - \operatorname{E}[C_{3,4} \mid \mathcal{D}] + \operatorname{E}[C_{3,4} \mid \mathcal{D}] - \widehat{C}_{3,4})^2 \mid \mathcal{D}] \\
&= \operatorname{Var}(C_{3,4} \mid \mathcal{D}) + (\operatorname{E}[C_{3,4} \mid \mathcal{D}] - \widehat{C}_{3,4})^2 \\
&\quad + 2 \operatorname{E}[(C_{3,4} - \operatorname{E}[C_{3,4} \mid \mathcal{D}])(\operatorname{E}[C_{3,4} \mid \mathcal{D}] - \widehat{C}_{3,4}) \mid \mathcal{D}] \\
&= (\sigma_3^2 f_2 + f_3^2 \sigma_2^2) C_{3,2} + (f_2 f_3 - \widehat{f}_2 \widehat{f}_3)^2 C_{3,2}^2 \\
&\quad + 2 \operatorname{E}[C_{3,4} - \operatorname{E}[C_{3,4} \mid \mathcal{D}] \mid \mathcal{D}](\operatorname{E}[C_{3,4} \mid \mathcal{D}] - \widehat{C}_{3,4}) \\
&= (\sigma_3^2 f_2 + f_3^2 \sigma_2^2) C_{3,2} + (f_2 f_3 - \widehat{f}_2 \widehat{f}_3)^2 C_{3,2}^2
\end{aligned}$$

where the second term cannot be estimated by simply replacing $f_2 f_3$ by $\widehat{f}_2 \widehat{f}_3$ since this would amount to estimating a positive term by zero.

## Problem 5

If $X \sim N(\lambda, \sigma^2)$, then, with $Z \sim N(0,1)$,

$$\begin{aligned}
\operatorname{VaR}_{0.005}(X) &= \operatorname{VaR}_{0.005}(\lambda + \sigma Z) = F^{-1}_{-d(0,1)(\lambda + \sigma Z)}(0.995) \\
&= d(0,1)\big( -\lambda + \sigma \Phi^{-1}(0.995) \big),
\end{aligned}$$

i.e. $\operatorname{VaR}_{0.005}(X) \leq 0$ if and only if $\lambda \geq \sigma \Phi^{-1}(0.995)$. Here, with $X = R_0/d(0,1) - C_1$ we get $\lambda = R_0/d(0,1) - \mu$, i.e. $R_0/d(0,1) - \mu \geq \sigma \Phi^{-1}(0.995)$. Therefore $R_0 = d(0,1)(\mu + \sigma \Phi^{-1}(0.995))$ is the smallest capital that ensures solvency. The capital provider's acceptability condition is

$$\frac{d(0,1) \operatorname{E}[R_0/d(0,1) - C_1]}{R_0 - L_0} = 1 + \eta$$

which gives

$$L_0 = d(0,1) \operatorname{E}[C_1] + \frac{\eta}{1+\eta}(R_0 - d(0,1) \operatorname{E}[C_1]) = d(0,1)\mu + \frac{\eta}{1+\eta} d(0,1) \sigma \Phi^{-1}(0.995)$$