Department of Mathematics, Stockholm University

Exam in MT7027, Risk models and reserving in non-life insurance, April 26, 2025, 14:00–19:00.

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Allowed aids: None.

Return: Communicated via course forum.

Arguments and computations should be clear and easy to follow.

Possibly the following approximations can be useful $1-\Phi(1) \approx 0.159$, $1-\Phi(2) \approx 0.023$, $1-\Phi(3) \approx 0.001$, $\Phi^{-1}(0.95) \approx 1.64$, $\Phi^{-1}(0.975) \approx 1.96$, $\Phi^{-1}(0.995) \approx 2.58$, Φ denotes the standard normal distribution function.

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Problem 1

For a certain insurance product there are n identical contract, each with a probability p of generating a single claim during a one-year period, and a zero probability of multiple claims. The claim size is assumed exponentially distributed with density $f(x) = \lambda e^{-\lambda x}$. Suppose that the number of claims and the claim sizes are independent. A reinsurance company offers to sell a one-year XL reinsurance with a lower cutoff point l and no upper limit. Let R denote the total claims cost for the reinsurance company in case the insurance company buys the XL reinsurance contract. Determine $\mathrm{E}[R]$. (10 p)

Problem 2

An insurance company expects a Poisson(λ)-distributed number of new customers for the next year. Each arriving customer is either a low-risk or a high-risk customer. A low-risk customer experiences a Poisson(λ_l)-distributed number of claims, a high-risk customer experiences a Poisson(λ_h)-distributed number of claims. Suppose the fraction of low-risk insurance customers in the population is p and that new customers can be seen as drawn randomly from the population, and that they experience accidents independently from each other. Determine the mean and variance for the number of claims generated by new customers next year. (10 p)

Problem 3

For $i \in \{0, \ldots, 4\}$ and $j \in \{1, \ldots, 4\}$, let $C_{i,j}$ be the cumulative claim amount for the first j development years due to accidents during accident year i. Assume that $\mathcal{D} = \{C_{i,j} : i+j \leq 5\}$ are known whereas $\{C_{i,j} : i+j > 5\}$ are yet unknown. Assume Mack's distribution-free chain ladder. For $i + j > 5$, let $\widehat{C}_{i,j}$ denote the predictor of $C_{i,j}$ based on data $\mathcal{D}$.

	1	2	3	4
0	$C_{0,1}$	$C_{0,2}$	$C_{0,3}$	$C_{0,4}$
1	$C_{1,1}$	$C_{1,2}$	$C_{1,3}$	$C_{1,4}$
2	$C_{2,1}$	$C_{2,2}$	$C_{2,3}$	$\mathbf{C_{2,4}}$
3	$C_{3,1}$	$C_{3,2}$	$\mathbf{C_{3,3}}$	$\mathbf{C_{3,4}}$
4	$C_{4,1}$	$\mathbf{C_{4,2}}$	$\mathbf{C_{4,3}}$	$\mathbf{C_{4,4}}$

Table 1: Cumulative claim amounts $C_{i,j}$.

(a) Compute $\mathrm{E}[C_{3,4} \mid \mathcal{D}]$. (5 p)

(b) Compute $\mathrm{Var}(\widehat{C}_{3,4} \mid \mathcal{D})$. (5 p)

Problem 4

Suppose that, for $i = 0, 1, \ldots,$

$$C_{i,1} = \exp(\mu_0 + \nu_0 Z_{i,1}) \text{ and } C_{i,j+1} = C_{i,j} \exp(\mu_j + \nu_j Z_{i,j+1}),\ j = 1, 2, \ldots,$$

where all $Z_{i,j}$ are independent and standard normally distributed. Mack's distribution-free chain ladder makes three assumptions. Explain, for each of these assumptions, whether the assumption holds or not. (10 p)

Problem 5

Consider an insurance company with current asset value A_0 invested in a risk-free one-year zero-coupon bond: one invested monetary unit today gives $g > 1$ units in one year. Suppose that the value in one year L_1 is the sum of claims costs during the year and the value of liabilities corresponding to claims payments beyond one year, all expressed in monetary units one year from today. Suppose that $\log(L_1)$ (the natural logarithm) is normally distributed with mean μ and variance σ^2. Determine an inequality in terms of A_0, g, μ and σ that corresponds to solvency, where solvency for a position with net worth X in one year corresponds to $\mathrm{VaR}_{0.005}(X) \leq 0$. (10 p)

Problem 1

Write $R = \sum_{k=1}^{N} \max(X_k - l, 0)$ and note that $\mathrm{E}[R] = \mathrm{E}[N]\,\mathrm{E}[\max(X - l, 0)] = np\,\mathrm{E}[\max(X - l, 0)]$.

$$\begin{aligned}\mathrm{E}[\max(X - l, 0)] &= \int_0^\infty \mathrm{P}(\max(X - l, 0) > y)dy = \int_0^\infty \mathrm{P}(X > l + y)dy \\ &= e^{-\lambda l}\int_0^\infty e^{-\lambda y}dy = e^{-\lambda l}/\lambda.\end{aligned}$$

Hence, $\mathrm{E}[R] = npe^{-\lambda l}/\lambda$.

Problem 2

Let Λ_k be a 2-point random variable that takes either value λ_l with probability p or λ_h with probability $1 - p$. Let $S = \sum_{k=1}^{N} M_k$, where N is independent of the iid pairs $(M_1, \Lambda_1), (M_2, \Lambda_2), \dots$. Hence,

$$\begin{aligned}\mathrm{E}[S] &= \mathrm{E}[\mathrm{E}[S \mid N]] = \mathrm{E}[N]\,\mathrm{E}[M], \\ \mathrm{Var}(S) &= \mathrm{E}[\mathrm{Var}(S \mid N)] + \mathrm{Var}(\mathrm{E}[S \mid N]) = \mathrm{E}[N]\,\mathrm{Var}(M) + \mathrm{Var}(N)\,\mathrm{E}[M]^2 \\ &= \mathrm{E}[N]\,\mathrm{E}[M^2],\end{aligned}$$

where the last equality follows from the Poisson property $\mathrm{E}[N] = \mathrm{Var}(N)$. Moreover,

$$\begin{aligned}\mathrm{E}[M] &= \mathrm{E}[\mathrm{E}[M \mid \Lambda]] = \mathrm{E}[\Lambda], \\ \mathrm{Var}(M) &= \mathrm{E}[\mathrm{Var}(M \mid \Lambda)] + \mathrm{Var}(\mathrm{E}[M \mid \Lambda]) = \mathrm{E}[\Lambda] + \mathrm{Var}(\Lambda).\end{aligned}$$

Hence,

$$\begin{aligned}\mathrm{E}[S] &= \lambda(p\lambda_l + (1 - p)\lambda_h), \\ \mathrm{Var}(S) &= \lambda(\mathrm{E}[\Lambda] + \mathrm{E}[\Lambda^2]) = \lambda(p(\lambda_l + \lambda_l^2) + (1 - p)(\lambda_h + \lambda_h^2))\end{aligned}$$

Problem 3

(a)

$$\begin{aligned}\mathrm{E}[C_{3,4} \mid \mathcal{D}] &= \mathrm{E}[C_{3,4} \mid C_{3,1}, C_{3,2}] = \mathrm{E}[\mathrm{E}[C_{3,4} \mid C_{3,1}, C_{3,2}, C_{3,3}] \mid C_{3,1}, C_{3,2}] \\ &= \mathrm{E}[f_3 C_{3,3} \mid C_{3,1}, C_{3,2}] = f_3 f_2 C_{3,2}\end{aligned}$$

(b) The predictor is $\widehat{C}_{3,4} = C_{3,2}\widehat{f}_2\widehat{f}_3$ which is a known number when conditioning on $\mathcal{D}$. Hence, $\mathrm{Var}(\widehat{C}_{3,4} \mid \mathcal{D}) = 0$.

Problem 4

By construction, all accident year are independent, which matches that property of Mack's CL.

$$\begin{aligned}\mathrm{E}[C_{i,j+1} \mid C_{i,1}, \dots, C_{i,j}] &= C_{i,j}\,\mathrm{E}[\exp(\mu_j + \nu_j Z_{i,j+1}) \mid C_{i,1}, \dots, C_{i,j}] \\ &= C_{i,j}\exp(\mu_j + \nu_j^2/2)\end{aligned}$$

which matches the property for the conditional expectation of Mack's CL.

$$\begin{aligned}\mathrm{Var}(C_{i,j+1} \mid C_{i,1},\dots,C_{i,j}) &= C_{i,j}^2 \,\mathrm{Var}(\exp(\mu_j + \nu_j Z_{i,j+1}) \mid C_{i,1},\dots,C_{i,j}) \\ &= C_{i,j}^2 \,\mathrm{Var}(\exp(\mu_j + \nu_j Z_{i,j+1}))\end{aligned}$$

which does not perfectly match the property for the conditional variance of Mack's CL because $C_{i,j}^2$ appears instead of $C_{i,j}$.

Problem 5

$X = A_1 - L_1 = A_0 g - \exp(\mu + \sigma Z)$ with Z standard normal.

$$\mathrm{VaR}_{0.005}(X) = g^{-1} F^{-1}_{L_1 - A_1}(0.995) = g^{-1}\Big(F^{-1}_{\exp(\mu+\sigma Z)}(0.995) - A_0 g\Big)$$

and $F^{-1}_{\exp(\mu+\sigma Z)}(0.995)$ can be found as the solution to $F_{\exp(\mu+\sigma Z)}(x) = 0.995$. Note that

$$F_{\exp(\mu+\sigma Z)}(x) = \mathrm{P}(\mu + \sigma Z \le \log(x)) = \Phi\left(\frac{\log(x) - \mu}{\sigma}\right)$$

which gives $x = \exp(\mu + \sigma\Phi^{-1}(0.995))$. Hence the inequality $\mathrm{VaR}_{0.005}(X) \le 0$ is equivalent to

$$g^{-1}\exp(\mu + \sigma\Phi^{-1}(0.995)) \le A_0$$