Department of Mathematics, Stockholm University

Exam in MT7054, Risk models and reserving in non-life insurance, March 19, 2026, 15:00–20:00.

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*Allowed aids*: None.

*Return*: Communicated via course forum.

Arguments and computations should be clear and easy to follow.

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## Problem 1

An insurance company's aggregate 1-year claims cost $S$ is given by

$$S = \begin{cases} 1 & \text{with probability } 0.80, \\ 2 & \text{with probability } 0.10, \\ 3 & \text{with probability } 0.05, \\ 4 & \text{with probability } 0.05. \end{cases}$$

Suppose a reinsurance company prices reinsurance contracts according to twice its expected claims cost. Consider a stop-loss contract with lower cutoff point $c \geq 1$ and no upper cutoff point. Compute and draw the graph of the price of this contract as a function of $c$. (10 p)

## Problem 2

For $i \in \{0, \dots, 4\}$ and $j \in \{1, \dots, 4\}$, let $C_{i,j}$ be the cumulative claim amount for the first $j$ development years due to accidents during accident year $i$. Assume that $\mathcal{D} = \{C_{i,j} : i + j \leq 5\}$ are known whereas $\{C_{i,j} : i + j > 5\}$ are yet unknown. Assume Mack's distribution-free chain ladder.

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 0 | $C_{0,1}$ | $C_{0,2}$ | $C_{0,3}$ | $C_{0,4}$ |
| 1 | $C_{1,1}$ | $C_{1,2}$ | $C_{1,3}$ | $C_{1,4}$ |
| 2 | $C_{2,1}$ | $C_{2,2}$ | $C_{2,3}$ | $\mathbf{C_{2,4}}$ |
| 3 | $C_{3,1}$ | $C_{3,2}$ | $\mathbf{C_{3,3}}$ | $\mathbf{C_{3,4}}$ |
| 4 | $C_{4,1}$ | $\mathbf{C_{4,2}}$ | $\mathbf{C_{4,3}}$ | $\mathbf{C_{4,4}}$ |

Table 1: Cumulative claim amounts $C_{i,j}$.

(a) Express the predictor $\widehat{C}_{3,4}$ in terms of elements in $\mathcal{D}$. (5 p)

(b) Compute $\operatorname{Var}(C_{3,4} \mid \mathcal{D})$. (5 p)

## Problem 3

Consider the same data as in Problem 2. Consider the modification of Mack's distribution-free chain ladder resulting from the following modification of the conditional variance property: $\operatorname{Var}(C_{i,j+1} \mid C_{i,1}, \dots, C_{i,j}) = \sigma_j^2$. Explain how this change of model assumption affects the expression for the predictor $\widehat{C}_{3,4}$. (10 p)

**Problem 4**

Consider the same data as in Problem 2. Consider the model $C_{i,j+1} = C_{i,j} \exp\big(\mu_j + \nu_j Z_{i,j+1}\big)$, where all variables $Z_{i,j}$ are independent with common probability density function

$$z \mapsto \frac{1}{\sqrt{2\pi}} \exp\left( - \frac{z^2}{2} \right).$$

The parameters of the model are estimated by maximum likelihood estimation.

(a) Compute $\mathrm{E}[C_{2,4} \mid \mathcal{D}]$. (5 p)

(b) Express the predictor $\widehat{C}_{2,4}$ in terms of elements in $\mathcal{D}$. (5 p)

**Problem 5**

Suppose that the current aggregate liabilities for an insurance company corresponds to a simple single-year cashflow $C_1$ at the end of the year. Suppose that $\mathrm{P}(C_1 = 0) = 0.95$ and $\mathrm{P}(C_1 = 1.1 \cdot 10^6) = 0.05$. Determine the current value of the liabilities according to cost-of-capital valuation with unlimited liability for the capital provider. Solvency corresponds to $\mathrm{VaR}_{0.005}(X) \leq 0$, where $X$ denotes a generic net value one year from today. Suppose that the one-year discount factor is $d(0,1) = 1$ and the cost-of-capital rate is $\eta = 0.05$. (10 p)

## Problem 1

The reinsurance premium is $p(c) = 2\,\mathrm{E}[\max(S-c,0)]$ and

$$\begin{aligned}\mathrm{E}[\max(S-c,0)] &= \mathrm{E}[(S-c)I\{S>c\}]\\ &= \mathrm{E}[SI\{S>c\}] - c\,\mathrm{E}[I\{S>c\}]\\ &= (\mathrm{E}[S \mid S>c] - c)\,\mathrm{P}(S>c).\end{aligned}$$

Note that

$$(\mathrm{E}[S \mid S>c], \mathrm{P}(S>c)) = \begin{cases} (4, 0.05) & \text{if } c \in [3,4), \\ (7/2, 0.1) & \text{if } c \in [2,3), \\ (11/4, 0.2) & \text{if } c \in [1,2). \end{cases}$$

The graph of $p(c)$ is obtained by linear interpolation between the points $(c, p(c))$:

$$(1, 7/10),\ (2, 3/10),\ (3, 1/10)\ (4, 0)$$

## Problem 2

(a) The predictor is $\widehat{C}_{3,4} = C_{3,2}\widehat{f}_2\widehat{f}_3$ with

$$\widehat{f}_2 = \frac{C_{0,3}+C_{1,3}+C_{2,3}}{C_{0,2}+C_{1,2}+C_{2,2}}, \quad \widehat{f}_3 = \frac{C_{0,4}+C_{1,4}}{C_{0,3}+C_{1,3}}$$

(b) Mack's model says that

$$\begin{aligned}&\mathrm{E}[C_{i,k+1} \mid C_{i,1},\dots,C_{i,k}] = f_k C_{i,k},\\ &\mathrm{Var}(C_{i,k+1} \mid C_{i,1},\dots,C_{i,k}) = \sigma_k^2 C_{i,k}\end{aligned}$$

and independence across accident years. Hence,

$$\begin{aligned}\mathrm{Var}(C_{3,4} \mid \mathcal{D}) &= \mathrm{Var}(C_{3,4} \mid C_{3,1}, C_{3,2})\\ &= \mathrm{E}[\mathrm{Var}(C_{3,4} \mid C_{3,1}, C_{3,2}, C_{3,3}) \mid C_{3,1}, C_{3,2}]\\ &\quad + \mathrm{Var}(\mathrm{E}[C_{3,4} \mid C_{3,1}, C_{3,2}, C_{3,3}] \mid C_{3,1}, C_{3,2})\\ &= \sigma_3^2\,\mathrm{E}[C_{3,3} \mid C_{3,1}, C_{3,2}] + f_3^2\,\mathrm{Var}(C_{3,3} \mid C_{3,1}, C_{3,2})\\ &= \sigma_3^2 f_2 C_{3,2} + f_3^2\sigma_2^2 C_{3,2}\end{aligned}$$

## Problem 3

Consider the regression equation $Y = f_2 A + \sigma_2\varepsilon$ written as

$$\begin{bmatrix} C_{0,3} \\ C_{1,3} \\ C_{1,3} \end{bmatrix} = f_2 \begin{bmatrix} C_{0,2} \\ C_{1,2} \\ C_{1,2} \end{bmatrix} + \sigma_2 \begin{bmatrix} \varepsilon_{0,3} \\ \varepsilon_{1,3} \\ \varepsilon_{1,3} \end{bmatrix}$$

Least squares estimation gives

$$\widehat{f}_2 = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}Y = \frac{C_{0,2}C_{0,3}+C_{1,2}C_{1,3}+C_{2,2}C_{2,3}}{C_{0,2}^2+C_{1,2}^2+C_{2,2}^2}$$

Similarly,

$$\widehat{f}_3 = \frac{C_{0,3}C_{0,4} + C_{1,3}C_{1,4}}{C_{0,3}^2 + C_{1,3}^2}$$

Since

$$\mathrm{E}[C_{3,4} \mid \mathcal{D}] = \mathrm{E}[\mathrm{E}[C_{3,4} \mid C_{3,2}, C_{3,3}] \mid C_{3,2}] = C_{3,2} f_2 f_3$$

we get the predictor $C_{3,2}\widehat{f}_2\widehat{f}_3$ with $\widehat{f}_2, \widehat{f}_3$ as expressed above.

**Problem 4**

$\mathrm{E}[C_{2,4} \mid \mathcal{D}] = C_{2,3}\exp(\mu_3 + \nu_3^2/2)$. For $j$ fixed, note that

$$X_1 := \log\frac{C_{0,4}}{C_{0,3}},\ X_2 := \log\frac{C_{1,4}}{C_{1,3}} \text{ are independent and } N(\mu_3, \nu_3^2).$$

The likelihood and log-likelihood are, with $n = 2$,

$$L = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\nu_3^2}} \exp\left( - \frac{(x_i - \mu_3)^2}{2\nu_3^2} \right), \quad l = -\frac{n}{2}\log(2\pi\nu_3^2) - \frac{1}{2\nu_3^2}\sum_{i=1}^{n}(x_i - \mu_3)^2$$

Solving $\partial l/\partial \mu_3 = 0$ and $\partial l/\partial \nu_3^2 = 0$ gives

$$\widehat{\mu_3} = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad \widehat{\nu_3^2} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \widehat{\mu_3})^2$$

Finally,

$$\widehat{C}_{2,4} = C_{2,3}\exp\left(\widehat{\mu_3} + \widehat{\nu_3^2}/2\right)$$

**Problem 5**

Since $0.05 > 0.005$,

$$R_0 = \mathrm{VaR}_{0.005}(-C_1) = d(0,1)F_{C_1}^{-1}(0.995) = 1.1 \cdot 10^6.$$

The capital provider's acceptability condition is

$$\frac{d(0,1)\,\mathrm{E}[R_0/d(0,1) - C_1]}{R_0 - L_0} = 1 + \eta$$

which gives, with $\mathrm{E}[C_1] = 5.5 \cdot 10^4$, $d(0,1) = 1$ and $\eta/(1+\eta) = 1/21$,

$$\begin{aligned} L_0 &= d(0,1)\,\mathrm{E}[C_1] + \frac{\eta}{1+\eta}(R_0 - d(0,1)\,\mathrm{E}[C_1]) \\ &= \frac{1}{1+\eta} d(0,1)\,\mathrm{E}[C_1] + \frac{\eta}{1+\eta} R_0 \\ &= \frac{20}{21} \cdot 5.5 \cdot 10^4 + \frac{1}{21} \cdot 1.1 \cdot 10^6 \\ &= \frac{22}{21} \cdot 10^5 \end{aligned}$$