Department of Mathematics, Stockholm University

Exam in MT7054 and MT7027, Risk models and reserving in non-life insurance, April 22, 2026, 08:00–13:00.

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Allowed aids: None.

Return: Communicated via course forum.

Arguments and computations should be clear and easy to follow.

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Problem 1

Suppose that the yearly claims cost for an insurance product is modelled as $S = \sum_{k=1}^{N} X_k$, where $N \sim \mathrm{Pois}(\lambda)$, $\lambda > 0$, is independent of the X_ks. The X_ks are independent with common distribution function $F(x) = 1-(1+x/\gamma)^{-\alpha}$, $\gamma > 0, \alpha > 1$, for $x > 0$. The insurer has bought a one-year XL reinsurance contract with retention level (cutoff point) $c > 0$. Compute the expected yearly claims cost for the reinsurer. (10 p)

Problem 2

For $i \in \{0, \dots, 4\}$ and $j \in \{1, \dots, 4\}$, let $C_{i,j}$ be the cumulative claim amount for the first j development years due to accidents during accident year i. Assume that $\mathcal{D} = \{C_{i,j} : i + j \leq 5\}$ are known whereas $\{C_{i,j} : i + j > 5\}$ are yet unknown.

	1	2	3	4
0	$C_{0,1}$	$C_{0,2}$	$C_{0,3}$	$C_{0,4}$
1	$C_{1,1}$	$C_{1,2}$	$C_{1,3}$	$C_{1,4}$
2	$C_{2,1}$	$C_{2,2}$	$C_{2,3}$	$\mathbf{C_{2,4}}$
3	$C_{3,1}$	$C_{3,2}$	$\mathbf{C_{3,3}}$	$\mathbf{C_{3,4}}$
4	$C_{4,1}$	$\mathbf{C_{4,2}}$	$\mathbf{C_{4,3}}$	$\mathbf{C_{4,4}}$

Table 1: Cumulative claim amounts $C_{i,j}$.

Assume that $(C_{k,1}, \dots, C_{k,4})$ and $(C_{l,1}, \dots, C_{l,4})$ are independent if $k \neq l$. Assume that there are constants $f_j > 0$ and $\sigma_j \geq 0$ such that

$$\begin{aligned} \mathrm{E}[C_{i,j+1} \mid C_{i,1}, \dots, C_{i,j}] &= f_j C_{i,j}, \\ \mathrm{Var}(C_{i,j+1} \mid C_{i,1}, \dots, C_{i,j}) &= \sigma_j^2 C_{i,j}^2. \end{aligned}$$

Compute $\mathrm{Var}(C_{3,4} \mid \mathcal{D})$. (10 p)

Problem 3

Consider the same model and data as in Problem 2. Derive an expression for the predictor $\widehat{C}_{2,4}$ with parameter estimation based on weighted least-squares estimation. (10 p)

Problem 4

Consider the data in Problem 2. Consider Mack's distribution-free chain ladder model. Compute the conditional mean-square error of prediction, conditional on data $\mathcal{D}$, for the predictor $\widehat{C}_{2,4}$. The answer should be expressed in terms of observed data and model parameters. (10 p)

Problem 5

Consider an insurer's liability cashflow (C_1, C_2, C_3), where C_t denotes the aggregate claims payment for year t, paid t years from today. Suppose that the current (time 0) continuously compounded interest rate is r for all maturity times and that in one year from today it has changed to $r + \Delta r$, where Δr is a random variable seen from today, independent of the liability cashflow. Suppose that the liability cashflow evolves according to $C_{t+1} = f_t C_t + \sigma_t \varepsilon_{t+1}$, where ε_{t+1} is a standard normally distributed random variable, independent of C_s for $s \leq t$. The parameters f_t and σ_t are assumed to be known at time 0. The new information that will be available at time 1 corresponds to observing C_1 and Δr.

Suppose that the insurer's assets correspond to an amount A_0 invested in 1-year risk-free bonds. Suppose that liability values are computed using traditional actuarial valuation (based on discounted expected cashflows). In order to assess solvency, the inequality $\mathrm{VaR}_{0.005}(X) \leq 0$ is assessed. Express X as a function known at time 0 evaluated at the pair $(C_1, \Delta r)$ of random variables observable at time 1. (10 p)

Problem 1

$$\begin{aligned}
\mathrm{E}\left[\sum_{k=1}^{N}\max(X_k-c,0)\right] &= \mathrm{E}\left[\mathrm{E}\left[\sum_{k=1}^{N}\max(X_k-c,0)\mid N\right]\right]\\
&= \mathrm{E}[N]\,\mathrm{E}[\max(X-c,0)] = \lambda\,\mathrm{E}[\max(X-c,0)]
\end{aligned}$$

Moreover,

$$\begin{aligned}
\mathrm{E}[\max(X-c,0)] &= \int_0^\infty \mathrm{P}(\max(X-c,0)>t)dt = \int_0^\infty \mathrm{P}(X>c+t)dt\\
&= \int_c^\infty \mathrm{P}(X>u)du = \int_c^\infty (1+u/\gamma)^{-\alpha}du\\
&= \gamma\int_{1+c/\gamma}^\infty x^{-\alpha}dx = \frac{\gamma}{\alpha-1}\left(1+\frac{c}{\gamma}\right)^{-\alpha+1}
\end{aligned}$$

Problem 2

$$\begin{aligned}
\mathrm{Var}(C_{3,4}\mid\mathcal{D}) &= \mathrm{Var}(C_{3,4}\mid C_{3,1},C_{3,2})\\
&= \mathrm{E}[\mathrm{Var}(C_{3,4}\mid C_{3,1},C_{3,2},C_{3,3})\mid C_{3,1},C_{3,2}]\\
&\quad + \mathrm{Var}(\mathrm{E}[C_{3,4}\mid C_{3,1},C_{3,2},C_{3,3}]\mid C_{3,1},C_{3,2})\\
&= \sigma_3^2\,\mathrm{E}[C_{3,3}^2\mid C_{3,1},C_{3,2}] + f_3^2\,\mathrm{Var}(C_{3,3}\mid C_{3,1},C_{3,2})\\
&= \sigma_3^2\big(\mathrm{Var}(C_{3,3}\mid C_{3,1},C_{3,2}) + \mathrm{E}[C_{3,3}\mid C_{3,1},C_{3,2}]^2\big) + f_3^2\sigma_2^2C_{3,2}^2\\
&= \sigma_3^2(\sigma_2^2+f_2^2)C_{3,2}^2 + f_3^2\sigma_2^2C_{3,2}^2
\end{aligned}$$

Problem 3

Consider the regression equation $\widetilde{Y} = f_3\widetilde{A} + \sigma_3\widetilde{D}\varepsilon$ written as

$$\begin{bmatrix} C_{0,4}\\ C_{1,4}\end{bmatrix} = f_3\begin{bmatrix} C_{0,3}\\ C_{1,3}\end{bmatrix} + \sigma_3\begin{bmatrix} C_{0,3} & 0\\ 0 & C_{1,3}\end{bmatrix}\begin{bmatrix} \varepsilon_{0,4}\\ \varepsilon_{1,4}\end{bmatrix}$$

With $Y := \widetilde{D}^{-1}\widetilde{Y}$ and $A := \widetilde{D}^{-1}\widetilde{A}$ we get the standard regression equation $Y = f_3A + \sigma_3\varepsilon$ and the least squares estimate $\widehat{f}_3 = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}Y$, where $A^{\mathrm{T}}A = 2$ and

$$A^{\mathrm{T}}Y = \widetilde{A}^{\mathrm{T}}\widetilde{D}^{-2}\widetilde{Y} = \begin{bmatrix} C_{0,3} & C_{1,3}\end{bmatrix}\begin{bmatrix} C_{0,3}^{-2} & 0\\ 0 & C_{1,3}^{-2}\end{bmatrix}\begin{bmatrix} C_{0,4}\\ C_{1,4}\end{bmatrix} = \frac{C_{0,4}}{C_{0,3}} + \frac{C_{1,4}}{C_{1,3}}$$

Since

$$\mathrm{E}[C_{2,4}\mid\mathcal{D}] = \mathrm{E}[C_{2,4}\mid C_{2,1},\dots,C_{2,3}] = C_{2,3}f_3$$

we get the predictor $C_{2,3}\widehat{f}_3$ with $\widehat{f}_3$ as expressed above.

Problem 4

$$\begin{aligned}\mathrm{E}[(C_{2,4}-\widehat{C}_{2,4})^2 \mid \mathcal{D}] &= \mathrm{Var}(C_{2,4} \mid \mathcal{D}) + (\mathrm{E}[C_{2,4} \mid \mathcal{D}] - \widehat{C}_{2,4})^2 \\ &= \sigma_3^2 C_{2,3} + (f_3 - \widehat{f}_3)^2 C_{2,3}^2,\end{aligned}$$

where

$$\widehat{f}_3 = \frac{C_{0,4}+C_{1,4}}{C_{0,3}+C_{1,3}}$$

Problem 5

$X = A_1 - L_1$, where $A_1 = A_0 e^r$ and

$$L_1 = C_1 + e^{-(r+\Delta r)\cdot 1}\,\mathrm{E}[C_2 \mid \mathcal{F}_1] + e^{-(r+\Delta r)\cdot 2}\,\mathrm{E}[C_3 \mid \mathcal{F}_1]$$

with

$$\begin{aligned}\mathrm{E}[C_2 \mid \mathcal{F}_1] &= \mathrm{E}[C_2 \mid C_1] = f_1 C_1, \\ \mathrm{E}[C_3 \mid \mathcal{F}_1] &= \mathrm{E}[\mathrm{E}[C_3 \mid C_1, C_2] \mid C_1] = \mathrm{E}[f_2 C_2 \mid C_1] = f_2 f_1 C_1\end{aligned}$$

Hence,

$$X = A_0 e^r - C_1\big(1 + f_1 e^{-(r+\Delta r)} + f_2 f_1 e^{-(r+\Delta r)\cdot 2}\big)$$