MATEMATISKA INSTITUTIONEN
STOCKHOLM UNIVERSITET
Avd. Matematik
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Tentamensskrivning i
Numerisk analys
Teoriprov
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Argue carefully. You are allowed to use intermediate results in the preceding problem(s) which you were not able to solve.

There are 6 problems and 16 points (p) each except the last problem with 20p.
Grades: A: $p \geq 90$; B: $80 \leq p \leq 89$; C: $70 \leq p \leq 79$; D: $60 \leq p \leq 69$; E: $50 \leq p \leq 59$;

(1) Consider the problem of solving the equation $Ax = b$ with

$$A = \begin{pmatrix} 2 & 10 & 0 & -1 \\ 0 & -1 & 1 & 5 \\ 5 & 1 & 0 & 0 \\ -1 & 0 & 10 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 30 \\ 25 \\ 10 \\ 70 \end{pmatrix}.$$

Show that the system of equations can be converted to an equivalent system with the coefficient matrix being strictly diagonally dominant. Provide a convergent iterative method. Argue why it converges.

(2) A numerical derivation gives

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \left(\frac{h^2}{3!} f^{(3)}(x) + \frac{h^4}{5!} f^{(5)}(x) + \cdots\right), \ h > 0.$$

Richardson's extrapolation gives

$$f'(x) = \frac{Af(x+2h) + Bf(x+h) + Cf(x-h) + Df(x-2h)}{12h} + O(h^\alpha).$$

Determine the coefficients A, B, C, D and the exponent α.

(3) Let $e^\top = (\underbrace{1, 1, ..., 1}_{n})$ and $b^\top = (1, 0, ..., 0)$ be vectors with n and m components respectively and $m > n$. Let the $m \times n$ matrix A have n columns $(1, \delta, 0, ...0)^\top$, $(1, 0, \delta, , ...0)^\top$, ... $(1, 0, ..., 0, ...\delta)^\top$ each with m components. Assume δ is very small, say 10^{-16}.

(a) Show that $A^\top A = \delta^2 I_n + ee^\top$ and hence it is positive semidefinite.
(b) Show that the maximum eigenvalue of $A^\top A$ is $\delta^2 + n$ and the smallest eigenvalue of $A \top A$ is δ^2. Compute further the condition number $\kappa_2(A^\top A)$.
(c) Argue the why the normal equation approach is not recommended for solving least squares problem (the over-determined system $Ax = b$). Suggest an alternative way to solve it.

(**Partial points will be given for solution with small m and n.**)

(4) Show that the error to compute the integral $I = \displaystyle\int_0^2 \frac{dx}{1+x^2}$ by trapezoidal rule T_n is

$$-\frac{h^2(b-a)}{12} f''(\xi_n). \quad \text{for some } \xi_n \text{ in the interval [0,2]}.$$

How large should n be so that

$$|I - T_n| \leq 5 \cdot 10^{-6}?$$

(5) Consider the initial value problem $y'(x) = -y$, $y(0) = 1$.

(a) Determine an explicit expression for y_n obtained by Euler's metod with step length h.
(b) For which values of h is the sequence $y_0, y_1, y_2, ...,$ bounded?
(c) Compute $\displaystyle\lim_{h \to 0} \frac{y(x, y) - e^{-x}}{h}$.

You have finished the exam if your homework point $p_h \geq 15$ (i.e. p=20). There are four parts in next problem. Do one part if $p_h \in [10, 15)$ (i.e. p=10); do two parts if $p_h \in [5, 10)$ (i.e. p=5). Note that all your p_h will be added.

(6) (a) Determine the coefficients A_0, A_0 and the points x_0 and x_1 in the formula

$$\int_{-1}^{1} f(x)dx \approx A_0 f(x_0) + A_1 f(x_1)$$

such that this is exact for polynomial of degree ≤ 3.

(b) What is the relation between the points x_0 and x_1 and the polynomial $p_2(x) = \frac{1}{2}(3x^2 - 1)$? Show that $1, x, p_2$ form an orthogonal basis in the vector space $P_2(-1, 1)$, the set of real polynomials of degree ≤ 2 equipped with the inner product $\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)g(x)dx$.

(c) Show that the formula derived in (a) is the same as that derived by Lagrange interpolating polynomial to approximate f using the zeros of $p_2(x)$ as interpolating points.

(d) Use your formula to approximate $\int_1^{3/2} x^2 \ln x \, dx$. (Leave the ln and square root as they are.)

To get the graded paper fill in the formula at
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