Exam on December 3, Numerical analysis I, 2021

(1) Consider the problem of solving the equation $Ax = b$ with

$$A = \begin{pmatrix} 2 & 10 & 0 & -1 \\ 0 & -1 & 1 & 5 \\ 5 & 1 & 0 & 0 \\ -1 & 0 & 10 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 30 \\ 25 \\ 10 \\ 70 \end{pmatrix}.$$

Show that the system of equations can be converted to an equivalent system with the coefficient matrix being strictly diagonally dominant. Provide a convergent iterative method. Argue why it converges.

Solution. By swapping the rows we have an equivalent system of equations

$$A = \begin{pmatrix} 5 & 1 & 0 & 0 \\ 2 & 10 & 0 & -1 \\ -1 & 0 & 10 & 0 \\ 0 & -1 & 1 & 5 \end{pmatrix}, \quad b = \begin{pmatrix} 10 \\ 30 \\ 70 \\ 25 \end{pmatrix}.$$

which is strictly diagonally dominant. This implies that Jacobi's method and Gauss-Seidel methods are convergent iterative methods. See for example DB p.192.

(2) A numerical derivation gives

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \left(\frac{h^2}{3!}f^{(3)}(x) + \frac{h^4}{5!}f^{(5)}(x) + \cdots\right), \ h > 0.$$

Richardson's extrapolation applied to compute the derivative $f'(x)$ gives

$$f'(x) = \frac{Af(x+2h) + Bf(x+h) + Cf(x-h) + Df(x-2h)}{12h} + O(h^\alpha).$$

Determine the coefficients A, B, C, D and the exponent α.

Solution.. Let $D(h) = \frac{f(x+h)-f(x-h)}{2h}$. Then

$$D(h) = f'(x) + \left(\frac{h^2}{3!}f^{(3)}(x) + \frac{h^4}{5!}f^{(5)}(x) + \cdots\right),$$
$$D(2h) = f'(x) + \left(\frac{(2h)^2}{3!}f^{(3)}(x) + \frac{(2h)^4}{5!}f^{(5)}(x) + \cdots\right),$$

Which yields

$$\frac{4D(h) - D(2h)}{3} = f'(x) + O(h^4).$$

So $\alpha = 4$. Witte the LHS explicitly, we have $A = -1, B = 8, C = -8, D = 1$.

(3) Let $e^\top = (\underbrace{1, 1, ..., 1}_{n})$ and $b^\top = (1, 0, ..., 0)$ be vectors with n and m components respectively and $m > n$. Let the $m \times n$ matrix A have n columns $(1, \delta, 0, ...0)^\top$, $(1, 0, \delta, , ...0)^\top$, ... $(1, 0, ..., 0, ...\delta)^\top$ each with m components. Assume δ is very small, say 10^{-16}.

(a) Show that $A^\top A = \delta^2 I_n + ee^\top$ and hence it is positive semidefinite.

(b) Show that the maximum eigenvalue of $A^\top A$ is $\delta^2 + n$ and the smallest eigenvalue of $A \top A$ is δ^2. Compute further the condition number $\kappa_2(A^\top A)$.

(c) Argue the why the normal equation approach is not recommended for solving least squares problem (the over-determined system $Ax = b$). Suggest an alternative way to solve it.

Solution. The matrix $A^\top A = \delta^2 I_n + ee^\top$ is positive definite because $\delta^2 I$ is positive definite and $ee^\top$ is positive semidefinite. Since $ee^\top$ is a rank one matrix, there is only one non-zero eigenvalues $e^\top e = n$. Thus the eigenvalues of $A^\top A = \delta^2 I_n + ee^\top$ are $\delta^2 + n$ and δ^2. So the first part of (b) follows. Then $\kappa_2(A^\top A) = \frac{\delta^2+n}{\delta^2}$. This will be very large if δ^2 is very small. So the normal equation $A^\top Ax = A^\top b$, to solve least squares problem, is very badly conditioned. To avoid this situation we can use orthogonalization methods. See examples 5.7.2 nd 5.7.4 in DB.

(4) Show that the error to compute the integral $I = \displaystyle\int_0^2 \frac{dx}{1+x^2}$ by trapezoidal rule T_n is

$$-\frac{h^2(b-a)}{12} f''(\xi_n). \quad \text{for some } \xi_n \text{ in the interval [0,2]}.$$

How large should n be so that

$$|I - T_n| \leq 5 \cdot 10^{-6}?$$

Solution. Let $E_n = I - T_n$. Now

$$E_n = -\frac{h^2(b-a)}{12} f''(\xi_n), \quad \text{for some } \xi_n \text{ in [0,2]}$$

where $a = 0$, $b = 2$ and $f(x) = \frac{1}{1+x^2}$. Differentiating f twice yields $f''(x) = \dfrac{-2+6x^2}{(1+x^2)^3}$. This gives

$$|f''(x)| \leq |-2+6x^2| \geq |-2-6x^2| \leq 2$$

Plugging this in E_n we obtain

$$|E_n| \leq \frac{2h^2}{12} \cdot 2 = \frac{h^2}{3}$$

which should be less than $5 \cdot 10^{-6}$, dvs $h^2/3 \leq 5 \cdot 10^{-6}$, i.e. $h \leq 0.003873$. Now

$$n = 2/h \geq 516.4 \quad \Longrightarrow \quad n \geq 517.$$

(5) Consider the initial value problem $y'(x) = -y$, $y(0) = 1$.
   (a) Determine an explicit expression for y_n obtained by Euler's metod with step length h.
   (b) For which values of h is the sequence $y_0, y_1, ...$ bounded?
   (c) Compute $\displaystyle\lim_{h\to 0} \frac{y(x,y) - e^{-x}}{h}$.

Solution. Plugging this particular function and the initial value in Euler's method we get $y_n = (1-h)^n$. The factor $|1-h| < 1$ gives step size $0 < 2$. Note in this case $h = 2$ will work too.

A straightforward computation yields

$$\frac{y(x,h) - e^{-x}}{h} = \frac{(1-h)^{x/h} - e^{-x}}{h} = \frac{e^{\frac{x\ln(1-h)}{h}} - e^{-x}}{h}$$

$$= \frac{e^{-x}(e^{x(\frac{\ln(1-h)}{h}+1)} - 1)}{h} = e^{-x} \cdot \frac{e^{x(\frac{\ln(1-h)}{h}+1)} - 1}{x(\frac{\ln(1-h)}{h}+1)} \cdot \frac{x(\frac{\ln(1-h)}{h}+1)}{h}$$

$$\to e^{-x} \cdot 1 \cdot x(-1/2) = -\frac{1}{2}xe^{-x} \quad \text{as } h \to 0$$

(6) (a) Derive the formula

$$\int_{-1}^{1} f(x)dx \approx A_0 f(x_0) + A_1 f(x_1)$$

such that this is exact for polynomial of degree ≤ 3.

(b) What is the relation between the points x_0 and x_1 and the polynomial $p_2(x) = \frac{1}{2}(3x^2 - 1)$? Show that $1, x, p_2$ form an orthogonal basis in the vector space $P_2(-1,1)$, the set of real polynomials of degree ≤ 2 equipped with the inner product $\langle f(x), g(x)\rangle = \int_{-1}^{1} f(x)g(x)dx$.

(c) Show that the formula derived in (a) is the same as that derived by Lagrange interpolating polynomial to approximate f using the zeros of $p_2(x)$ as interpolating points.

(d) Use your formula to approximate $\int_1^{3/2} x^2 \ln x\, dx$. (Leave the ln and square root as they are.)

Solution. (a) Let $q_1 = 1$, $q_2(x) = x$, $q_3(x) = x^2$, $q_3(x) = x^3$. By the requirement

$$2 = \int_{-1}^{1} 1dx = A_0 + A_1,$$

$$0 = \int_{-1}^{1} xdx = A_0x_0 + A_1x_1,$$

$$\frac{2}{3} = \int_{-1}^{1} x^2dx = A_0x_0^2 + A_1x_1^2,$$

$$0 = \int_{-1}^{1} x^3dx = A_0x_0^3 + A_1x_1^3, .$$

The second and the last equation imply $x_0^2 = x_1^2$. Since these are two distinct points we may assume $x_1 = -x_0$ implying $A_0 = A_1$. Then $A_0 = A_1 = 1$. Substituting them in the third equation equation we get $x_1 = -x_0 = \frac{1}{\sqrt{3}}$.

(b) x_0 and $x - 1$ are the zeros of p_2. It is easy to show that

$$\langle 1, x\rangle = \langle 1, p_2\rangle = \langle x, p_2\rangle = 0,$$

(c) Let $\ell(x) = f(x_0)\ell_0(x) + f(x_1)\ell_1(x)$ with $\ell_0(x) = \frac{x-x_1}{x_0-x_1}$ and $\ell_1(x) = \frac{x-x_0}{x_1-x_0}$. And

$$\int_{-1}^{1} \frac{x-x_1}{x_0-x_1}dx = -\frac{2x_0}{x_1-x_0} = 1, \int_{-1}^{1} \frac{x-x_0}{x_1-x_0}dx = \frac{2x_1}{x_1-x_0} = 1.$$

Approximating of f by ℓ and then integrating gives

$$\int_{-1}^{1} f(x)dx \approx f(x_0)\int_{-1} \ell_0(x)dx + f(x_1)\int_{-1}^{1} \ell_1(x)dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right),$$

as desired.

(d) To use the numerical integration formula we obtained, we have to change the variable so that the interval is $[-1,1]$: $t = a + bx$ where $x \in [1, 3/2]$ has to be transformed to $[-1,1]$, whose result is $a = -5, b = 4$. Then $t = -5 + 4x$ or $x = (t+5)/4$, and

$$\int_1^{3/2} x^2 \ln x\, dx = \frac{1}{4}\int_{-1}^{1} \left(\frac{t+5}{4}\right)^2 \ln\left(\frac{t+5}{4}\right) dt$$

$$\approx \frac{1}{4}\left(\left(\frac{-1/\sqrt{3}+5}{4}\right)^2 \ln\left(\frac{-1/\sqrt{3}+5}{4}\right) + \left(\frac{1/\sqrt{3}+5}{4}\right)^2 \ln\left(\frac{1/\sqrt{3}+5}{4}\right)\right)$$

$$= \frac{1}{96}\left((38 - 5\sqrt{3})\ln\frac{5 - \frac{1}{\sqrt{3}}}{4} + (38 + 5\sqrt{3})\ln\frac{5 - \frac{1}{\sqrt{3}}}{4}\right) = 0.192269$$

(Note that the integral is equal to $-\frac{19}{72} + \frac{9}{8}\ln\frac{3}{2} = 0.192259$.)