Department of Mathematics
Stockholm University

MM 5020 - Abstract Algebra

Spring 2023 - final exam

Examiner: Gregory Arone
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- **No** use of textbook, notes, or calculators is allowed.
- Unless told otherwise, you may quote results that were proved in class. When you do, state precisely the result that you are using.
- Be sure to justify your answers, and show clearly all steps of your solutions.
- In problems with multiple parts, results of earlier parts can be used in the solution of later parts, even if you do not solve the earlier parts

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1. For each of the following statements, determine if it is true or false. Give a brief justification or a counterexample.
   (a) (2 points) Every group of order 8 is abelian.
   (b) (2 points) Suppose x and y are elements of some group G. If $x^3 = y^3$ then $x = y$.
2. Let $\sigma \in S_7$ be the following permutation
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 7 & 6 & 1 & 4 & 3 \end{pmatrix}.$$
   (a) (1 point) Write σ in cycle notation (i.e., as a product of disjoint cycles).
   (b) (1 point) Find the order of σ.
   (c) (1 point) Is σ an even permutation?
   (d) (2 points) Find the order of σ^{10}.
3. (4 points) Suppose G is a non-abelian group of or order 2^n, for some n. Prove that G has an element of order 4.
4. (a) (3 points) Prove that a group of order 45 must be abelian.
   (b) (3 points) Prove that a group of order 224 can not be simple.
5. Let R, S be rings, and suppose $f: R \to S$ is a *surjective* homomorphism of rings. Recall that if $I \subset R$ then $f(I)$ denotes the image of I in S. Similarly, if $J \subset S$, then $f^{-1}(J)$ denotes the pre-image of J in R.
   (a) (2 points) Suppose M is a maximal ideal of S. Prove that $f^{-1}(M)$ is a maximal ideal of R.
   (b) (2 points) Suppose I is an ideal of R. Prove that $f(I)$ is an ideal of S.
   (c) (2 points) Show with examples that if f is not surjective, then neither (a) nor (b) need to hold.
6. Let $\mathbb{Q}[x]$ be the polynomial ring over the rationals.
   (a) (2 points) Find the greatest common divisor of the polynomials $x^4 - 1$ and $x^5 - x^3$ in $\mathbb{Q}[x]$.
   (b) (3 points) Is the ideal $(x^4 - 1, x^5 - x^3)$ a maximal ideal of $\mathbb{Q}[x]$?