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MA 5020 - Abstract Algebra
Final Exam (at home)

August 20, 2021

Instructions: You are allowed to consult the textbook. Notes and calculators are not allowed. Searching the internet for solutions is NOT ALLOWED. Unless told otherwise, you may quote results that you learned during the class. When you do, state precisely the result that you are using. Be sure to justify your answers, and show clearly all steps of your solutions. In problems with multiple parts, results of earlier parts can be used in the solution of later parts, even if you do not solve the earlier parts

On the first page, write the following

1. Your name
2. Your personal number
3. Write and sign the following pledge on the first page:

> *On my honor as a student, I have not received help or used inappropriate resources on this exam.*

After this, begin every problem on a new page (but you do not have to begin every part of a problem on a new page).
Unless you have made provisions for extra time, the exam must be uploaded no later 15:00.

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1. Here and throughout the test, $\mathbb{Z}/n$ denotes the cyclic group of order n, S_n is the group of permutations of $\{1, \ldots, n\}$ and $A_n \subset S_n$ is the subgroup of even permutations.
   (a) [3 pts] Which of the following groups of order 60 are isomorphic to each other? Give a brief and clear justification

   1. $\mathbb{Z}/60$, 2. A_5, 3. $\mathbb{Z}/10\times\mathbb{Z}/6$, 4. $\mathbb{Z}/3\times\mathbb{Z}/5\times\mathbb{Z}/4$, 5. $\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/3\times\mathbb{Z}/5$, 6. $A_4\times\mathbb{Z}/5$.

   (b) [1 pt] Does there exist an injective group homomorphism $\mathbb{Z}/21 \hookrightarrow A_{10}$?
   (c) [1 pt] Does there exist an injective group homomorphism $Q_8 \hookrightarrow S_5$? Here Q_8 is the quaternion group.

2. Let G be a group, and $X \subset G$ a subset (not necessarily a subgroup). Recall that the centralizer of X, denoted $C_G(X)$, is defined as follows
$$C_G(X) = \{g \in G \mid gx = xg \text{ for all } x \in X\}.$$
   You can use without proof that $C_G(X)$ is always a subgroup of G.
   (a) [2 pts] Suppose N is a normal subgroup of G. Prove that $C_G(N)$ is a normal subgroup of G.
   (b) [3 pts] Let A, B be (not necessarily normal) subgroups of G. Recall that $AB = \{ab \mid a \in A, b \in B\}$. Prove that $C_G(AB) = C_G(A) \cap C_G(B)$.

3. Let $\mathrm{GL}_3(\mathbb{R})$ be the group of invertible 3×3 matrices. Define a function
$$f\colon \mathrm{GL}_3(\mathbb{R}) \to \mathrm{GL}_3(\mathbb{R})$$
   by the following formula
$$f(A) = \frac{A}{\sqrt[3]{\det(A)}}.$$

(a) [2 pts] Is f a homomorphism?

(b) [2 pts] Is f an injective function?

(c) [2 pts] is f a surjective function?

Let Z_3 consist of matrices of the form $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$, where $a \neq 0$. It is not difficult to check that Z_3 is a normal subgroup of $\mathrm{GL}_3(\mathbb{R})$. You can assume this without proof. Recall also that $\mathrm{SL}_3(\mathbb{R})$ is the group of 3×3 matrices of determinant 1.

(d) [2 pts] Prove that there is a group isomorphism

$$\mathrm{GL}_3(\mathbb{R})/Z_3 \cong \mathrm{SL}_3(\mathbb{R}).$$

4. Let p, q be prime numbers satisfying $3 < p$, $p \equiv 2 (\mathrm{mod}\ 3)$, $q = 2p + 1$. For example, it could be that $p = 11$ and $q = 23$.

   Let G be a group with $3pq$ elements.

   (a) [1 pt] Prove that G has a normal q-Sylow subgroup.

   (b) [4 pts] Prove that G has a normal 3-Sylow subgroup.

5. Let $R = \{a + b\sqrt{7} \mid a, b \in \mathbb{Z}\}$. You can use without proof that R is a subring of the real numbers. Let

$$I = \{a + b\sqrt{7} \in R \mid a \text{ is divisible by } 7\}$$

   (a) [2 pts] Prove that I is an ideal of R.

   (b) [2 pts] Prove that I is a *prime* ideal.

   (c) [2 pts] Describe the quotient ring R/I.

   (d) [1 pt] Is I a maximal ideal?