Department of Mathematics
Stockholm University

MA 5020 - Abstract Algebra

Summer 2022 - Make up exam

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- You may use the text (Dummit and Foote).
- You may **not** use class notes and/or any notes and study guides you have created.
- You may **not** use a calculator, a cell phone or computer.
- You may quote results that are proved in the book. When you do, state precisely the result that you are using, or give a precise pointer to the book.
- Be sure to justify your answers, and show clearly all steps of your solutions.
- In problems with multiple parts, results of earlier parts can be used in the solution of later parts, even if you do not solve the earlier parts

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1. Let S_5 be the group of permutations of the set $\{1, 2, 3, 4, 5\}$. Let $H \subset S_5$ be the subset consisting of permutations σ that satisfy $\sigma(3) = 3$.
   (a) (2 points) Prove that H is a subgroup of S_5.
   (b) (1 point) Find the number of elements in H.
   (c) (2 points) Is H a normal subgroup of S_5?
2. (a) (2 points) Let G be a finite group and let $\mathbb{Z}$ denote the additive group of integers. Prove that there are no non-trivial homomorphisms from G to $\mathbb{Z}$.
   (b) (2 points) How many group homomorphisms are there from $\mathbb{Z}/12$ to $\mathbb{Z}/15$?
3. Let p be a prime.
   (a) (2 points) Suppose G is any group and $N \lhd G$ is a normal subgroup of **index** p. Let $K \subset G$ be any subgroup. Prove that either $K \subset N$ or $KN = G$.
   (b) (3 points) Suppose P is a p-group and $N \lhd P$ is a normal subgroup of **order** p. Prove that $N \subset Z(P)$, i.e., N is in the center of P.
4. (a) (3 points) Prove that every group of order 1225 is abelian. For your convenience: $1225 = 5^2 \cdot 7^2$.
   (b) (3 points) Prove that a group of order 224 can not be simple. For your convenience: $224 = 32 \cdot 7$.
5. (3 points) Let $\mathbb{F}$ be a field.
   (a) (2 points) Prove that there is an isomorphism of rings $\mathbb{F}[x, y]/(x - y^2) \cong \mathbb{F}[z]$.
   (b) (3 points) Prove that the rings $\mathbb{F}[x, y]/(x - y^2)$ and $\mathbb{F}[x, y]/(x^2 - y^2)$ are not isomorphic.
6. Let R be a commutative ring with a unit. Suppose that I and J are co-maximal ideals of R.
   (a) (3 points) Prove that I and J^2 are co-maximal ideals.
   (b) (2 points) Is the assumption that R has a unit necessary in part (a)? Justify your answer with either an argument or a counterexample.