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MA 5020 - Abstract Algebra
Final Exam

Oct 26, 2021

Instructions: Textbooks, notes and calculators are not allowed. You may quote results that you learned during the class. When you do, state precisely the result that you are using. Unless explicitly instructed otherwise, be sure to justify your answers, and show clearly all steps of your solutions. In problems with multiple parts, results of earlier parts can be used in the solution of later parts, even if you do not solve the earlier parts

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1. (a) [2 pts] True or false: if a permutation σ has order 2 then it is an odd permutation. Prove or give a counterexample.

   (b) [2 pts] How many permutations $\sigma \in S_8$ are there, that satisfy $\sigma(1234)(567)\sigma^{-1} = (3572)(486)$? Note: you are not required to list them, just to say how many there are, with a brief and clear justification.

2. Let G be a group, and $H \subset G$ a subgroup. Define the *core* of H as follows

$$\operatorname{Core}_G(H) = \bigcap_{g \in G} gHg^{-1}.$$

   (a) [1 pt] True or false: $\operatorname{Core}_G(H)$ is a normal subgroup of G. No justification required.

   (b) [2 pts] Describe the core of a 2-Sylow subgroup of S_4.

   (c) [2 pts] Suppose that H has index n in G. Prove that the quotient group $G/\operatorname{Core}_G(H)$ is isomorphic to a subgroup of S_n.

3. (a) [2 pts] Let P and Q be two Sylow subgroups of G, for distinct primes p and q. Suppose that $n_p = n_q = 1$ (where n_p denotes, as usual, the number of p-Sylow subgroups of G). Prove that for every $x \in P$ and $y \in Q$, $xy = yx$.

   (b) [2 pts] Suppose G has $p^m q^n$ elements, where p and q are distinct primes. Let P and Q be a p-Sylow and q-Sylow subgroup of G. Suppose that P is contained in the center of G. Prove that G is isomorphic to $P \times Q$.

4. [5 pts] Prove that a group of order $132 = 3 \cdot 4 \cdot 11$ is not simple.

5. Let R, S be integral domains (i.e., commutative rings with a $1 \neq 0$, and no zero divisors). Let $f\colon R \to S$ be a ring homomorphism. You may use without proof the fact that if I is an ideal of S, then $f^{-1}(I)$ is an ideal of R.

   (a) [2 pts] Show that either $f(1) = 1$, or $f(r) = 0$ for all $r \in R$.
   Assume that $f(1) = 1$ in parts (b)-(d) below.

   (b) [2 pts] Suppose I is a principal ideal of S. Does it follow that $f^{-1}(I)$ is a principal ideal of R? Prove or give a counterexample.

   (c) [2 pts] Suppose I is a prime ideal of S. Does it follow that $f^{-1}(I)$ is a prime ideal of R? Prove or give a counterexample.

(d) [2 pts] Suppose that I is a maximal ideal of S. Does it follow that $f^{-1}(I)$ is a maximal ideal of R? Prove or give a counterexample.

6. [4 pts] Let $p(x) = x^3 + ax^2 + bx + 1$, where $a, b \in \mathbb{Z}$. Let (p) be the ideal generated by p in $\mathbb{Q}[x]$. Prove that one, and only one, of the following possibilities must hold

   1. $\mathbb{Q}[x]/(p)$ is a field
   2. $a = b$ or $a + b = -2$.