Department of Mathematics
Stockholm University

MA 5020 - Abstract Algebra

Summer 2024 - final exam

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- **No** use of textbook, notes, or calculators is allowed.
- Unless told otherwise, you may quote results that you learned during the class. When you do, state precisely the result that you are using.
- Be sure to justify your answers, and show clearly all steps of your solutions.

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1. Let S_n denote the symmetric group on n letters, and $\mathbb{Z}/n$ the cyclic group of order n. For each of the following statements, determine if it is true or false. Give a brief justification or a counterexample.
   (a) (2 points) All subgroups of S_4 of order 8 are isomorphic to each other.
   (b) (2 points) If two subgroups of S_4 are each isomorphic to $\mathbb{Z}/4$ then they are conjugate.
   (c) (2 points) If two subgroups of S_4 are each isomorphic to $\mathbb{Z}/2\times\mathbb{Z}/2$ then they are conjugate.
2. (4 points) How many abelian groups of order 500 are there, up to isomorphism? Describe each one of them explicitly, as a product of cyclic groups (There may be more than one way to formulate the answer. It is enough to give one description of each group).
3. (a) (2 points) Let Q be a normal p-subgroup of a finite group G. Prove that Q is contained in every p-Sylow subgroup of G.
   (b) (3 points) Prove that a group of order 132 must have a normal p-Sylow subgroup for some prime p that divides 132.
4. (5 points) Let R be a commutative ring with unit. Let $\mathbb{Z}$ denote the ring of integers. Suppose that there is a non-zero ring homomorphism $\phi\colon R \to \mathbb{Z}$. Prove that R has infinitely many maximal ideals.
5. (5 points) Let $\mathbb{R}$ and $\mathbb{C}$ be the fields of real and complex numbers respectively. Prove that there is an isomorphism of rings $\mathbb{R}[x]/(x^4-1) \cong \mathbb{R}\times\mathbb{R}\times\mathbb{C}$. Construct an explicit isomorphism.
6. (5 points) Let $\mathbb{K}/\mathbb{F}$ be an algebraic field extension. Suppose that $\mathbb{F} \subset R \subset \mathbb{K}$, where R is a *subring* of $\mathbb{K}$ containing $\mathbb{F}$. Prove that R is in fact a subfield of $\mathbb{K}$.