Department of Mathematics
STOCKHOLM UNIVERSITY
Examiner: A. Sola

Written examination
Matematics III Abstract Algebra
17 March 2026

No textbooks or notes allowed. Please explain your reasoning and write out detailed arguments. At least 12 points are needed for a passing grade.

1. **(4 points in total)** (a) Show (providing a complete argument) that if G is a group and the subgroup H has index 2 then H is normal in G. (2p)

   (b) Must a subgroup of index 3 be normal? Is it possible for a subgroup of order 3 to be normal? Give proofs or counterexamples. (2p)

2. **(4 points in total)** Let G be a group and let $A \neq \emptyset$ be a subset of G.

   (a) Define the *centralizer* $C_G(A)$ of A in G and prove that $C_G(A)$ is a subgroup of G. (1p)

   (b) Let the dihedral group D_8 act on itself by conjugation. Which conjugacy classes form the center $Z(D_8)$? Compute $|C_{D_8}(g)|$ for $g \notin Z(D_8)$. (2p)

   (c) Determine the conjugacy classes of D_8. (1p)

3. **(4 points in total)** Show that a group of order 30 has a normal subgroup which is isomorphic to the cyclic group Z_{15}.

4. **(4 points in total)** Let R be a ring with $1 \neq 0$. A non-zero element $a \in R$ is said to be *nilpotent* if there exists $n \in \mathbb{N}$ such that $a^n = 0$.

   (a) Let a be nilpotent. Prove that that $1 + a$ is a unit of R. (2p)

   (b) Does the set of nilpotent elements of R form an ideal? (2p)

5. **(4 points in total)** An ideal I in a commutative ring R is said to be a *primary ideal* of R if

   - I is a proper ideal, and
   - whenever $a, b \in R$ with $ab \in I$ but $a \notin I$, then there exists $n \in \mathbb{N}$ such that $b^n \in I$.

   (a) Give an example of an ideal that is primary but not prime. (1p)

   (b) Prove that an ideal is primary if and only if the quotient ring R/I is non-trivial and has the property that every zero divisor is nilpotent. (3p)

6. **(4 points in total)**

   (a) State *Eisenstein's criterion* for the ring $\mathbb{Z}[x]$ of polynomials with integer coefficients. (1p)

   (b) Let p be a prime and consider the polynomial

   $$R_p(x) = \frac{x^p - 1}{x - 1}.$$

   Is R_p irreducible as an element of $\mathbb{Z}[x]$? (3p)

   (Hint: you may find it helpful to introduce a change of variables.)