MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Make up assignment
MM5020 Abstract Algebra
7.5 hp
August 6, 2025

Please read carefully the general instructions:

- During the exam any textbook, class notes, or any other supporting material is forbidden.
- In particular, calculators are not allowed during the exam.
- In all your solutions show your reasoning, explaining carefully what you are doing. Justify your answers. A correct answer without proper justification will not award full points.
- Use natural language, not just mathematical symbols.
- Use clear and legible writing. Write preferably with a ball-pen or a pen (black or dark blue ink).
- A maximum score of 24 points can be achieved.

GOOD LUCK!

Throughout this exam, for a positive integer n, S_n denotes the symmetric group on n letters, A_n is the alternating group on n letters, and $\mathbb{Z}/n$ is the cyclic group of order n. If n is prime and we want to emphasize the field structure of $\mathbb{Z}/n$, we denote it by $\mathbb{F}_n$.

1. Let G be a group and consider H a subgroup of index 2.

   (a) (2 pts) Write an argument that shows that H is normal in G (To get full points it is not enough to cite the textbook, lectures and homework: you have to provide an argument).

   For an element $h \in H$, denote by $[h]_G$ its conjugacy class in G.

   (b) (2 pts) Show that $[h]_G \subseteq H$.

   (c) (2 pts) Prove the following: If $\mathrm{C}_G(h) \leq H$, then $[h]_G$ splits as the union of two conjugacy classes of H.

2. This exercise aims to show that A_5 is the only simple group of order 60.

   (a) (2 pts) Let G be a simple subgroup of S_n, show that either $G < A_n$ or $|G| \leq 2$.

   (b) (2 pts) Suppose now that G is a simple group of order 60, show that $n_5 = 6$.

   (c) (2 pts) Deduce from (a) and (b) that a simple group of order 60, G, is isomorphic to a subgroup of A_6. Show that $[A_6 : G] = 6$ (You can use without proof that $|A_6| = 360$.)

   (d) (2 pts) In the notation of (c), let S be the set of left cosets of G in A_6. Then G acts on S by left multiplication. Show that the action is nontrivial and admits one fixed point - that is that there is $s \in S$ such that $g \cdot s = s$ for all $g \in G$. Deduce that $G < S_5$ and conclude the proof. (You can use without proof that A_6 is simple)

3. (2 pts) List all the abelian groups of order 24 up to isomorphism (to get full points you should provide ONE group for every isomorphism class).

4. (2 pts) Let R be a unitary commutative ring and consider $I \subseteq R$ an ideal. Given $\varphi : R \to S$ a SURJECTIVE homomorphism of unitary commutative rings, show that $\varphi(I)$ is an ideal of S.

5. Let $\alpha = \sqrt[2]{2 + \sqrt{2}} \in \mathbb{C}$.

   (a) (2 pts) Find the minimal polynomial of α over $\mathbb{Q}$ (to get full points you also have to show that this is indeed the minimal polynomial).

   (b) (2 pts) Compute $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ and find a basis for $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$.

   (c) (2 pts) Express α^{-2} as a $\mathbb{Q}-$linear combination of the elements of the basis provided in (b).