Abstract Algebra - Re-Exam

October 7^{th} 2025

Exercise 1:

Let G be a group, $H < G$.

If $xyx^{-1}y^{-1} \in H \ \forall x, y \in G$, then in particular $ghg^{-1} \in hH = H$ $\forall g \in G$ and $h \in H < G$, hence H is normal in G and

$\forall x, y \in G \quad xyx^{-1}y^{-1} \in H \implies \pi(xyx^{-1}y^{-1}) = 1 \in G/H$

$\implies \pi(x)\pi(y) = \pi(y)\pi(x) \in G/H$

$\implies G/H$ is ab.

where $\pi : G \to G/H$ is the canonical projection

Conversely, if H is normal in G and G/H is ab. then

$xHyH = yHxH \quad \forall x, y \in G$

$\iff (xy)H = (yx)H \quad \forall x, y \in G$

$\iff xyx^{-1}y^{-1} \in H \quad \forall x, y \in G$

Exercise 2.

a) Let $H < G$ be resp a gp and a subgp, and assume $[G:H] = n$

Let $H \backslash G$ be the left cosets for the action of H on G (by conjuga:). Then G acts on $H \backslash G$ via

$$g \cdot \tilde{g}H = (g\tilde{g})H$$

and this action reads as a permuta: of the n elem^t of $H \backslash G$ and defines a group hom $G \to S_n$ whose kernel is given

This defines ... by elements of G that fix all the cosets, namely $g\tilde{g}H = \tilde{g}$ $\forall \tilde{g} \in$ ...

In part $gH = id\, H$ hence $g \in H$. whence $H \subset \ker \varphi$.

b) G is finite and $[G:H] = p$, the smallest prime dividing $|G|$

This def. a gp homomorphism $G \longrightarrow S_p$ whose kernel is denoted by $K \supset H$. Then G/K is isomorphic to a subgp of S_p hence $\# G/K \mid p!$ As $\# G/K \mid \# G$ and p is the smallest prime that divides $\# G$ we actually has $\# G/K = p$

$$\# G/K = [G:H][H:K] = p[H:K]$$

$\Rightarrow [H:K] = 1 \Rightarrow H = K$.

But since K is normal so is H.

Exercise 3:

Let G be a group and H be a subgroup of G

a) $$\begin{aligned} H \times G &\longrightarrow G \\ (h, g) &\longmapsto gh^{-1} \end{aligned}$$ is a group action of H on G as

- $gh^{-1} \in G$ $\forall (h,g) \in H \times G$,
- $\forall h_1, h_2 \in H$ and $\forall g \in G$, $$\begin{aligned} h_1 \cdot (h_2 \cdot g) &= h_1 (g h_2^{-1}) \\ &= g h_2^{-1} h_1^{-1} \\ &= g (h_1 h_2)^{-1} \\ &= (h_1 h_2) \cdot g \end{aligned}$$
- $\forall g \in G$ $id \cdot g = g$.

b) Let $g \in G$, $$\begin{aligned} G_g &= \{ h \in H \mid hg = g \} \\ &= \{ h \in H \mid h = 1 \} = \{1\}. \end{aligned}$$

Exercise 4:

a) Let G be a simple group of order $2^4 \cdot 5^n$ for $n \geq 4$.

The number of 5 Sylow divides $2^4 \cdot 5^n$ and is congruent to 1 (mod

thus $n_5 = 2^4 = 16$.

b) G acts on its 5-Sylow by conjugation (as they are all conjugate) and this def a morphism

$$\varphi : G \longrightarrow S_{16}$$

whose kernel is trivial as $g \in \ker \varphi \iff gPg^{-1} = P \;\; \forall P \in Syl_5$

hence $g \in N_G(P) \;\; \forall P \in Syl_5$. But $\bigcap_{P \in Syl_5} N_G(P)$ is normal in G

as if $x \in N_G(P) \; \forall P \in Syl_5$, let $g \in G$

$$g x P x^{-1} g^{-1} = g P' g^{-1} = P'' \quad \text{where } P, P', P'' \in Syl_5$$

because all 5-Sylow are conjugate.

But G is simple hence $\bigcap N_G(P) = \{1\}$ and G is isomorphic to a subgroup of S_{16} via (1^{rst} iso thm applied to φ)

So $\# G \mid \# S_{16} = (2^4)!$, hence a contradiction. Therefore

G is not simple.

Exercise 5:

- First of all A_5, and $S_3 \times \mathbb{Z}/10\mathbb{Z}$ are NOT abelian while $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z}$, $\mathbb{Z}/60\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z}$ are.

- So we are left to investigate whether A_5 and $S_3 \times \mathbb{Z}/10\mathbb{Z}$ are iso on the one hand, same for $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z}$ and $\mathbb{Z}/60\mathbb{Z}$.

is not (for instance

• A_5 is a simple group while $S_3 \times \mathbb{Z}/10\mathbb{Z}$. $\{e\} \times \mathbb{Z}/10\mathbb{Z}$ is a normal subgroup of $S_3 \times \mathbb{Z}/10\mathbb{Z}$ as it is the kernel of $\pi : S_3 \times \mathbb{Z}/10\mathbb{Z} \longrightarrow S_3$ which is a gp ise.

$$(x, y) \longmapsto x$$

• $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z}$ has a an element of order 60, namely $[3]_4 \times [14]_{15}$ while there are no such element in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z}$ as the order of $(a, b, c) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z}$ is given by the least common multiple of its components. The max possible order here is therefore $\mathrm{lcm}(2, 2, 15) = 30$.

So $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z}$ are not iso.

Same for $\mathbb{Z}/60\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z}$.

• $\mathrm{Gcd}(4, 15) = 1$ hence $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z}$ is a cyclic group of order 60, it is therefore iso to $\mathbb{Z}/60\mathbb{Z}$

An iso is given by

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\varphi"] \arrow[d] & (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/15\mathbb{Z}) \\
\mathbb{Z}/60\mathbb{Z} \arrow[ur, "\cong"'] &
\end{tikzcd}$$

$$x \longmapsto ([x]_4, [x]_{15})$$

as $\ker \varphi = \{x \in \mathbb{Z} \mid [x]_4 = [0]_4 \text{ and } [x]_{15} = [0]_{15}\}$

$= \{x \in \mathbb{Z} \mid x \text{ is a mult. of } 4 \text{ and } 15\}$

$= 60\mathbb{Z}$.

Exercise 6:

We are actually asked to show that $x^3 - 3x^2 - x - 3$ is irr. in $\mathbb{F}_5[x]$ (this is because $\mathbb{F}_5$ is a PID)

$$\left(x^3 - 3x^2 - x - 3\right) \in \mathbb{F}_5[x]$$

$$\wr\wr$$

$$P(x) = x^3 + 2x^2 + 4x + 2 \quad \text{in } \mathbb{F}_5[x]$$

Let $a \in \mathbb{F}_5$ be a root, then $a^3 + 2a^2 + 4a + 2 = 0$

$$\iff a^3 = -2(a^2 + 2a + 1)$$

$$\Rightarrow -2 \mid a^3 \Rightarrow a \text{ is either } [2]_5 \text{ or } [4]_5$$

in any case these two are not roots of $P(x)$ (just do a direct check!).

Exercise 7:

1) As K is a field, $K[y]$ is integral and one can build its field of fractions $K(y) = K[y]/(y)$
It indeed contains K as a subfield as K identifies with constant polynomials.

2) For $K(y)$ to be alg. over K, y must be alg over K meaning that y is the root of a non zero polyn. equa. with coeff in K, which is never the case here.