

No textbooks or notes allowed. Please explain your reasoning and write out detailed arguments. At least 12 points are needed for a passing grade.

1. (4 points in total)

- (a) Define the center of a group G . (1p)
- (b) Is it possible for the center of G to be equal to G ? Can the center of a group with more than one element be trivial? Give proofs or counterexamples. (1p)
- (c) Compute the center of the dihedral group D_8 of order 8. (2p)

2. (4 points in total) Let G be a group and let A and B be subgroups of G .

- (a) Show that the product AB is a subgroup if one of A and B is normal. (2p)
- (b) Show that AB is normal if both A and B are normal. (2p)

3. (4 points in total) (a) Define the notion of a Sylow p -subgroup of a group G . (1p).

- (b) Determine the Sylow p -subgroups of the symmetric group S_3 . Are any of these Sylow p -subgroups normal? (3p)

4. (4 points in total)

Let R be a commutative ring.

- (a) Define, for two ideals $I, J \subset R$, the product ideal IJ . (1p)
- (b) Prove that IJ is contained in the ideal $I \cap J$. (2p)
- (c) Can $IJ = I \cap J$? Must $IJ = I \cap J$ hold? Give proofs or counterexamples. (1p)

5. (4 points in total)

- (a) What does it mean to say that an element r in an integral domain R is irreducible? What does it mean to say that r is prime in the integral domain R ? (1p)
- (c) Prove that a prime element of an integral domain is irreducible. (2p)
- (d) Does the converse to (c) hold? Give a proof or a counterexample. (1p)

6. (4 points in total)

- (a) State *Eisenstein's criterion* for the ring $R[x]$, where R is an integral domain. (1p)
- (b) Let $R = \mathbb{Q}[x]$ and let n be a positive integer. Prove that $t^n - x$ is irreducible in $R[t]$. (4p)