Department of Mathematics
Stockholm University

MA 5020 - Abstract Algebra

Summer 2024 - final exam

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- **No** use of textbook, notes, or calculators is allowed.
- Unless told otherwise, you may quote results that you learned during the class. When you do, state precisely the result that you are using.
- Be sure to justify your answers, and show clearly all steps of your solutions.

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1. (a) (2 points) Let G be a group and let $g \in G$ be an element. Suppose that g satisfies $g^3 = e$ and $g^7 = e$, where e is the identity element of G. Prove that $g = e$.

   (b) (3 points) Let G be a group and H a subgroup of G. Show that there is a bijection between the set of left cosets of H and the set of right cosets of H.

2. (a) (3 points) How many isomorphisms are there from the group $\mathbb{Z}/6 \times \mathbb{Z}/5$ to $\mathbb{Z}/3 \times Z/10$?

   (b) (2 points) How many isomorphisms are there from the group $\mathbb{Z}/4 \times \mathbb{Z}/6$ to $\mathbb{Z}/8 \times \mathbb{Z}/3$?

3. (a) (2 points) Prove that there is no simple group of order 312.

   (b) (3 points) Let G be a finite group, let $N \lhd G$ be a normal subgroup, let p be a prime, and let P be a Sylow p-subgroup of G. Prove that $N \cap P$ is a Sylow p-subgroup of N.

4. (5 points) Let $\mathbb{Z}/5[x]$ be the ring of polynomials with coefficients in integers modulo 5. Find $\gcd(x^2 - x - 2, x^3 - 7x + 6)$ in $\mathbb{Z}/5[x]$. Furthermore, find two polynomials $p(x), q(x) \in \mathbb{Z}/5[x]$ such that
$$p(x)(x^2 - x - 2) + q(x)(x^3 - 7x + 6) = \gcd(x^2 - x - 2, x^3 - 7x + 6).$$

5. (5 points) Suppose R is a PID, and $S \subset R$ is a subring containing the unit of R. Is S necessarily a PID? Prove, or give a counterexample.

6. (5 points) Suppose $\mathbb{F} \subset \mathbb{K}$ are fields and $\alpha \in \mathbb{K}$ is an element for which $[\mathbb{F}(\alpha) : \mathbb{F}]$ is odd. Prove that $\mathbb{F}(\alpha^2) = \mathbb{F}(\alpha)$.