MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Make up assignment
MM5020 Abstract Algebra
7.5 hp
March 15th, 2025

Please read carefully the general instructions:

- During the exam any textbook, class notes, or any other supporting material is forbidden.
- In particular, calculators are not allowed during the exam.
- In all your solutions show your reasoning, explaining carefully what you are doing. Justify your answers. A correct answer without proper justification will not award full points.
- Use natural language, not just mathematical symbols.
- Use clear and legible writing. Write preferably with a ball-pen or a pen (black or dark blue ink).
- A maximum score of 24 points can be achieved.

GOOD LUCK!

1. Let G be a group and consider H a subgroup of index 2.

   (a) (2 pts) Write an argument that shows that H is normal in G (To get full points it is not enough to cite the textbook, lectures and homework: you have to provide an argument).

   For an element $h \in H$, denote by $[h]$ its conjugacy class in G.

   (b) (2 pts) Show that $[h] \subseteq H$.

   (c) (2 pts) Prove the following: If $\mathrm{C}_G(h) \not\subseteq H$, then $[h]$ is a conjugacy class of H.

2. Let G be a group acting transitively on a set X. For $x \in X$, denote by G_x the stabilizer of x, and consider the set
$$\Delta_x := \{y \in X \mid g \cdot y = y \text{ for all } g \in G_x\}.$$

   (a) Show that, for all $g \in \mathrm{N}_G(G_x)$, $g \cdot \Delta_x \subseteq \Delta_x$.

   (b) From point (a), we know that the action of G on X induces an action of $\mathrm{N}_G(G_x)$ on Δ_x. Show that this action is transitive. (**Hint:** It is enough to show that, for $y \in \Delta_x$, if $g \cdot y = x$, then $g \in \mathrm{N}_G(G_x)$)

3. (2 pts) Show that there is no simple group of order $870 = 29 \cdot 5 \cdot 3 \cdot 2$.

4. (2 pts) List all the abelian groups of order 60.

5. Let R be a unitary commutative ring and consider $I \subseteq R$ an ideal.

   (a) (2 pts) Show that
$$\sqrt{I} := \{a \in R \mid a^m \in I \text{ for some } m \in \mathbb{Z}, m \geq 0\}$$
   is an ideal of R.

   (b) (2 pts) Show that if I is a prime ideal then $\sqrt{I} = I$.

6. Let $\alpha = \sqrt[3]{2+\sqrt{2}} \in \mathbb{C}$.

   (a) (2 pts) Find the minimal polynomial of α over $\mathbb{Q}$ (to get full points you also have to show that this is indeed the minimal polynomial).

   (b) (2 pts) Compute $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ and find a basis for $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$.

   (c) (2 pts) Express α^{-3} as a $\mathbb{Q}-$linear combination of the elements of the basis provided in (b).