MM5020 Exam March 15 2025

Exercise 1

(a) A group $H < G$ is normal if for all $g \in G$
$gH = Hg$

The index of $H < G$ is the number of left and right coset of H in G. If $[G:H]=2$
Then there are just two cosets
This Now $H = h \cdot H$ for $h \in H$ is a coset
si $= H \cdot h$ both left & right
Since cosets give a partition we have that for $g \notin H$
$gH = G \setminus H = Hg$ and H is normal

(b) $[h] = \{ g \in G \mid g = \alpha h \alpha^{-1} \text{ for some } \alpha \in G \}$

Since $h \in H$ and $H \triangleleft G$ we have that
$\alpha h \alpha^{-1} \in H$ for all $\alpha \in G$. Thus
$[h] \subseteq H$

(c) We want to show that for $\alpha \in G$ there is a $\beta \in H$ such that
$\alpha h \alpha^{-1} = \beta h \beta^{-1}$

If $\alpha \in H$ we just take $\alpha = \beta$
Suppose then that $\alpha \notin H$ let $\gamma \in C_G(h) \setminus H$
then, since $[G:H]=2$ we have that
$\alpha H = \gamma H$ and $\alpha \cdot \gamma^{-1} \in H$

now

$$(\alpha\gamma^{-1}) h (\alpha\gamma^{-1})^{-1} = \alpha\gamma^{-1} h \gamma \alpha^{-1} =$$

$$= \alpha h \alpha^{-1}$$

since $\gamma \in C_G(h)$

thus we take $\beta = \alpha\gamma^{-1}$ and we are done.

Exercise 2

(a) We want to show that for $g \in N_G(G_x)$ and $y \in \Delta_x$ $\quad g \cdot y \in \Delta_x$

This is equivalent to showing that $g \cdot y$ is fixed by G_x

thus let $h \in G_x$ and consider

$$h \cdot (g \cdot y) = (hg) \cdot y$$

since $g \in N_G(G_x)$, we have that $g^{-1}hg \in G_x$

In particular $hg = gh'$ for some $h' \in G_x$

Thus $h \cdot (g \cdot y) = (gh') \cdot y$

$$= g(h' \cdot y)$$

$$= g \cdot y \quad \text{since } y \text{ is fixed by elements of } G_x$$

(b) Let $y \in \Delta_x$ we want to show that $x = g \cdot y$ with $g \in N_G(G_x)$. Since G acts transitive on X we have that there is an element $h \in G$ such that

$$h \cdot y = x$$

Let us show that $h \in N_G(G_x)$. To this aim let $\alpha \in G_x$ we want to show that $h\alpha h^{-1} \in G_x$.

Now

$$(h\alpha h^{-1})\cdot x = (h\alpha h^{-1})\cdot(h\cdot y)$$

$$= (h\alpha)\cdot y$$

$$= h\cdot(\alpha\cdot y) = h\cdot y = x$$

$y \in \Delta_x$ so $\alpha \cdot y = y$ for $\alpha \in G_x$.

Exercise 3 Suppose G is simple

$n_{29} \equiv 1 \mod 29$ and $n_{29} | 30$

If $n_{29} \neq 1$ then $n_{29} = 30$.

This means that there are 30×28 elements of order 30

$n_3 \equiv 1 \mod 3$ and $n_3 | 10 \cdot 29$

$n_3 \neq 1 \Rightarrow n_3 \geq 10$ which means that there are at least 20 elements of order 3

n_5 and n_2 are both at least 6 or 3

Which yields 6×4 24 elements of order 5

and 3 elements of order 2

$\Rightarrow$ there are too many elements since $30 \times 28 + 6 \times 4 + 3 + 20 > 30 \cdot 29$.

Exercise 4 $\quad 60 = 4 \cdot 3 \cdot 5$

The possibilities are

- $\mathbb{Z}/5 \times \mathbb{Z}/3 \times \mathbb{Z}/4 \cong \mathbb{Z}/60$
- $\mathbb{Z}/5 \times \mathbb{Z}/3 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \cong \mathbb{Z}/30 \times \mathbb{Z}/2$

Exercise 5

(a) $0 = 0^1$ so $0 \in \sqrt{I}$

Let $a, b \in \sqrt{I}$ we want to show that $\alpha a + \beta b \in \sqrt{I}$ for all $\alpha, \beta \in R$

Since $a \in \sqrt{I}$ then $a^m \in I$ for some $m \in \mathbb{Z}$, $m > 0$

In the same way $b^k \in I$ for some $k \in \mathbb{Z}$, $k > 0$

$$(\alpha a + \beta b)^{m+k} = \sum_{i=0}^{m+k} \binom{m+k}{i} \alpha^i a^i \beta^{m+k-i} b^{m+k-i}$$

$$= \sum_{i=0}^{m} \binom{m+k}{i} \alpha^i a^i \beta^{m+k-i} \cdot b^{m+k-i}$$

$$+ \sum_{i=m+1}^{m+k} \binom{m+k}{i} \alpha^i \beta^{m+k-i} b^{m+k-i} a^i$$

since if $i \leq m$ $m+k-i \geq k$ which means that

$$b^{m+k-i} = b^n \cdot b^k \in I$$

if $i > m$ then $a^i = a^n a^m \in I$

We deduce that $(\alpha a + \beta b)^{m+k} \in I$ and so

$(\alpha a + \beta b) \in \sqrt{I}$ as we wanted

(b) For any ideal $\sqrt{I} \supseteq I$

Since $a \in I \Rightarrow a^n \in I$ so $\forall n \in \mathbb{Z}, n>0$

and so $a \in \sqrt{I}$

We shall prove the other inclusion where I is prime. Let $a \in \sqrt{I}$ then for some $n>0$ $a^n \in I$

We show by induction on n that $a \in I$

- If $n=1$ there is nothing to prove
- ~~If~~ Assume the results is true for $m=k$.

$I \ni a^{k+1} = a a^k$

Since I is prime either $a \in I$ or $a^k \in I$

The induction assumption warrant at $a^k \in I$ if the latter so $a \in I$.

Exercise 6

(a) $\alpha = \sqrt[3]{2+\sqrt{2}}$

$\Rightarrow \alpha^3 = 2+\sqrt{2} \iff \alpha^3 - 2 = \sqrt{2}$

$\Rightarrow \alpha^6 - 4\alpha^3 + 4 = 2$

$\alpha^6 - 4\alpha^3 + 2 = 0$

We have shown that $\sqrt[3]{2+\sqrt{2}}$ is a root of
$x^6 - 4x^3 + 2$
which is monic. To show that this is indeed the minimal polynomial of α we have to show that this is irreducible, which it is by Eisenstein with $p=2$

$[\mathbb{Q}(\alpha):\mathbb{Q}] = \deg$ of minimal poly of α
$= 6$

$1\ \alpha\ \alpha^2\ \alpha^3\ \alpha^4\ \alpha^5$ yield a basis

$\alpha^6 - 4\alpha^3 = -2$

$\Rightarrow \alpha^3(\alpha^3 - 4) = -2$

$\alpha^3 \underbrace{\left[\frac{-1}{2}(\alpha^3 - 4)\right]}_{} = 1$

this is the inverse of α^3 which is $2 - \frac{1}{2}\alpha^3$ in the coordinate of the basis

Rmk

$1\ \alpha^{-1} \dots \alpha^{-5}$ yields also a basis. To see this it is enough to show that they are α linearly independent. To this aim. let

$$\sum_{i=0}^{5} c_i \alpha^{-i} = 0$$ be a $\mathbb{Q}$ linear combination yielding 0.

$$0 = \alpha^5 \left(\sum_{i=0}^{5} c_i \alpha^{-i} \right) = \sum_{i=0}^{5} c_i \alpha^{5-i}$$

$$= \sum_{k=0}^{5} c_i \alpha^k$$

Since the α^i yields a basis we have that $c_i = 0$ for all i proving the claim.

In this basis α^{-3} is just α^{-3} as a combination of the element of the basis.