Department of Mathematics
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Written examination
Matematics III Abstract Algebra
06 August 2026

No textbooks or notes allowed. Please explain your reasoning and write out detailed arguments. At least 12 points are needed for a passing grade.

1. **(4 points in total)**

   (a) Let G be a group and $H \leq G$ be a subgroup. Give a definition of the normalizer of H in G. (1p)

   (b) Show that $N_G(H)$ is a subgroup of G. Show that H is a normal subgroup of $N_G(H)$. (1p)

   (c) Show that $N_G(H)$ is the largest subgroup of G in which H is normal, in the sense that if $H \leq K$ is normal in K, then K is a subgroup of $N_G(H)$. (1p)

   (d) Show that for any subgroup $K \leq N_G(H)$, the set $KH = \{kh \mid k \in K, h \in H\}$ is a group in which H is normal. (1p)

2. **(3 points in total)** Let G be a finite group of even order $|G| = 2n$, with $n \in \mathbb{N}$.

   (a) Let $H < G$ be a subgroup of order n. Show that H is normal in G. (2p)

   (b) Suppose that there exists two subgroups $H_1, H_2 < G$ of order n such that $H_1 \cap H_2 = \{1\}$, where $1 \in G$ denotes the identity element of G. Show that there exists $a \in G$ with $a \notin H_1$ and $a \notin H_2$ such that $G = H_1 \cup H_2 \cup \{a\}$. (1p)

3. **(5 points in total)**

   The goal of this problem is to show that any group of order 12 contains a characteristic subgroup. (Recall that a subgroup $H < G$ is a characteristic subgroup if for any automorphism φ of G, $\varphi(H) = H$.)

   (a) Give an example of a group of order 12. (1p)

   (b) Give the definition of a Sylow p-subgroup of a finite group. (1p)

   (c) Show that a normal Sylow p-subgroup is a characteristic subgroup. (1p)

   (d) Let G be a group of order 12.

   (i) Suppose that G has a unique Sylow 3-subgroup $P < G$. Show that P is a characteristic subgroup. (1p)

   (ii) Suppose that G has at least two Sylow 3-subgroups. Show then that there are at least 8 elements of order 3 in G. Deduce that G has a unique Sylow 2-subgroup $H < G$, and that H is characteristic. (1p)

4. **(4 points in total)**

   Let R be an integral domain which is finite. The goal of this problem is to show that R is a field.

   (a) Let $r \in R \setminus \{0\}$. Show that the function

   $$\varphi(x) = rx$$

   defines an automorphism of the additive group $(R, +)$. (2p)

   (b) Deduce that R is a field. What are the possible orders of R? (2p)

5. **(4 points in total)**

Let R be a unique factorisation domain (UFD) with field of franctions $K = \mathrm{Frac}(R)$. Let

$$f = a_d X^d + a_{d-1} X^{d-1} + \cdots + a_0 \in R[X]$$

with $a_0 \neq 0$ and $a_d \neq 0$.

(a) Show that if $r = \frac{\alpha}{\beta} \in K$ is a root of f, with $\alpha, \beta \in R$ coprime, then $\alpha | a_0$ and $\beta | a_d$. (2p)

(b) Is the polynomial $X^3 - 4X^2 - \frac{9}{2}X - \frac{5}{2}$ irreducible over $\mathbb{Q}$? (2p)

6. **(4 points in total)**

Let R be a commutative ring with identity $1 \neq 0$, and let $I, J \subset R$ be ideals. Denote by

$$\pi_I \colon R \longrightarrow R/I \quad \text{and} \quad \pi_J \colon R \to R/J$$

the two projections onto the associated quotient rings.

(a) Show that $\pi_I(J) = \frac{I+J}{I} \subset R/I$, and that $\pi_J(I) = \frac{I+J}{J} \subset R/J$. (1p)

b) Deduce that $(R/I)/\pi_I(J)$ and $(R/J)/\pi_J(I)$ are isomorphic. (1p)

We now want to apply this result to give a criterion for when a prime ideal $(p) \subset \mathbb{Z}$ remains prime upon a ring extension. Let $f \in \mathbb{Z}[X]$ be an irreducible monic polynomial and $p \in \mathbb{Z}$ be a prime number.

c) Show that $(\mathbb{Z}[X]/(f))/(p)$ is isomorphic to $\mathbb{F}_p[X]/(\overline{f})$. (Recall that $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ and we write $\overline{f}$ for the reduction of f modulo p.) (1p)

(d) Conclude that $(p) \subset \mathbb{Z}[X]/(f)$ is a prime ideal if and only if $\overline{fx}$ is irreducible in $\mathbb{F}_p[X]$. (1p)