MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Sofia Tirabassi

Exam in
MM5020 - Abstract Algebra
7.5 hp
April 17th, 2024

Please read carefully the general instructions:

- During the exam any textbook, class notes, or any other supporting material is forbidden.
- In particular, calculators are not allowed during the exam.
- In all your solutions show your reasoning, explaining carefully what you are doing. Justify your answers. A correct answer without proper justification will not award full points.
- Use natural language, not just mathematical symbols.
- Use clear and legible writing. Write preferably with a ball-pen or a pen (black or dark blue ink).
- A maximum score of 30 points can be achieved.

GOOD LUCK!

1. For each of the following statements, determine if it is true or false. Give a brief justification or a counterexample.

   (a) (2 pts) Let G be a group and $H \lhd G$. If G/H is abelian, then $Z(G) > H$.

   (b) (2 pts) $\mathbb{Q}/\mathbb{Z}$ (where the operation in both groups is the usual addition) has elements of infinite order.

2. Recall that a subgroup H of a group G is *characteristic* if, for every $\sigma \in \mathrm{Aut}(G)$, we have that $\sigma(H) = H$.

   (a) (1 pt) Show that characteristic subgroups are normal.

   (b) (2 pts) Show that if H is characteristic in N and $N \lhd G$, then $H \lhd G$.

   (c) (2 pts) Show that if $P \in \mathrm{Syl}_p(G)$, then P is characteristic in its normalizer $N_G(P)$.

3. Let $G < S_n$ act transitively on $\{1, 2, \ldots, n\}$.

   (a) (3 pts) If $G_1 = \{g \in G \mid g \cdot 1 = 1\}$, show that $[G : G_1] = n$.

   (b) (2 pts) If G is abelian, then $|G| = n$.

4. Show the following statements:

   (a) (3 pts) There is no simple group of order $312 = 2^3 \cdot 39$.

   (b) (2 pts) There is no simple group of order $200 = 2^3 \cdot 5^2$.

5. Let R be a unitary commutative ring and consider $\mathfrak{J}$, the intersection of all the maximal ideals of R.

   (a) (1 pt) Show that $\mathfrak{J}$ is an ideal of R.

   (b) (2 pts) Show that if $1 - ax$ is not a unit in R for some $a \in R$, then x cannot be in $\mathfrak{J}$.

   (c) (2 pts) Conversely, suppose that $x \notin \mathfrak{J}$. Show that $1 - ax$ is not a unit for a in R. (***Hint:*** *if $x \notin \mathfrak{m}$ for a maximal ideal $\mathfrak{m}$, what can we say about $x + \mathfrak{m}$ in $R/\mathfrak{m}$?*)

6. Consider the polynomial $p(x) = x^3 + x + 1$ in $\mathbb{Z}/5\mathbb{Z}[x]$.

   (a) (2 pts) Explain why $\mathbb{Z}/5\mathbb{Z}[x]/(p(x))$ is a field.
   (**Hint:** you can use the following fact:*a polynomial of degree 2 or 3 is irreducible over a field Fif, and only if it has no roots in F.*)

   (b) (2 pt) Let $\alpha = x + (p(x))\ \mathbb{Z}/5\mathbb{Z}[x]/(p(x))$, and consider the field $F = \mathbb{Z}/5\mathbb{Z}(\alpha)$. Show that $p(x)$ is the minimal polynomial of α over $\mathbb{Z}/5\mathbb{Z}$. Provide a basis of F over $\mathbb{Z}/5\mathbb{Z}$.

   (c) (2 pt) With α as in the previous point, show that $\alpha^4 + \alpha = 4\alpha^2$. Express α^5 as a linear combination of the elements in the chosen basis.